Research Article

A Study on Conservative C-Bochner Curvature Tensor in K-Contact and Kenmotsu Manifolds Admitting Semisymmetric Metric Connection

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The paper deals with the study on conservative C-Bochner curvature tensor in K-contact and Kenmotsu manifolds admitting semisymmetric metric connection, and it is shown that these manifolds are η -Einstein with respect to Levi-Civita connection, and the results are illustrated with examples.

1. Introduction

In 1924, Friedmann and Schouten [1] introduced the idea of semisymmetric linear connection on a differentiable manifold. In 1932, Hayden [2] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semisymmetric metric connection on a Riemannian manifold was published by Yano [3] in 1970. After that the properties of semisymmetric metric connection have been studied by many authors like Amur and Pujar [4], Bagewadi [5], Sharfuddin and Hussain [6], De and Pathak [7], and so forth.

A K-contact manifold is a differentiable manifold with a contact metric structure such that ξ is a Killing vector field ([8, 9]). These are studied by many authors like ([8–11]). The notion of Kenmotsu manifolds was defined by Kenmotsu [12]. Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kaehler manifold N with warping function $f(t) = se^t$, where s is a nonzero constant. For example it is hyperbolic space (-1) . Kenmotsu manifolds were studied by many authors such as Binh et al. [13], Bagewadi and Venkatesha [14].

In this paper we study conservative C-Bochner curvature tensor in K-contact and Kenmotsu manifolds admitting semisymmetric metric connection and obtain results on K-contact and Kenmotsu manifolds with respect to Levi-Civita connection.

2. Preliminaries

An n -dimensional differential manifold M is said to have an almost contact structure (ϕ, ξ, η) if it carries a tensor field ϕ of type $(1,1)$, a vector field ξ , and 1-form η on M , respectively, such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0. \quad (2.1)$$

Thus a manifold M equipped with this structure is called an almost contact manifold [9] and is denoted by (M, ϕ, ξ, η) . If g is a Riemannian metric on an almost contact manifold M such that,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad (2.2)$$

where X and Y are vector fields defined on M , then M is said to have an almost contact metric structure (ϕ, ξ, η, g) and M with this structure is called an almost contact metric manifold and is denoted by (M, ϕ, ξ, η, g) .

If on (M, ϕ, ξ, η, g) the exterior derivative of 1-form η satisfies,

$$d\eta(X, Y) = g(X, \phi Y), \quad (2.3)$$

then (ϕ, ξ, η, g) is said to be a contact metric structure and M equipped with a contact metric structure is called contact metric manifold.

If moreover ξ is Killing vector field, then M is called a K-contact manifold [8, 9]. A K-contact manifold is called Sasakian [9], if the relation

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \quad (2.4)$$

holds, where ∇ denotes the operator of covariant connection with respect to g .

An almost contact metric manifold, which satisfies the following conditions,

$$\begin{aligned} (\nabla_X \phi)Y &= -\eta(Y)\phi X - g(X, \phi Y)\xi, \\ \nabla_X \xi &= X - \eta(X)\xi, \end{aligned} \quad (2.5)$$

where ∇ denotes the Riemannian connection of g hold, (M, ϕ, ξ, η, g) is called a Kenmotsu manifold.

In a K-contact manifold M , the following properties hold:

$$\nabla_X \xi = -\phi X, \quad (2.6)$$

$$g(R(\xi, X)Y, \xi) = g(X, Y) - \eta(X)\eta(Y), \quad (2.7)$$

$$R(\xi, X)\xi = -X + \eta(X)\xi, \quad (2.8)$$

$$S(X, \xi) = (n-1)\eta(X), \quad (2.9)$$

where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator of M , respectively.

In a Kenmotsu manifold M , the following properties hold [12]:

$$\nabla_X \xi = X - \eta(X)\xi, \quad (2.10)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.11)$$

$$S(X, \xi) = -(n-1)\eta(X), \quad (2.12)$$

$$Q\xi = -(n-1)\xi, \quad (2.13)$$

where R is the Riemannian curvature tensor, S is the Ricci tensor, and Q is the Ricci operator of M , respectively.

