Research Article

Two-Step Modified Newton Method for Nonlinear Lavrentiev Regularization

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A two step modified Newton method is considered for obtaining an approximate solution for the nonlinear ill-posed equation $F(x) = f$ when the available data are f^δ with $\|f - f^\delta\| \leq \delta$ and the operator F is monotone. The derived error estimate under a general source condition on $x_0 - \hat{x}$ is of optimal order; here x_0 is the initial guess and $\hat{x}$ is the actual solution. The regularization parameter is chosen according to the adaptive method considered by Perverzev and Schock (2005). The computational results provided endorse the reliability and effectiveness of our method.

1. Introduction

Recently George and Elmahdy [1] studied iterative methods for solving the ill-posed operator equation:

$$F(x) = f, \quad (1.1)$$

where $F : D(F) \subseteq X \rightarrow X$ is a nonlinear monotone operator (i.e., $\langle F(x) - F(y), x - y \rangle \geq 0$, for all $x, y \in D(F)$) and X is a real Hilbert space. The convergence analysis in [1] is based on suitably constructed majorizing sequences. Recall that a sequence (t_n) in $\mathbb{R}$ is said to be a majorizing sequence of the sequence $(x_n) \in X$ if

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \quad \forall n \geq 0. \quad (1.2)$$

Throughout this paper, the inner product and the corresponding norm on the Hilbert space X are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively, $D(F)$ is the domain of F and $F'(\cdot)$ is the Fréchet derivative of F .

In application, usually only noisy data f^δ are available, such that

$$\|f - f^\delta\| \leq \delta. \quad (1.3)$$

Then the problem of recovery of $\hat{x}$ from noisy equation $F(x) = f^\delta$ is ill posed, in the sense that a small perturbation in the data can cause large deviation in the solution.

For monotone operators, one usually uses the Lavrentiev regularization method (see [2–5]) for solving (1.1). In this method, the regularized approximation x_α^δ is obtained by solving the operator equation

$$F(x) + \alpha(x - x_0) = f^\delta. \quad (1.4)$$

It is known (cf. [5], Theorem 1.1) that (1.4) has a unique solution $x_\alpha^\delta \in B_r(\hat{x}) := \{x \in X : \|x - \hat{x}\| < r\} \subset D(F)$ for any $\alpha > 0$ provided that $r = \|x_0 - \hat{x}\| + \delta/\alpha$.

The optimality of the Lavrentiev method was proved in [5] under a general source condition on $x_0 - \hat{x}$. However the main drawback here is that, the regularized equation (1.4) remains nonlinear and one may have difficulties in solving them numerically.

Thus in the last few years, more emphasis was put on the investigation of iterative regularization methods (see [6–12]). In this paper, we considered a modified form of the method considered in [1], but we analyse the method as a two step modified Newton Lavrentiev method (TSMNLM). The proposed analysis is motivated by the two step directional Newton method (TSDNM) considered in [13], by Argyros and Hilout, for approximating a zero x^* of a differentiable function F defined on a convex subset $\mathfrak{D}$ of a Hilbert space H with values in $\mathbb{R}$. The TSMNLM for approximating the zero x_α^δ of $F(x) - f^\delta + \alpha(x - x_0) = 0$ is defined as

$$y_{n,\alpha}^\delta = x_{n,\alpha}^\delta - R_\alpha(x_0)^{-1} \left[F(x_{n,\alpha}^\delta) - f^\delta + \alpha(x_{n,\alpha}^\delta - x_0) \right], \quad (1.5)$$

$$x_{n+1,\alpha}^\delta = y_{n,\alpha}^\delta - R_\alpha(x_0)^{-1} \left[F(y_{n,\alpha}^\delta) - f^\delta + \alpha(y_{n,\alpha}^\delta - x_0) \right], \quad (1.6)$$

where $x_{0,\alpha}^\delta := x_0$ and $R_\alpha(x_0) = F'(x_0) + \alpha I$. Here the regularization parameter α is chosen from the finite set $D_M = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \dots < \alpha_M\}$.

