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Research Article

A Comparison Principle for Some Types of Elliptic Equations

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In this paper a comparison principle between a continuous viscosity supersolution and a continuous viscosity subsolution is presented. The operator of interest is a fully nonlinear uniformly elliptic one with a gradient term which could be noncontinuous and grow like some BMO functions, as shown in the last section.

1. Introduction

The aim of this paper is to study some fully nonlinear uniformly elliptic equations, where the gradient term could be noncontinuous and growing like some BMO functions. Given an equation $F(X, p, t, x) = 0$, in the case of classical solutions, the comparison principle states the following:

(i) let $u, v \in C^2(\Omega)$ be, respectively, a sub and a supersolution of the equation, if $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω,

as proven in [1] for convex operator and in [2] for uniformly elliptic ones.

Some years later, Jensen in [3], using his known approximation functions, proved such a kind of principle between a viscosity subsolution and a viscosity supersolution, both in $W^{1,\infty}(\Omega)$, for operators which grow linearly in the gradient term and could be uniformly elliptic and nonincreasing in the t variable or degenerate elliptic and decreasing in t. In the same time, in [4], Trudinger was able to compare solutions which are $C(\overline{\Omega}) \cap C^{0,1}(\Omega)$ and $C(\overline{\Omega}) \cap C^{0,\alpha}(\Omega)$.

Then Jensen et al. [5] extended these results considering a zero order term and sub- and supersolutions which are only BUC(Ω). Soon after, Ishii in [6] and Jensen in [3]

independently proved a comparison principle for only continuous bounded functions, where in the first paper the author considers continuous degenerate operators of Isaacs type, which grow linearly in the p variable, while the second concerns uniformly elliptic operators which are Lipschitz in the gradient term and nonincreasing in t.

Then Ishii and Lions in [7] obtain this kind of result between bounded viscosity sub- and supersolution for strictly elliptic operators which grow quadratically in the p variable and are nonincreasing in t. In the same article, these two authors weakened the structure conditions on F and compared continuous bounded functions where at least one has to be locally Lipschitz; then, this result was sharped by Crandall in [8] (see also [9]).

Crandall et al., in their pioneering paper [10] were able to prove such a kind of results between viscosity solutions for degenerate elliptic equations, nonincreasing in t, extending the results obtained before (see also [11, 12]). Then Koike and Takahashi in their work [13] compared L^p-viscosity sub- and supersolutions, when at least one of them is L^p-strong.

In the last years, Bardi and Mannucci in [14] prove a comparison principle for fully nonlinear degenerate elliptic equations that satisfy some conditions of partial nondegeneracy, with linear growth in the gradient term (see also [15]) and Sirakov, [16], has the same result for fully nonlinear equations of Hamilton-Jacobi-Bellman-Isaacs type with unbounded ingredients and the most quadratic growth in p.

Also, it is interesting to mention the series of papers by Birindelli and Demengel, [17–19], where they investigate on singular fully nonlinear equations.

The paper is organized as follows: in the first section some auxiliary results are stated; the second one is characterized by an overview on inf and supconvex envelope; the proof of the main result is given in the third section; finally, in the last one, some examples which justify the interest on this kind of operators are listed.

2. Preliminaries and Auxiliary Results

First of all, it is useful to give some definitions. We say that P is a paraboloid of opening M when

$$P(x) = l_0 + l(x) \pm \frac{M}{2}|x|^2, \tag{2.1}$$

where M is a positive constant, l_0 is a constant, and $l(x)$ is a linear function. P is a convex paraboloid if there is the + sign in (2.1), concave otherwise.

Given two functions u and v on an open set A, v touches u from the above in $x_0 \in A$ when

$$\begin{gathered} u(x) \leq v(x) \quad \forall x \in A, \\ u(x_0) = v(x_0). \end{gathered} \tag{2.2}$$

In this case, one could also say that u touches v from below.