Let M be an n -dimensional Riemannian manifold of class C^∞ with metric tensor g and let ∇ be the Levi-Civita connection on M . A linear connection $\tilde{\nabla}$ in an almost contact metric manifold M is said to be a semisymmetric connection if the torsion tensor T of the connection $\tilde{\nabla}$ satisfies

$$T(X, Y) = \pi(Y)X - \pi(X)Y, \quad (2.14)$$

where π is a 1-form on M with ρ as associated vector field, that is, $\pi(X) = g(X, \rho)$ for any differentiable vector field X on M .

A semisymmetric connection $\tilde{\nabla}$ is called semisymmetric metric connection if it further satisfies $\tilde{\nabla}g = 0$.

In an almost contact manifold semisymmetric metric connection is defined by identifying the 1-form of (2.14) with the contact-form η , that is,

$$T(X, Y) = \eta(Y)X - \eta(X)Y, \quad (2.15)$$

with ξ as associated vector field, that is, $g(X, \xi) = \eta(X)$.

The relation between the semisymmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of M has been obtained by Yano [3], which is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi. \quad (2.16)$$

The above condition satisfies K-contact and Kenmotsu manifolds also.

We denote R , S , and r by curvature tensor, Ricci tensor, and scalar curvature with respect to Levi-Civita connection and correspondingly $\tilde{R}$, $\tilde{S}$, and $\tilde{r}$ with respect to semisymmetric metric connection. If B denotes C-Bochner curvature tensor [15] with respect to Levi-Civita connection $\tilde{B}$ with respect to semisymmetric metric connection is given by

$$\begin{aligned}
\tilde{B}(X, Y)Z &= \tilde{R}(X, Y)Z + \frac{1}{n+3} \\
&\times \left[g(X, Z)\tilde{Q}Y - \tilde{S}(Y, Z)X - g(Y, Z)\tilde{Q}X + \tilde{S}(X, Z)Y \right. \\
&\quad + g(\phi X, Z)\tilde{Q}\phi Y - \tilde{S}(\phi Y, Z)\phi X - g(\phi Y, Z)\tilde{Q}\phi X + \tilde{S}(\phi X, Z)\phi Y \\
&\quad + 2\tilde{S}(\phi X, Y)\phi Z + 2g(\phi X, Y)\tilde{Q}\phi Z + \eta(Y)\eta(Z)\tilde{Q}X - \eta(Y)\tilde{S}(X, Z)\xi \\
&\quad \left. + \eta(X)\tilde{S}(Y, Z)\xi - \eta(X)\eta(Z)\tilde{Q}Y \right] \\
&- \frac{\tilde{D} + n - 1}{n+3} [g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X + 2g(\phi X, Y)\phi Z] \\
&+ \frac{\tilde{D}}{n+3} [\eta(Y)g(X, Z)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y - \eta(X)g(Y, Z)\xi] \\
&- \frac{\tilde{D} - 4}{n+3} [g(X, Z)Y - g(Y, Z)X],
\end{aligned} \tag{2.17}$$

where $\tilde{D} = (n - 1 + \tilde{r}) / (n + 1)$.

Differentiate (2.17) covariantly with respect to $\tilde{\nabla}$ and then contracting we get