The organization of this paper is as follows. Convergence analysis of TSMNLM is given in Section 2, error bounds under an a priori and under an adaptive choice of the regularization parameter is given in Section 3 and Section 4 deals with implementation of the adaptive choice rule. Example and results of computational experiments are given in Section 5. Finally the paper ends with conclusion in Section 6.

2. Convergence Analysis for TSMNLM

We need the following assumptions for the convergence analysis of TSMNLM.

Assumption 2.1 (see [4]). F possesses a locally uniformly bounded Fréchet derivative $F'(\cdot)$ at all x in the domain $D(F)$.

Assumption 2.2 (cf. [4], Assumption 3). There exists a constant $k_0 > 0$ such that for every $x, u \in D(F)$ and $v \in X$, there exists an element $\Phi(x, u, v) \in X$ satisfying

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \quad \|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\| \quad (2.1)$$

for all x, u in $D(F)$ and $v \in X$.

Let

$$e_{n,\alpha}^\delta := \|y_{n,\alpha}^\delta - x_{n,\alpha}^\delta\|, \quad \forall n \geq 0, \quad (2.2)$$

$$\delta_0 < \frac{3\alpha_0}{4k_0}, \quad \rho < \frac{\sqrt{3/2 - 2k_0\delta_0/\alpha_0} - 1}{k_0}, \quad (2.3)$$

$$\gamma_\rho := \frac{\delta_0}{\alpha_0} + \frac{k_0}{2}\rho^2 + \rho.$$

Hereafter, for convenience, we use the notations x_n, y_n , and e_n for $x_{n,\alpha}^\delta, y_{n,\alpha}^\delta$, and $e_{n,\alpha}^\delta$, respectively.

Remark 2.3. Note that if $\|x_0 - \hat{x}\| \leq \rho$, then by Assumption 2.2 it follows that

$$\begin{aligned} e_0 &= \|y_0 - x_0\| = \|R_\alpha(x_0)^{-1}(f^\delta - F(x_0))\| \\ &= \|R_\alpha(x_0)^{-1}(f^\delta - f + F(\hat{x}) - F(x_0) - F'(x_0)(\hat{x} - x_0) + F'(x_0)(\hat{x} - x_0))\| \\ &\leq \|R_\alpha(x_0)^{-1}(f^\delta - f)\| + \left\| R_\alpha(x_0)^{-1} \int_0^1 [F'(\hat{x} + t(\hat{x} - x_0)) - F'(x_0)] dt (\hat{x} - x_0) \right\| \\ &\quad + \|R_\alpha(x_0)^{-1} F'(x_0)(\hat{x} - x_0)\| \\ &\leq \frac{\delta}{\alpha} + \frac{k_0}{2}\rho^2 + \rho \\ &\leq \frac{\delta_0}{\alpha_0} + \frac{k_0}{2}\rho^2 + \rho \\ &= \gamma_\rho. \end{aligned} \quad (2.4)$$

Let

$$q = k_0 r. \quad (2.5)$$

Then $\gamma_\rho / (1 - q) < r$, if

$$r \in \left(\frac{1 - \sqrt{1 - 4k_0\gamma_\rho}}{2k_0}, \frac{1 + \sqrt{1 - 4k_0\gamma_\rho}}{2k_0} \right). \quad (2.6)$$

Theorem 2.4. Let q and r be as in (2.5) and (2.6), respectively, and let γ_ρ be as in (2.3). Let e_n be as in (2.2) and let y_n and x_n be as in (1.5) and (1.6), respectively, with $\delta \in [0, \delta_0]$ and $\alpha \in D_M$. Then