Consider the following:

$$\zeta_\delta = \left\{ x \in \Omega \mid \text{such that for } p \in \overline{B(0,\delta)} \;\; w(z) = w(x) + p(z-x) \;\; \forall z \in \Omega \right\}. \tag{2.3}$$

From [3], we have the following.

Lemma 2.1. *Assume that $w \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ and that $D^2_\lambda w \geq -K_0$ (in the sense of distribution) for all direction λ. If w has an interior maximum then there exist two constants $c_0 > 0$ and $\delta_0 > 0$ such that*

$$meas(\zeta_\delta) \geq c_0 \delta^n \quad \forall \delta < \delta_0. \tag{2.4}$$

Then some lemmas from [3] are needed for the sequel.

Lemma 2.2. *Let $w \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ and assume that*

$$D^2_\lambda w \geq -K_0 \text{ (in the sense of distributions)} \quad \forall \text{ direction } \lambda. \tag{2.5}$$

Then there exists a function $M \in L^1(\Omega; \mathcal{S}(n))$ and a matrix valued measure $\Gamma \in \mathfrak{M}(\Omega; \mathcal{S}(n))$ such that

(1) $D^2 w = M + \Gamma$ *(in distributional sense),*

(2) Γ *is singular with respect to Lebesgue measure,*

(3) $\Gamma(S)$ *is positive semidefinite for all Borel subsets S di Ω,*

(4) $M(x)(\xi,\xi) \geq -K_0|\xi|^2 \;\; \forall \xi \in \mathbb{R}^n$*, for almost every $x \in \Omega$.*

Lemma 2.3. *Let $w \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ and assume that*

$$D^2_\lambda w \geq -K_0 \text{ (in the sense of distribution)} \quad \forall \text{ direction } \lambda. \tag{2.6}$$

If w has an interior maximum then there exists a constant $\delta_0 > 0$ such that for $D^2 w = M + \Gamma$ (as in the previous lemma) as

$$\int_{\zeta_\delta} \left[\frac{(trace(M(x)))^-}{|Dw(x)|} \right]^n dx = \infty \quad \text{per } 0 < \delta < \delta_0. \tag{2.7}$$

Lemma 2.4. *Assume that $w \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ and that (2.5) holds. If $D^2 w = M + \Gamma$ is the decomposition of Lemma 2.2, then for almost every $x \in \Omega$ we have*

$$w(x+z) - w(x) - Dw(x)(z) - \frac{1}{2} M(x)(z,z) = o\left(|z|^2\right). \tag{2.8}$$

Finally, set

$$\begin{aligned} Q(x,y) &= \text{distance}((x,y),\text{graph}(u)), \\ \Omega_\varepsilon &= \{x \in \mathbb{R}^n \mid \text{distance}(x,\mathbb{R}^n \setminus \Omega) > \varepsilon\}, \\ O &= \left\{(x,y) \in \mathbb{R}^{n+1} \mid x \in \Omega_\varepsilon, Q(x,y) \le \varepsilon\right\}, \\ O^+ &= \{(x,y) \in O \mid y > u(x)\}, \\ O^- &= \{(x,y) \in O \mid y < u(x)\}. \end{aligned} \tag{2.9}$$

3. Sup and Inf Convex Envelope

The aim of this paper is to consider equations of the following form:

$$F(X,p,t) + H(p) = 0, \tag{3.1}$$

where F and H are such that the following hold:

(1) F is a continuous function on $\mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R}$;

(2) there exist two constants c_0 and c_1 such that

$$F(M,p,t) - F(N,q,t) \ge c_0 \text{trace}(M-N) - c_1|p-q| \tag{3.2}$$

for all $M \ge N$ and $(p,q,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$;

(3) $F(M,p,t) \le F(M,p,s)$ for all $t > s$ and $(M,p) \in \mathcal{S}^n \times \mathbb{R}^n$;