$$\begin{aligned}
(\text{Div} \cdot \tilde{B}) &= \frac{n+2}{n+3} \left[(\tilde{\nabla}_X \tilde{S})(Y, Z) - (\tilde{\nabla}_Y \tilde{S})(X, Z) \right] + \frac{1}{n+3} \\
&\times \left[(\tilde{\nabla}_{\phi Y} \tilde{S})(\phi X, Z) - (\tilde{\nabla}_{\phi X} \tilde{S})(\phi Y, Z) + 2(\tilde{\nabla}_{\phi Z} \tilde{S})(\phi X, Y) - (\tilde{\nabla}_{\xi} \tilde{S})(X, Z)\eta(Y) \right. \\
&\quad \left. - \tilde{S}(X, Z)(\tilde{\nabla}_{\xi} \eta)(Y) - (\tilde{\nabla}_{\xi} \tilde{S})(Y, Z)\eta(X) + \tilde{S}(Y, Z)(\tilde{\nabla}_{\xi} \eta)(X) \right] \\
&+ \frac{(n-1)}{2(n+1)(n+3)} \left[(\tilde{\nabla}_Y \tilde{r})g(\phi X, \phi Z) - (\tilde{\nabla}_X \tilde{r})g(\phi Y, \phi Z) \right] \\
&+ \frac{n}{(n+1)(n+3)} \left[(\tilde{\nabla}_{\phi Y} \tilde{r})g(\phi X, Z) - (\tilde{\nabla}_{\phi X} \tilde{r})g(\phi Y, Z) + 2(\tilde{\nabla}_{\phi Z} \tilde{r})g(\phi X, Y) \right] \\
&- \frac{\tilde{r} + (n-1)(n+2)}{(n+1)(n+3)} \left[\eta(\tilde{\nabla}_{\phi X} \xi)g(\phi Y, Z) - \eta(\tilde{\nabla}_{\phi Y} \xi)g(\phi X, Z) - 2\eta(\tilde{\nabla}_{\phi Z} \xi)g(\phi X, Y) \right] \\
&+ \frac{\tilde{r} + (n-1)}{(n+1)(n+3)} \left[(\tilde{\nabla}_{\xi} \eta)(Y)g(X, Z) - (\tilde{\nabla}_{\xi} \eta)(X)g(Y, Z) + (\tilde{\nabla}_Y \eta)(Z)\eta(X) \right. \\
&\quad \left. - (\tilde{\nabla}_X \eta)(Z)\eta(Y) + \{ (\tilde{\nabla}_Y \eta)(X) - (\tilde{\nabla}_X \eta)(Y) \} \eta(Z) \right] \\
&+ \frac{(\tilde{\nabla}_{\xi} \tilde{r})}{(n+1)(n+3)} [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)].
\end{aligned} \tag{2.18}$$

3. Relation between R, S, r and $\tilde{R}, \tilde{S}, \tilde{r}$ in a K-Contact Manifold

A relation between the curvature tensor R and $\tilde{R}$ of type $(1, 3)$ of the connections ∇ and $\tilde{\nabla}$ by using (2.16) is given by

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + [g(\phi Y, Z)X - g(\phi X, Z)Y] + [g(Y, Z)\phi X - g(X, Z)\phi Y] \\ &\quad + [g(\phi X, \phi Z)Y - g(\phi Y, \phi Z)X] + [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi. \end{aligned} \quad (3.1)$$

From (3.1), it follows that

$$\tilde{S}(Y, Z) = S(Y, Z) + (n-2)g(\phi Y, Z) - (n-2)g(Y, Z) + (n-2)\eta(Y)\eta(Z), \quad (3.2)$$

where $\tilde{S}$ denotes the Ricci tensor with respect to semisymmetric metric connection and S denotes the Ricci tensor. On contacting (3.2), we get

$$\tilde{r} = r - (n-1)(n-2), \quad (3.3)$$

where $\tilde{r}$ and r are scalar curvatures with respect to semisymmetric metric connection and Levi-Civita connection.

In a Riemannian manifold M , ξ is a Killing vector field in K-contact manifold, that is, S and r are invariant under it, that is,

$$(L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0, \quad (3.4)$$

$$(L_\xi S)(X, Y) = (L_\xi g)(QX, Y) = 0, \quad (3.5)$$

$$(L_\xi r) = 0, \quad (3.6)$$

where L is Lie-derivative. We know that

$$\begin{aligned} (\nabla_\xi S)(Y, Z) &= \xi S(Y, Z) - S(\nabla_\xi Y, Z) - S(Y, \nabla_\xi Z) \\ &= (L_\xi S)(Y, Z) - S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi), \end{aligned} \quad (3.7)$$

by using (3.5) in (3.7) and by virtue of (2.6), we have

$$(\nabla_\xi S)(Y, Z) = 0. \quad (3.8)$$

Now in a K-contact manifold $\tilde{L}$, $\tilde{S}$, and $\tilde{r}$ are with respect to semisymmetric metric connection, that is,

$$\begin{aligned} (\tilde{L}_\xi g)(X, Y) &= 2[g(X, Y) - \eta(X)\eta(Y)] \neq 0, \\ (\tilde{L}_\xi \tilde{S})(X, Y) &= 2(n-2)g(\phi X, Y) + 2g(X, Y) - 2\eta(X)\eta(Y) \neq 0, \\ (\tilde{L}_\xi \tilde{r}) &\neq 0. \end{aligned} \quad (3.9)$$

This shows that it is not K-contact with respect to a semisymmetric metric connection.