$$(a) \|x_n - y_{n-1}\| \leq q \|y_{n-1} - x_{n-1}\|;$$

$$(b) \|y_n - x_n\| \leq q^2 \|y_{n-1} - x_{n-1}\|;$$

$$(c) e_n \leq q^{2n} \gamma_\rho;$$

$$(d) x_n, y_n \in B_r(x_0).$$

Proof. Observe that if $x_n, y_n \in B_r(x_0)$, then by Assumption 2.2 we have

$$\begin{aligned} x_n - y_{n-1} &= y_{n-1} - x_{n-1} - R_\alpha(x_0)^{-1} (F(y_{n-1}) - F(x_{n-1}) + \alpha(y_{n-1} - x_{n-1})) \\ &= R_\alpha(x_0)^{-1} [R_\alpha(x_0)(y_{n-1} - x_{n-1}) - (F(y_{n-1}) - F(x_{n-1})) - \alpha(y_{n-1} - x_{n-1})] \\ &= R_\alpha(x_0)^{-1} \int_0^1 [F'(x_0) - F'(x_{n-1} + t(y_{n-1} - x_{n-1}))] \times (y_{n-1} - x_{n-1}) dt \\ &= R_\alpha(x_0)^{-1} F'(x_0) \int_0^1 \Phi(x_0, x_{n-1} + t(y_{n-1} - x_{n-1}), y_{n-1} - x_{n-1}) dt, \end{aligned} \quad (2.7)$$

and hence

$$\|x_n - y_{n-1}\| \leq k_0 r \|y_{n-1} - x_{n-1}\|. \quad (2.8)$$

Again observe that if $x_n, y_n \in B_r(x_0)$, by Assumption 2.2 and (2.8), we have

$$\begin{aligned} y_n - x_n &= x_n - y_{n-1} - R_\alpha(x_0)^{-1} (F(x_n) - f^\delta + \alpha(x_n - x_0)) \\ &= R_\alpha(x_0)^{-1} [R_\alpha(x_0)(x_n - y_{n-1}) - (F(x_n) - F(y_{n-1})) - \alpha(x_n - y_{n-1})] \\ &= R_\alpha(x_0)^{-1} \int_0^1 [F'(x_0) - F'(y_{n-1} + t(x_n - y_{n-1}))] dt (x_n - y_{n-1}) \\ &= R_\alpha(x_0)^{-1} F'(x_0) \int_0^1 \Phi(x_0, y_{n-1} + t(x_n - y_{n-1}), x_n - y_{n-1}) dt, \end{aligned} \quad (2.9)$$

and hence

$$\|y_n - x_n\| \leq k_0 r \|x_n - y_{n-1}\| \leq q^2 \|y_{n-1} - x_{n-1}\|. \quad (2.10)$$

Thus if $x_n, y_n \in B_r(x_0)$, then (a) and (b) follows from (2.8) and (2.10), respectively. Now using induction we will prove that $x_n, y_n \in B_r(x_0)$. Note that $x_0, y_0 \in B_r(x_0)$ and hence by (2.8)

$$\begin{aligned}
 \|x_1 - x_0\| &\leq \|x_1 - y_0\| + \|y_0 - x_0\| \\
 &\leq (1 + q)e_0 \\
 &\leq \frac{e_0}{1 - q} \\
 &\leq \frac{\gamma_\rho}{1 - q} \\
 &< r,
 \end{aligned} \tag{2.11}$$

that is, $x_1 \in B_r(x_0)$; again by (2.10)

$$\begin{aligned}
 \|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\| \\
 &\leq q^2 e_0 + (1 + q)e_0 \\
 &\leq \frac{e_0}{(1 - q)} \\
 &\leq \frac{\gamma_\rho}{1 - q} \\
 &< r,
 \end{aligned} \tag{2.12}$$

that is, $y_1 \in B_r(x_0)$. Suppose $x_k, y_k \in B_r(x_0)$ for some $k > 1$. Then since

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\|, \tag{2.13}$$

we will first find an estimate for $\|x_{k+1} - x_k\|$. Note that by (a) and (b) we have

$$\begin{aligned}
 \|x_{k+1} - x_k\| &\leq \|x_{k+1} - y_k\| + \|y_k - x_k\| \\
 &\leq (q + 1)\|y_k - x_k\| \leq (1 + q)q^{2k}e_0.
 \end{aligned} \tag{2.14}$$

Therefore by (2.13) we have

$$\begin{aligned}
 \|x_{k+1} - x_0\| &\leq (1 + q) \left[q^{2k} + q^{2(k-1)} + \cdots + 1 \right] e_0 \\
 &\leq (1 + q) \frac{1 - q^{2k+1}}{1 - q^2} e_0
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{e_0}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r.
\end{aligned} \tag{2.15}$$