(4) there exists a positive function g on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$|H(p) - H(q)| \le \gamma g(p,q)|p-q| \tag{3.3}$$

for $\gamma > 0$, all $p,q \in \mathbb{R}^n$, where g has to satisfy the following: if $u,v \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$, $u-v$ has an interior maximum, and $D^2u - D^2v \ge -K_0$ (in the sense of distribution), then there exists a constant $\delta_0 > 0$ such that

$$\int_{\zeta_\delta} \left[\frac{(\text{trace}((M-N)(x)))^-}{g(Du(x),Dv(x))|Du(x)-Dv(x)|}\right]^n dx = \infty \tag{3.4}$$

for $0 < \delta < \delta_0$ and M, N are the functions defined in Lemma 2.2.

We say that the structure condition holds if and only if (2.1)–(4.4) are fulfilled. Define, as in [20], the convex envelope of a function.

Definition 3.1. Let Ω be a bounded domain of $\mathbb{R}^n$, A a subset of Ω such that $\overline{A} \subset \Omega$, $u \in C(\overline{\Omega})$. We call, respectively, sup and inf convex envelope of u as in the following objects:

$$u^{\varepsilon}(x) = \sup_{\overline{A}} \left\{ u(y) - \frac{1}{2\varepsilon}|x-y|^2 \right\},$$
$$u_{\varepsilon}(x) = \inf_{\overline{A}} \left\{ u(y) + \frac{1}{2\varepsilon}|x-y|^2 \right\}. \tag{3.5}$$

Now it is possible to give some properties of the sup convex envelope, noting that similar ones hold for the inf convex envelope.

Proposition 3.2 (see [20]). *Let Ω be a bounded domain of $\mathbb{R}^n$, A a subset such that $\overline{A} \subset \Omega$, $u \in C(\overline{\Omega})$ and $x_0, x_1 \in A$. Then*

(1) $\exists x_0^* \in \overline{A}$: $u^{\varepsilon}(x_0) = u(x_0^*) - (1/2\varepsilon)|x_0^* - x_0|^2$,

(2) $u^{\varepsilon}(x_0) \geq u(x_0)$,

(3) $|u^{\varepsilon}(x_0) - u^{\varepsilon}(x_1)| \leq (3/\varepsilon)\ diam(A)|x_0 - x_1|$,

(4) $0 < \varepsilon < \varepsilon' \Rightarrow u^{\varepsilon}(x_0) \leq u^{\varepsilon'}(x_0)$,

(5) $|x_0^* - x_0|^2 \leq \varepsilon osc_A u$,

(6) $0 < u^{\varepsilon}(x_0) - u(x_0) \leq u(x_0^*) - u(x_0)$.

Theorem 3.3 (see [20]). *Let A be an open set such that $\overline{A} \subset \Omega$, we have*

(1) *$u^{\varepsilon} \in C(A)$ e $u^{\varepsilon} \downarrow u$ uniformly in A for $\varepsilon \to 0$;*

(2) *for all $x_0 \in A$ there exists a concave paraboloid of opening $2/\varepsilon$ which touches u^{ε} from below in $x_0 \in A$.*

Then u^{ε} is $C^{1,1}$ from below in A.
In particular u^{ε} is pointwise differentiable to the second order for almost every $x \in A$.

Before going further, it is useful to give the definition of viscosity solution.

Definition 3.4. A *viscosity subsolution* of $F(D^2u, Du, u, x) + H(Du) = 0$ is a function $u \in \mathrm{USC}(\Omega)$ such that $\forall x_0 \in \Omega$ and $\forall \phi \in C^2(\Omega)$. If $u - \phi$ has a local maximum in x_0, then the following holds

$$F\left(D^2\phi(x_0), D\phi(x_0), u(x_0), x_0\right) + H\left(D\phi(x_0)\right) \geq 0. \tag{3.6}$$

A *viscosity supersolution* of $F(D^2u, Du, u, x) + H(Du) = 0$ is a function $u \in \mathrm{LSC}(\Omega)$ such that $\forall x_0 \in \Omega$ and $\forall \phi \in C^2(\Omega)$. If $u - \phi$ has a local minimum in x_0, then the following holds

$$F\left(D^2\phi(x_0), D\phi(x_0), u(x_0), x_0\right) + H\left(D\phi(x_0)\right) \leq 0. \tag{3.7}$$

A continuous function u is a *viscosity solution* of $F(D^2u, Du, u, x) + H(Du) = 0$ if and only if u is both a viscosity sub- and supersolution.