Now from (3.2), we have

$$\begin{aligned} (\tilde{\nabla}_W \tilde{S})(Y, Z) &= (\nabla_W S)(Y, Z) + 2(n-2)g(Y, W)\eta(Z) - 2(n-2)\eta(Y)g(Z, \phi W) \\ &\quad - 2(n-2)\eta(Y)\eta(Z)\eta(W) - \eta(Y)S(W, Z) - \eta(Z)S(Y, W) \\ &\quad + (n-1)g(W, Y)\eta(Z) + (n-1)g(W, Z)\eta(Y). \end{aligned} \quad (3.10)$$

Put $W = \xi$ in (3.10), we have

$$\begin{aligned} (\tilde{\nabla}_\xi \tilde{S})(Y, Z) &= (\nabla_\xi S)(Y, Z), \\ (\tilde{\nabla}_\xi \tilde{r}) &= (\nabla_\xi r). \end{aligned} \quad (3.11)$$

But

$$(\nabla_\xi S)(Y, Z) = 0 \implies (\tilde{\nabla}_\xi \tilde{S})(Y, Z) = 0, \quad (3.12)$$

$$(\nabla_\xi r) = 0 \implies (\tilde{\nabla}_\xi \tilde{r}) = 0. \quad (3.13)$$

Put $Z = \xi$ in (3.10), we have

$$\begin{aligned} (\tilde{\nabla}_W \tilde{S})(Y, \xi) &= S(Y, \phi W) - S(Y, W) + (n-1)g(Y, W) - (n-1)g(Y, \phi W) \\ &\quad + 2(n-2)g(\phi Y, \phi W). \end{aligned} \quad (3.14)$$

4. K-Contact Manifold Admitting Semisymmetric Metric Connection with $\text{Div} \cdot \tilde{B} = 0$

Considering $\text{Div} \cdot \tilde{B} = 0$ in (2.18) and putting $X = \xi$ in the equation; using (3.2), (3.3), (3.12), (3.13), and (3.14) and by virtue of (2.1), (2.6), and (2.16), we get

$$\begin{aligned} & \frac{n+2}{n+3} [-S(\phi Y, Z) + (3n-5)g(Z, \phi Y) + S(Y, Z) - (n-1)g(Y, Z)] \\ & - \frac{(n-1)}{2(n+1)(n+3)} [(\nabla_{\xi} r)g(\phi Y, \phi Z)] + \frac{r - (n-1)(n-3)}{(n+1)(n+3)} [g(Y, \phi Z) + g(Y, Z) - \eta(Y)\eta(Z)] \\ & + \frac{(\nabla_{\xi} r)}{(n+1)(n+3)} [\eta(Y)\eta(Z) - g(Y, Z)] = 0. \end{aligned} \quad (4.1)$$

In a K-contact manifold, $\xi[r] = 0$, that is, $\nabla_{\xi} r = 0$ then the above equation reduces to

$$\begin{aligned} & \frac{n+2}{n+3} [-S(\phi Y, Z) + (3n-5)g(Z, \phi Y) + S(Y, Z) - (n-1)g(Y, Z)] \\ & + \frac{r - (n-1)(n-3)}{(n+1)(n+3)} [g(Y, \phi Z) + g(Y, Z) - \eta(Y)\eta(Z)] = 0. \end{aligned} \quad (4.2)$$

Interchanging Y and Z in the above equation then we have

$$\begin{aligned} & \frac{n+2}{n+3} [-S(\phi Z, Y) + (3n-5)g(Y, \phi Z) + S(Z, Y) - (n-1)g(Z, Y)] \\ & + \frac{r - (n-1)(n-3)}{(n+1)(n+3)} [g(Z, \phi Y) + g(Z, Y) - \eta(Y)\eta(Z)] = 0. \end{aligned} \quad (4.3)$$