So by induction $x_n \in B_r(x_0)$ for all $n \geq 0$. Again by (a), (b), and (2.15) we have

$$\begin{aligned}
\|y_{k+1} - x_0\| &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \\
&\leq q^{2k+2}e_0 + (1+q)\left[q^{2k} + q^{2(k-1)} + \dots + 1\right]e_0 \\
&\leq (1+q)\frac{1-q^{2k+3}}{1-q^2}e_0 \\
&\leq \frac{e_0}{1-q} \\
&\leq \frac{\gamma_\rho}{1-q} \\
&< r.
\end{aligned} \tag{2.16}$$

Thus $y_{k+1} \in B_r(x_0)$ and hence by induction $y_n \in B_r(x_0)$ for all $n \geq 0$. This completes the proof of the Theorem. $\square$

The main result of this section is the following theorem.

Theorem 2.5. *Let y_n and x_n be as in (1.5) and (1.6), respectively, and assumptions of Theorem 2.4 hold. Then (x_n) is Cauchy sequence in $B_r(x_0)$ and converges to $x_\alpha^\delta \in \overline{B_r(x_0)}$. Further $F(x_\alpha^\delta) + \alpha(x_\alpha^\delta - x_0) = f^\delta$ and*

$$\|x_n - x_\alpha^\delta\| \leq \frac{q^{2n}\gamma_\rho}{(1-q)}. \tag{2.17}$$

Proof. Using the relation (b) and (c) of Theorem 2.4, we obtain

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \sum_{i=0}^{m-1} \|x_{n+i+1} - x_{n+i}\| \\
&\leq \sum_{i=0}^{m-1} (1+q)e_{n+i} \\
&\leq \sum_{i=0}^{m-1} (1+q)q^{2(n+i)}e_0
\end{aligned}$$

$$\begin{aligned}
&\leq (1+q) \frac{(q^{2n} - q^{2n+2m})}{1-q^2} e_0 \\
&\leq \frac{q^{2n}}{1-q} e_0 \\
&\leq \frac{q^{2n}}{1-q} \gamma_\rho.
\end{aligned} \tag{2.18}$$

Thus x_n is a Cauchy sequence in $B_r(x_0)$ and hence it converges, say, to $x_\alpha^\delta \in \overline{B_r(x_0)}$. Observe that

$$\begin{aligned}
\|F(x_n) - f^\delta + \alpha(x_n - x_0)\| &= \|R_\alpha(x_0)(x_n - y_n)\| \\
&\leq \|R_\alpha(x_0)\| \|x_n - y_n\| \\
&\leq \frac{1}{\alpha} q^{2n} \gamma_\rho.
\end{aligned} \tag{2.19}$$

Now by letting $n \rightarrow \infty$ in (2.19), we obtain $F(x_\alpha^\delta) - f^\delta + \alpha(x_\alpha^\delta - x_0) = 0$. This completes the proof. $\square$

3. Error Bounds under Source Conditions

The objective of this section is to obtain an error estimate for $\|x_n - \hat{x}\|$ under the following assumption on $x_0 - \hat{x}$.

Assumption 3.1 (see [4]). There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \rightarrow (0, \infty)$ with $a \geq \|F'(\hat{x})\|$ satisfying $\lim_{\lambda \rightarrow 0} \varphi(\lambda) = 0$ and $v \in X$ with $\|v\| \leq 1$ such that

$$\begin{aligned}
x_0 - \hat{x} &= \varphi(F'(\hat{x}))v, \\
\sup_{\lambda \geq 0} \frac{\alpha \varphi(\lambda)}{\lambda + \alpha} &\leq c_\varphi \varphi(\alpha), \quad \forall \lambda \in (0, a].
\end{aligned} \tag{3.1}$$

Remark 3.2. It can be seen that functions

$$\varphi(\lambda) = \lambda^\nu, \quad \lambda > 0 \tag{3.2}$$

for $0 < \nu \leq 1$ and

$$\varphi(\lambda) = \begin{cases} \left(\ln \frac{1}{\lambda}\right)^{-p}, & 0 < \lambda \leq e^{-(p+1)}, \\ 0, & \text{otherwise,} \end{cases} \tag{3.3}$$

for $p \geq 0$ satisfy Assumption 3.1 (see [14]).