Remember the following.

(i) USC(Ω) is the set of upper semicontinuous function u in Ω such that $u < \infty$.

(ii) LSC(Ω) is the set of lower semicontinuous function v in Ω such that $v > -\infty$.

Now, to complete this section, note that, as stated in the following theorem (see [5, 20]), the convex envelope of a viscosity solution is a viscosity solution of the same equation.

Theorem 3.5. *Let $u, v \in C(\Omega)$ be bounded functions which are, respectively, viscosity subsolution and supersolution of $F(M, p, t) + H(p) = 0$. If F is uniformly elliptic and nonincreasing then there exist two Lipschitz continuous and bounded functions u^* e v_* and an open set A_1, with $\overline{A}_1 \subset A$, such that u^* is semiconvex, v_* is semiconcave on A_1 which are, respectively, viscosity subsolution and supersolution of $F(M, p, t) + H(p) = 0$ in A_1.*

4. Comparison Principle

Now it is possible to prove the comparison principle

Theorem 4.1 (Comparison Principle). *Let $u, v \in C(\overline{\Omega})$. Assume that u is a viscosity supersolution and v is a viscosity subsolution of*

$$F\left(D^2 w(x), Dw(x), w(x)\right) + H(Dw(x)) = 0. \tag{4.1}$$

Suppose that $u \geq v$ on $\partial\Omega$. If F and H satisfy the structure condition, then

$$u \geq v \quad \text{in } \Omega. \tag{4.2}$$

Proof. Suppose that the contrary holds true as

$$u < v \quad \text{in } \Omega. \tag{4.3}$$

Define $\tilde{v} = v_\varepsilon^+ - \varepsilon$ and $\tilde{u} = u_\varepsilon^- + \varepsilon$.

By Theorem 3.5, $\tilde{v}$ and $\tilde{u}$ are, respectively, viscosity sub- and supersolution of the previous equation. By the properties of sup and inf convex envelope we know that for all direction λ

$$D_\lambda^2 \tilde{u} \leq \frac{K}{\varepsilon}, \quad D_\lambda^2 \tilde{v} \geq -\frac{K}{\varepsilon} \quad \text{in } \Omega_\varepsilon (\text{in the sense of distribution}). \tag{4.4}$$

Let $\tilde{w} = \tilde{v} - \tilde{u}$.

Note that $w \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ and satisfies

$$D_\lambda^2 \tilde{w} \geq -K_0 \quad \text{in } \Omega_\varepsilon \ \forall \text{ direction } \lambda \text{ (in the sense of distribution)}. \tag{4.5}$$

Take

$$\tilde{\zeta}_\delta = \left\{ x \in \Omega \mid \text{for some } p \in \overline{B(0;\delta)},\ \tilde{w}(z) \leq \tilde{w}(x) + p \cdot (z - x) \quad \forall z \in \Omega \right\} \tag{4.6}$$

from Lemma 2.1, for some $\delta_0 > 0$, we have

$$\operatorname{meas}\left(\tilde{\zeta}_\delta\right) \geq c_0 \delta^n \quad \forall \delta < \delta_0. \tag{4.7}$$

Now, from Lemma 2.2, the following hold

$$\begin{aligned} D^2\tilde{v} &= \widetilde{M}^+ + \tilde{\Gamma}^+, \\ D^2\tilde{u} &= \widetilde{M}^- + \tilde{\Gamma}^-, \end{aligned} \tag{4.8}$$

so $\widetilde{M} = \widetilde{M}^+ - \widetilde{M}^-$ and $\tilde{\Gamma} = \tilde{\Gamma}^+ - \tilde{\Gamma}^-$ give a decomposition of $D^2\tilde{w}$.