Adding these equation (4.2) and (4.3), we have

$$S(Y, Z) = \left[\frac{(n-1)(n^2 + 4n - 1) - r}{(n+1)(n+2)} \right] g(Y, Z) + \left[\frac{r - (n-1)(n-3)}{(n+1)(n+2)} \right] \eta(Y)\eta(Z). \quad (4.4)$$

Then the above equation is written as $S(Y, Z) = \alpha g(Y, Z) + \beta \eta(Y)\eta(Z)$, where

$$\alpha = \left[\frac{(n-1)(n^2 + 4n - 1) - r}{(n+1)(n+2)} \right], \quad \beta = \left[\frac{r - (n-1)(n-3)}{(n+1)(n+2)} \right]. \quad (4.5)$$

On contracting (4.4), we get

$$R_1 = \frac{(n-1)[n(n^2 + 4n - 1) - r - (n-3)]}{(n+1)(n+2)}. \quad (4.6)$$

Hence we state the following theorem.

Theorem 4.1. *If in a K-contact manifold the C-Bochner curvature tensor with respect to semisymmetric metric connection is conservative, then the manifold is η -Einstein with respect to Levi-Civita connection and the scalar curvature of such a manifold is given in (4.6).*

5. Example for K-Contact Manifold

Consider the 3-dimensional manifold $M = \{(x, y, z) : (x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3 . Let (E_1, E_2, E_3) be linearly independent at each point of M

$$E_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}. \quad (5.1)$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(E_1, E_2) &= g(E_2, E_3) = g(E_1, E_3) = 0, \\ g(E_1, E_1) &= g(E_2, E_2) = g(E_3, E_3) = 1, \end{aligned} \quad (5.2)$$

where g is given by $g = [(1 - y^2)dx \otimes dx + (1 - x^2)dy \otimes dy + dz \otimes dz]$. Let ξ be the vector field, η be the 1-form, and ϕ be the $(1, 1)$ tensor field defined by

$$\begin{aligned} \xi &= \frac{\partial}{\partial z}, & \eta &= dz + xdy - ydx, \\ \phi E_1 &= -E_2, & \phi E_2 &= E_1, & \phi E_3 &= 0. \end{aligned} \quad (5.3)$$

The linearity property of ϕ and g yields that

$$\begin{aligned} \eta(E_3) &= 1, & \phi^2 U &= -U + \eta(U)E_3, \\ g(\phi U, \phi W) &= g(U, W) - \eta(U)\eta(W), \end{aligned} \quad (5.4)$$

for any vector fields U, W on M . Thus for $E_3 = \xi$. The structure (ϕ, ξ, η, g) defines on M . By definition of Lie bracket, we have

$$[E_1, E_2] = -2E_3, \quad [E_1, E_3] = [E_2, E_3] = 0. \quad (5.5)$$

Let ∇ be Levi-Civita connection with respect to the above metric g given by Koszul formula, that is

$$\begin{aligned} 2g(\nabla_X Y, Z) = & X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \quad (5.6)$$

Then by Koszula formula, we have

$$\begin{aligned} \nabla_{E_1} E_1 &= 0, & \nabla_{E_2} E_2 &= 0, & \nabla_{E_3} E_3 &= 0, \\ \nabla_{E_1} E_2 &= -E_3, & \nabla_{E_2} E_1 &= -E_3, & \nabla_{E_2} E_3 &= -E_1, \\ \nabla_{E_1} E_3 &= E_2, & \nabla_{E_3} E_1 &= E_2, & \nabla_{E_3} E_2 &= -E_1. \end{aligned} \quad (5.7)$$

Clearly one can see that (ϕ, ξ, η, g) is a K-contact structure.