We will be using the error estimates in the following proposition, which can be found in [5], for our error analysis.

Proposition 3.3 (cf. [5], Proposition 3.1). *Let $\hat{x} \in D(F)$ be a solution of (1.1) and let $F : D(F) \subseteq X \rightarrow X$ be a monotone operator in X . Let x_α be the unique solution of (1.4) with f in place of f^δ and let x_α^δ be the unique solution of (1.4). Then*

$$\begin{aligned} \|x_\alpha^\delta - x_\alpha\| &\leq \frac{\delta}{\alpha}, \\ \|x_\alpha - \hat{x}\| &\leq \|x_0 - \hat{x}\|. \end{aligned} \tag{3.4}$$

The following theorem can be found in [1].

Theorem 3.4 (see, [1], Theorem 3.1). *Let x_α be the unique solution of (1.4) with f in place of f^δ . Let the assumptions in the Proposition 3.3 and the Assumptions 2.1, 2.2, and 3.1 be satisfied. Then*

$$\|x_\alpha - \hat{x}\| \leq (k_0 r + 1) c_\varphi \varphi(\alpha). \tag{3.5}$$

Combining the estimates in Proposition 3.3, Theorems 2.5, and 3.4 we obtain the following.

Theorem 3.5. *Let x_n be as in (1.6) and let the assumptions in Proposition 3.3, Theorems 2.5, and 3.4 be satisfied. Then*

$$\|x_n - \hat{x}\| \leq \frac{q^{2n} \gamma_\rho}{(1-q)} + \max\{1, (k_0 r + 1) c_\varphi\} \left(\varphi(\alpha) + \frac{\delta}{\alpha} \right). \tag{3.6}$$

Let

$$\overline{C} := \max\left\{ \frac{\gamma_\rho}{1-q} + 1, (k_0 r + 1) c_\varphi \right\}, \tag{3.7}$$

and let

$$n_\delta := \min\left\{ n : q^{2n} \leq \frac{\delta}{\alpha} \right\}. \tag{3.8}$$

Theorem 3.6. *Let x_α^δ be the unique solution of (1.4) and let x_n be as in (1.6). Let the assumptions in Theorem 3.5 be satisfied. Let $\overline{C}$ be as in (3.7) and let n_δ be as in (3.8). Then*

$$\|x_{n_\delta} - \hat{x}\| \leq \overline{C} \left(\varphi(\alpha) + \frac{\delta}{\alpha} \right). \tag{3.9}$$

3.1. A Priori Choice of the Parameter

Note that the error estimate $\varphi(\alpha) + \delta/\alpha$ in (3.9) is of optimal order if $\alpha := \alpha_\delta$ satisfies, $\varphi(\alpha_\delta)\alpha_\delta = \delta$.

Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$, $0 < \lambda \leq a$, we have $\delta = \alpha_\delta\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. In view of the above observations and (3.9), we have the following.

Theorem 3.7. *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$, and let the assumptions in Theorem 3.6 hold. For $\delta > 0$, let $\alpha := \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Let n_δ be as in (3.8). Then*

$$\|x_{n_\delta} - \hat{x}\| = O\left(\psi^{-1}(\delta)\right). \quad (3.10)$$

3.2. An Adaptive Choice of the Parameter

In this subsection, we will present a parameter choice rule based on the adaptive method studied in [3, 9]. Let

$$D_M(\alpha) := \left\{ \alpha_i = \mu^i \alpha_0, i = 0, 1, \dots, M \right\}, \quad (3.11)$$

where $\mu > 1$, $\alpha_0 > 0$, and let

$$n_i := \min \left\{ n : q^{2n} \leq \frac{\delta}{\alpha_i} \right\}. \quad (3.12)$$

Then for $i = 0, 1, \dots, M$, we have

$$\left\| x_{n_i} - x_{\alpha_i}^\delta \right\| \leq \frac{\delta}{\alpha_i}, \quad \forall i = 0, 1, \dots, M. \quad (3.13)$$

Let $x_i := x_{n_i, \alpha_i}^\delta$. In this paper, we select $\alpha = \alpha_i$ from $D_M(\alpha)$ for computing x_i for each $i = 0, 1, \dots, M$.