From Lemma 2.4, for almost every $x \in \tilde{\zeta}_\delta$, there exist $D\tilde{u}$ and $D\tilde{v}$ and

$$\begin{aligned} \tilde{v}(x+z) - \tilde{v}(x) - D\tilde{v}(x)(z) - \frac{1}{2}\widetilde{M}^+(x)(z,z) &= o\left(|z|^2\right), \\ \tilde{u}(x+z) - \tilde{u}(x) - D\tilde{v}(x)(z) - \frac{1}{2}\widetilde{M}^-(x)(z,z) &= o\left(|z|^2\right). \end{aligned} \tag{4.9}$$

Moreover, by the definition of $\tilde{\zeta}_\delta$, we have

$$\widetilde{M}^- \succ \widetilde{M}^+ \quad \text{for almost every } x \in \tilde{\zeta}_\delta. \tag{4.10}$$

Applying the definition of viscosity subsolution and supersolution, it is possible to write

$$F\left(\widetilde{M}^+(x), D\tilde{v}(x), \tilde{v}(x)\right) + H(D\tilde{v}(x)) \geq F\left(\widetilde{M}^-(x), D\tilde{u}(x), \tilde{u}(x)\right) + H(D\tilde{u}(x)). \tag{4.11}$$

By (4.4) of Lemma 2.2, we have

$$\frac{K}{\varepsilon} \succ \widetilde{M}^-(x) \succ \widetilde{M}^+(x) \succ -\frac{K}{\varepsilon} \tag{4.12}$$

for almost every $x \in \tilde{\zeta}_\delta$.

Since F and H satisfy the structural condition, for almost every $x \in \tilde{\zeta}_\delta$, we obtain

$$\begin{aligned} &F\left(\widetilde{M}^+(x), D\tilde{v}(x), \tilde{v}(x)\right) + H(D\tilde{v}(x)) \\ &\quad \leq F\left(\widetilde{M}^-(x), D\tilde{u}(x), \tilde{u}(x)\right) + H(D\tilde{u}(x)) \\ &\qquad - \left(c_0 \ \operatorname{trace}\left(\left(\widetilde{M}^- - \widetilde{M}^+\right)(x)\right) - c_1|D\tilde{u}(x) - D\tilde{v}(x)|\right) \\ &\qquad + \gamma g(D\tilde{u}(x), D\tilde{v}(x))|D\tilde{u} - D\tilde{v}|. \end{aligned} \tag{4.13}$$

Computing

$$c_0 \operatorname{trace}\left(\left(\widetilde{M}^- - \widetilde{M}^+\right)(x)\right) - c_1|D\tilde{u}(x) - D\tilde{v}(x)| - \gamma g(D\tilde{u}(x), D\tilde{v}(x))|D\tilde{u} - D\tilde{v}|, \tag{4.14}$$

we can show that it is nonnegative. In fact, from Lemma 2.3 (fixed a constant C) we have

$$\left(\operatorname{trace}\left(\widetilde{M}(x)\right)\right)^- \geq C|D\tilde{w}(x)| \quad \text{for every } x \in \tilde{\zeta}_\delta \setminus \widetilde{E_1}, \tag{4.15}$$

where meas $(\widetilde{E_1}) = 0$; while from the structural condition

$$\left(\operatorname{trace}\left(\widetilde{M}(x)\right)\right)^- \geq Cg(D\tilde{u}(x), D\tilde{v}(x))|D\tilde{w}(x)| \quad \text{for every } x \in \tilde{\zeta}_\delta \setminus \widetilde{E_2}, \tag{4.16}$$