The Ricci tensor $S(X, Y)$ is given by

$$\begin{aligned} S(X, Y) &= \sum_{i=1}^3 g(R(X, E_i)E_i, Y) \\ &= g(R(X, E_1)E_1, Y) + g(R(X, E_2)E_2, Y) + g(R(X, E_3)E_3, Y). \end{aligned} \quad (5.8)$$

The nonzero components of $R(X, E_i)E_i$, where $i = 1, 2, 3$, and by virtue of (5.7) we have

$$\begin{aligned} R(E_2, E_1)E_1 &= -3E_2, & R(E_3, E_1)E_1 &= E_3, \\ R(E_1, E_2)E_2 &= -3E_1, & R(E_3, E_2)E_2 &= E_3, \\ R(E_1, E_3)E_3 &= E_1, & R(E_2, E_3)E_3 &= E_2. \end{aligned} \quad (5.9)$$

Using these in (5.8), we have

$$S(X, Y) = -2g(X, Y) + 4\eta(X)\eta(Y). \quad (5.10)$$

This shows that R^3 is an η -Einstein. This is an example of K-contact manifold which is an η -Einstein.

If $X = Y = Z = E_i$, in (2.18) and by virtue of (3.2), we obtain $(\text{Div} \cdot \tilde{B}) = 0$. Thus Theorem 4.1 holds true.

However, if $X \neq Y \neq Z = E_i$, in (2.18) and by virtue of (3.2), we obtain $(\text{Div} \cdot \tilde{B}) \neq 0$. Hence in general, if $X = \sum_{i=1}^3 a_i E_i$, $Y = \sum_{i=1}^3 b_i E_i$ and $Z = \sum_{i=1}^3 c_i E_i$, then $(\text{Div} \cdot \tilde{B})(X, Y)Z \neq 0$. In this case the converse of Theorem 4.1 does not hold true.

6. Relation between R, S, r and $\tilde{R}, \tilde{S}, \tilde{r}$ in a Kenmotsu Manifold

A relation between the curvature tensor R and $\tilde{R}$ of type $(1, 3)$ of the connections ∇ and $\tilde{\nabla}$ by using (2.16) is given by

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + [g(X, Z)Y - g(Y, Z)X] + 2[g(\phi X, \phi Z)Y - g(\phi Y, \phi Z)X] \\ &\quad + 2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi. \end{aligned} \quad (6.1)$$

From (6.1), it follows that

$$\tilde{S}(Y, Z) = S(Y, Z) - (n-1)g(Y, Z) - 2(n-2)g(\phi Y, \phi Z), \quad (6.2)$$

where $\tilde{S}$ denotes the Ricci tensor with respect to semisymmetric metric connection and S denotes the Ricci tensor. On contacting (6.2), we get

$$\tilde{r} = r - (n-1)(3n-4), \quad (6.3)$$

where $\tilde{r}$ and r are scalar curvatures with respect to semisymmetric metric connection and Levi-Civita connection.

In a Kenmotsu manifold M , ξ is a unit vector field in Kenmotsu manifold, then the following properties hold

$$\begin{aligned} (L_\xi g)(X, Y) &= 2[g(X, Y) - \eta(X)\eta(Y)], \\ (L_\xi S)(X, Y) &= 2[S(X, Y) + (n-1)\eta(X)\eta(Y)], \\ (L_\xi r) &= (\text{Div } \xi)r, \end{aligned} \quad (6.4)$$

where L is a Lie derivative. We know that

$$\begin{aligned} (\nabla_\xi S)(Y, Z) &= \xi S(Y, Z) - S(\nabla_\xi Y, Z) - S(Y, \nabla_\xi Z) \\ &= (L_\xi S)(Y, Z) - S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi). \end{aligned} \quad (6.5)$$

By using (6.4) in (6.5) and by virtue of (2.10) we have

$$(\nabla_\xi S)(Y, Z) = 0, \quad (6.6)$$

$$(\nabla_\xi r) = 0. \quad (6.7)$$

Now in a Kenmotsu manifold $\tilde{L}$, $\tilde{S}$, and $\tilde{r}$ are with respect to semisymmetric metric connection, then we define the properties like

$$\begin{aligned} (\tilde{L}_\xi g)(X, Y) &= 4[g(X, Y) - \eta(X)\eta(Y)], \\ (\tilde{L}_\xi \tilde{S})(X, Y) &= 4[S(X, Y) - (n-1)g(\phi X, \phi Y) - 2(n-2)g(\phi X, \phi Y) \\ &\quad + (n-1)\eta(X)\eta(Y)], \\ \tilde{L}_\xi \tilde{r} &= (\text{Div } \tilde{\xi})\tilde{r}. \end{aligned} \quad (6.8)$$