Theorem 3.8 (cf. [9], Theorem 4.3). *Assume that there exists $i \in \{0, 1, 2, \dots, M\}$ such that $\varphi(\alpha_i) \leq \delta/\alpha_i$. Let assumptions of Theorems 3.6 and 3.7 hold and let*

$$\begin{aligned} l &:= \max \left\{ i : \varphi(\alpha_i) \leq \frac{\delta}{\alpha_i} \right\} < M, \\ k &:= \max \left\{ i : \|x_i - x_j\| \leq 4\bar{C} \frac{\delta}{\alpha_j}, j = 0, 1, 2, \dots, i-1 \right\}. \end{aligned} \quad (3.14)$$

Then $l \leq k$ and

$$\|\hat{x} - x_k\| \leq c\psi^{-1}(\delta), \quad (3.15)$$

where $c = 6\bar{C}\mu$.

4. Implementation of Adaptive Choice Rule

Following steps are involved in implementing the adaptive choice rule.

- (i) Choose $\alpha_0 > 0$ such that $\delta_0 < 3\alpha_0/4k_0$ and $\mu > 1$.
- (ii) Choose $\alpha_i := \mu^i \alpha_0$, $i = 0, 1, 2, \dots$

Finally the adaptive algorithm associated with the choice of the parameter specified in Theorem 3.8 involves the following steps.

4.1. Algorithm

Step 1. Set $i = 0$.

Step 2. Choose $n_i = \min\{n : q^{2n} \leq \delta/\alpha_i\}$.

Step 3. Solve $x_i := x_{n_i, \alpha_i}^\delta$ by using the iteration (1.5) and (1.6).

Step 4. If $\|x_i - x_j\| > 4\bar{c} \frac{\delta}{\alpha_j}$, $j < i$, then take $k = i - 1$ and return x_k .

Step 5. Else set $i = i + 1$ and go to Step 2.

5. Numerical Example

In this section, we consider the example considered in [4] for illustrating the algorithm considered in Section 4.1. We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X with $\dim V_n = n + 1$. Precisely we choose V_n as the linear span of $\{v_1, v_2, \dots, v_{n+1}\}$, where $v_i, i = 1, 2, \dots, n + 1$ are the linear splines in a uniform grid of $n + 1$ points in $[0, 1]$.

Example 5.1 (see, [4], Section 4.3). Let $F : D(F) \subseteq L^2(0, 1) \rightarrow L^2(0, 1)$ defined by

$$F(u) := \int_0^1 k(t, s) u^3(s) ds, \quad (5.1)$$

where

$$k(t, s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1, \\ (1-s)t, & 0 \leq t \leq s \leq 1. \end{cases} \quad (5.2)$$

Then for all $x(t), y(t) : x(t) > y(t)$,

$$\langle F(x) - F(y), x - y \rangle = \int_0^1 \left[\int_0^1 k(t, s) (x^3 - y^3)(s) ds \right] (x - y)(t) dt \geq 0. \quad (5.3)$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3 \int_0^1 k(t, s)(u(s))^2 w(s) ds. \quad (5.4)$$

Note that for $u, v > 0$,

$$\begin{aligned} (F'(v) - F'(u))w &= \left(3 \int_0^1 k(t, s)(u(s))^2 ds \right) \\ &\quad \times \left[\frac{\int_0^1 k(t, s)((v(s))^2 - (u(s))^2) w(s) ds}{\int_0^1 k(t, s)(u(s))^2 ds} \right] \\ &:= F'(u)\Phi(v, u, w), \end{aligned} \quad (5.5)$$

where $\Phi(v, u, w) = (\int_0^1 k(t, s)((v(s))^2 - (u(s))^2) w(s) ds) / (\int_0^1 k(t, s)(u(s))^2 ds)$.