where meas$(\widetilde{E_2}) = 0$. Then, for $\tilde{E} = E_1 \cup E_2$, since meas$(\tilde{E}) = 0$, we have

$$\begin{aligned} F\left(\widetilde{M}^+(x), D\tilde{v}(x), \tilde{v}(x)\right) + H(D\tilde{v}(x)) < F\left(\widetilde{M}^-(x), D\tilde{u}(x), \tilde{u}(x)\right) + H(D\tilde{u}(x)) \\ \text{for almost every } x \in \tilde{\zeta}_\delta \setminus \tilde{E} \subset \tilde{\zeta}_\delta, \end{aligned} \tag{4.17}$$

which contradicts (4.3). So $u \geq v$ in Ω. □

Remark 4.2. Note that in the last line it is essential that $\|D\tilde{u}\|_{L^\infty}$ and $\|D\tilde{v}\|_{L^\infty}$ are finite.

5. Examples

It is possible to give some examples for $g(Du, Dv)$. Assume that $u, v \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ and let $D^2u = M^+ + \Gamma^+$ and $D^2v = M^- + \Gamma^-$, as in Lemma 2.2.

(1) $g(Du, Dv) = |Du - Dv|^{\alpha-1}$ $(\alpha > 1)$.

Consider

$$\int_{\zeta_\delta} \left[\frac{(\operatorname{trace}(M^+ - M^-))^-}{|Du - Dv|^\alpha}\right]^n \geq \int_{\zeta_\delta} \left[\frac{1}{(|Du| + |Dv|)^{\alpha-1}}\right]^n \left[\frac{(\operatorname{trace}(M^+ - M^-))^-}{|Du - Dv|}\right]^n. \tag{5.1}$$

Since $Du, Dv \in L^\infty$ then

$$\int_{\zeta_\delta} \left[\frac{(\operatorname{trace}(M^+ - M^-))^-}{|Du - Dv|^\alpha}\right]^n \geq C_1 \int_{\zeta_\delta} \left[\frac{(\operatorname{trace}(M^+ - M^-))^-}{|Du - Dv|}\right]^n = \infty, \tag{5.2}$$

where the last equality is given by Lemma 2.3.

(2) $g(Du, Dv) = |\log|Du - Dv + K_0|||Du - Dv|$.

We have

$$\int_{\zeta_\delta}\left[\frac{(\operatorname{trace}(M^+-M^-))^-}{\left|\log|Du-Dv+K_0|\right||Du-Dv|^2}\right]^n \geq \int_{\zeta_\delta}\left[\frac{1}{\left|\log|Du-Dv+K_0|\right|}\right]^n\left[\frac{(\operatorname{trace}(M^+-M^-))^-}{|Du-Dv|^2}\right]^n=\infty \tag{5.3}$$

since $|\log|Du-Dv+K_0||\leq C<\infty$.

(3) $g(Du,Dv)=|\log|Du-Dv+K_0||(|Du|+|Dv|)$.

Arguing as in the previous example, we can obtain the result.

(4) $g(Du,Dv)=|\log|Du-Dv+K_0||$.

In fact, suppose

$$\int_{\zeta_\delta}\left[\frac{(\operatorname{trace}(M^+-M^-))^-}{\left|\log|Du-Dv+K_0|\right||Du-Dv|}\right]^n<\infty, \tag{5.4}$$

then

$$\int_{\zeta_\delta}\left[\frac{(\operatorname{trace}(M^+-M^-))^-}{\left|\log|Du-Dv+K_0|\right|(|Du|+|Dv|)|Du-Dv|}\right]^n \leq \left\|\left[\frac{1}{(|Du|+|Dv|)}\right]^n\right\|_{L^\infty}\int_{\zeta_\delta}\left[\frac{(\operatorname{trace}(M^+-M^-))^-}{\left|\log|Du-Dv+K_0|\right||Du-Dv|}\right]^n<\infty, \tag{5.5}$$

which is a contradiction to the previous example.

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