Now from (6.2), we have

$$\begin{aligned} (\tilde{\nabla}_X \tilde{S})(Y, Z) &= (\nabla_X S)(Y, Z) - (n-1)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y)] \\ &\quad - [\eta(Y)S(X, Z) + \eta(Z)S(X, Y)] + 4(n-2) \\ &\quad \times [\eta(Y)g(\phi X, \phi Z) + g(\phi X, \phi Y)\eta(Z)]. \end{aligned} \quad (6.9)$$

Putting $X = \xi$ in the above equation and by virtue of (6.6) then we have

$$(\tilde{\nabla}_\xi \tilde{S})(Y, Z) = 0, \quad (6.10)$$

$$(\tilde{\nabla}_\xi \tilde{r}) = 0. \quad (6.11)$$

Again putting $Z = \xi$ in (6.9), we have

$$(\tilde{\nabla}_X \tilde{S})(Y, \xi) = -2S(Y, X) - 2(n-1)g(Y, X) + 4(n-2)g(\phi Y, \phi X). \quad (6.12)$$

7. Kenmotsu Manifold Admitting Semisymmetric Metric Connection with $\text{Div} \cdot \tilde{B} = 0$

Considering $\text{Div} \cdot \tilde{B} = 0$ in (2.18) and putting $X = \xi$ in the equation; using (6.2), (6.3), (6.10), and (6.12) and by virtue of (2.10), (2.16) and on simplification we get

$$\begin{aligned} &\frac{n+2}{n+3} [2S(Y, Z) + 2(n-1)g(Y, Z) - 4(n-2)g(\phi Y, \phi Z)] \\ &\quad - \frac{(n-1)}{2(n+1)(n+3)} [(\nabla_\xi r)g(\phi Y, \phi Z)] + \frac{r - (n-1)(3n-4) + (n-1)}{(n+1)(n+3)} \\ &\quad \times [2g(Y, Z) - 2\eta(Y)\eta(Z)] - \frac{(\nabla_\xi r)}{(n+1)(n+3)} [g(\phi Y, \phi Z)] = 0. \end{aligned} \quad (7.1)$$

In a Kenmotsu manifold M , $\nabla_{\xi}r = 0$ then the above equation reduces to

$$\begin{aligned} & \frac{n+2}{n+3} [2S(Y, Z) + 2(n-1)g(Y, Z) - 4(n-2)g(\phi Y, \phi Z)] \\ & + \frac{r - (n-1)(3n-5)}{(n+1)(n+3)} [2g(Y, Z) - 2\eta(Y)\eta(Z)] = 0. \end{aligned} \quad (7.2)$$

On simplifying the above equation we get

$$\begin{aligned} S(Y, Z) = & \left[(n-3) - \frac{r - (n-1)(3n-5)}{(n+1)(n+2)} \right] g(Y, Z) \\ & + \left[-2(n-2) + \frac{r - (n-1)(3n-5)}{(n+1)(n+2)} \right] \eta(Y)\eta(Z). \end{aligned} \quad (7.3)$$

Then the above equation is written as $S(Y, Z) = \alpha g(Y, Z) + \beta \eta(Y)\eta(Z)$, where

$$\alpha = \left[(n-3) - \frac{r - (n-1)(3n-5)}{(n+1)(n+2)} \right], \quad \beta = \left[-2(n-2) + \frac{r - (n-1)(3n-5)}{(n+1)(n+2)} \right]. \quad (7.4)$$

On contracting (7.3), we get

$$r_1 = \left[(n-4)(n-1) - \frac{(n-1)[r - (n-1)(3n-5)]}{(n+1)(n+2)} \right]. \quad (7.5)$$

Hence we state the following theorem.