Observe that

$$\begin{aligned} \Phi(v, u, w) &= \frac{\int_0^1 k(t, s)((v(s))^2 - (u(s))^2) w(s) ds}{\int_0^1 k(t, s)(u(s))^2 ds} \\ &= \frac{\int_0^1 k(t, s)(u(s) + v(s))(v(s) - u(s)) w(s) ds}{\int_0^1 k(t, s)(u(s))^2 ds}. \end{aligned} \quad (5.6)$$

So Assumption 2.2 satisfies with $k_0 \geq \|(\int_0^1 k(t, s)(u(s) + v(s)) ds) / (\int_0^1 k(t, s)(u(s))^2 ds)\|$.

In our computation, we take $y(t) = (t - t^{11})/110$ and $y^\delta = y + \delta v_{n+1}$. Then the exact solutions

$$\hat{x}(t) = t^3. \quad (5.7)$$

We use

$$x_0(t) = t^3 + \frac{3}{56}(t - t^8) \quad (5.8)$$

as our initial guess, so that the function $x_0 - \hat{x}$ satisfies the source condition:

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))1, \quad (5.9)$$

where $\varphi(\lambda) = \lambda$. Thus we expect to obtain the rate of convergence $O(\delta^{1/2})$.

Observe that while performing numerical computation on finite dimensional subspace (V_n) of X , one has to consider the operator $P_n F'(x_0) P_n$ instead of $F'(x_0)$, where P_n is the

Figure 1: Curves of the exact and approximate solutions.

orthogonal projection on to V_n . Thus incurs an additional error $\|P_n F'(x_0)P_n - F'(x_0)\| = O(\|F'(x_0)(I - P_n)\|)$.

Let $\|F'(x_0)(I - P_n)\| \leq \varepsilon_n$. For the operator $F'(x_0)$ defined in (5.4), $\varepsilon_n = O(n^{-2})$ (cf. [15]). Thus we expect to obtain the rate of convergence $O((\delta + \varepsilon_n)^{1/2})$.

We choose $\alpha_0 = (1.5)\delta$, $\mu = 1.5$ and $q = 0.51$. The results of the computation (four decimal places) are presented in Table 1. The plots of the exact solution and the approximate solution obtained are given in Figures 1 and 2.

The last column of Table 1 shows that the error $\|x_k - \hat{x}\|$ is of $O((\delta + \varepsilon_n)^{1/2})$.

6. Conclusion

We considered a two step method for obtaining an approximate solution for a nonlinear ill-posed operator equation $F(x) = f$ where $F : D(F) \subseteq X \rightarrow X$ is a monotone operator defined on a real Hilbert space X when the available data is f^δ with $\|f - f^\delta\| \leq \delta$. The

Figure 2: Curves of the exact and approximate solutions.

Table 1

n	k	n_k	$\delta + \varepsilon_n$	α_k	$\ x_k - \hat{x}\ $	$\ x_k - \hat{x}\ /(\delta + \varepsilon_n)^{1/2}$
8	2	3	0.1016	0.3428	0.2634	0.8266
16	2	3	0.1004	0.3388	0.1962	0.6191
32	2	3	0.1001	0.3378	0.1429	0.4518
64	2	3	0.1000	0.3376	0.1036	0.3275
128	2	3	0.1000	0.3375	0.0755	0.2387
256	2	3	0.1000	0.3375	0.0560	0.1772
512	2	3	0.1000	0.3375	0.0430	0.1360
1024	2	3	0.1000	0.3375	0.0347	0.1096

proposed method converges to a solution of the normal equation $F(x) + \alpha(x - x_0) = f^\delta$ and hence is an approximation for the solution $\hat{x}$, for properly chosen parameter α . We obtained optimal order error estimate by choosing the regularization parameter α according to the adaptive method considered by Pereverzev and Schock [3]. The computational results provided endorse the reliability and effectiveness of our method.

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