Theorem 7.1. *If in a Kenmotsu manifold the C-Bochner curvature tensor with respect to semisymmetric metric connection is conservative, then the manifold is η -Einstein with respect to Levi-Civita connection, and the scalar curvature of such a manifold is given in (7.5).*

8. Example for Kenmotsu Manifold

Let $M = \{(x, y, z) \in \mathbb{R}^3\}$. Let (E_1, E_2, E_3) be linearly independent vector fields given by

$$E_1 = e^{-z} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad E_2 = e^{-z} \left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad E_3 = \frac{\partial}{\partial z}. \quad (8.1)$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(E_1, E_2) &= g(E_2, E_3) = g(E_1, E_3) = 0, \\ g(E_1, E_1) &= g(E_2, E_2) = g(E_3, E_3) = 1, \end{aligned} \quad (8.2)$$

where g is given by $g = (e^{2z}/2)(dx \otimes dx + dy \otimes dy) + dz \otimes dz$. Let ξ be the vector field, η be the 1-form, and ϕ be the $(1,1)$ tensor field defined by

$$\begin{aligned} \eta &= dz, & \xi &= E_3 = \frac{\partial}{\partial z}, \\ \phi E_1 &= E_2, & \phi E_2 &= -E_1, & \phi E_3 &= 0. \end{aligned} \quad (8.3)$$

The linearity property of ϕ and g yields that

$$\begin{aligned} \eta(E_3) &= 1, & \phi^2 U &= -U + \eta(U)E_3, \\ g(\phi U, \phi W) &= g(U, W) - \eta(U)\eta(W), \end{aligned} \quad (8.4)$$

for any vector fields U, W on M . By definition of Lie bracket, we have

$$[E_1, E_2] = 0, \quad [E_2, E_3] = E_2, \quad [E_1, E_3] = E_1. \quad (8.5)$$

Let ∇ be the Levi-Civita connection with respect to above metric g which is given by Koszula formula (5.6), and by virtue of it we have

$$\begin{aligned} \nabla_{E_1} E_3 &= E_1, & \nabla_{E_2} E_3 &= E_2, & \nabla_{E_3} E_3 &= 0, \\ \nabla_{E_1} E_2 &= 0, & \nabla_{E_2} E_2 &= -E_3, & \nabla_{E_3} E_2 &= 0, \\ \nabla_{E_1} E_1 &= -E_3, & \nabla_{E_2} E_1 &= 0, & \nabla_{E_3} E_1 &= 0. \end{aligned} \quad (8.6)$$

Clearly one can see that (ϕ, ξ, η, g) is a Kenmotsu structure.

The nonzero components of $R(X, E_i)E_i$, where $i = 1, 2, 3$, and by virtue of (8.6) we have

$$\begin{aligned} R(E_1, E_2)E_2 &= -E_1, & R(E_1, E_3)E_3 &= -E_1, \\ R(E_2, E_1)E_1 &= -E_2, & R(E_2, E_3)E_3 &= -E_2, \\ R(E_3, E_1)E_1 &= -E_3, & R(E_3, E_2)E_2 &= -E_3. \end{aligned} \quad (8.7)$$

The Ricci tensor $S(X, Y)$ is given in (5.8) by virtue of (8.7), and we have

$$S(X, Y) = -2g(X, Y) + 0 \cdot \eta(X)\eta(Y). \quad (8.8)$$

This shows that R^3 is an η -Einstein. This is an example of Kenmotsu manifold which is an η -Einstein.

If $X = Y = Z = E_i$, in (2.18) and by virtue of (6.2), we obtain $(\text{Div} \cdot \tilde{B}) = 0$. Thus Theorem 7.1 holds true.

However, if $X \neq Y \neq Z = E_i$, in (2.18) and by virtue of (6.2), we obtain $(\text{Div} \cdot \tilde{B}) \neq 0$. Hence in general, if $X = \sum_{i=1}^3 a_i E_i$, $Y = \sum_{i=1}^3 b_i E_i$ and $Z = \sum_{i=1}^3 c_i E_i$, then $(\text{Div} \cdot \tilde{B})(X, Y)Z \neq 0$. In this case the converse of Theorem 7.1 does not hold true.

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