International Scholarly Research Network
ISRN Algebra
Volume 2012, Article ID 729356, 11 pages
doi:10.5402/2012/729356

Research Article

Commutativity Theorems for $*$-Prime Rings with Differential Identities on Jordan Ideals

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Received 20 November 2012; Accepted 12 December 2012

Academic Editors: P. Damianou, M. Fontana, and M. Siles Molina

In this paper we explore commutativity of $*$-prime rings in which derivations satisfy certain differential identities on Jordan ideals. Furthermore, examples are given to demonstrate that our results cannot be extended to semiprime rings.

1. Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. R is 2-*torsion free* if $2x = 0$ yields $x = 0$. We recall that R is *prime* if $aRb = 0$ implies $a = 0$ or $b = 0$. A ring with involution $(R, *)$ is $*$-prime if $aRb = aRb^* = 0$ yields $a = 0$ or $b = 0$. It is easy to check that a $*$-prime ring is semiprime. Moreover, every prime ring having an involution $*$ is $*$-prime but the converse does not hold, in general. For example, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{\text{ex}}$, defined by $*_{\text{ex}}(x, y) = (y, x)$, is $*_{\text{ex}}$-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings.

In all that follows $Sa_*(R) = \{x \in R \mid x^* = x \text{ or } x^* = -x\}$ will denote the set of symmetric or skew-symmetric elements of R. For $x, y \in R$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$. An additive subgroup J of R is a *Jordan ideal* if $x \circ r \in J$ for all $x \in J$ and $r \in R$. Moreover, if $J^* = J$, then J is called a $*$-Jordan ideal. We will use without explicit mention the fact that if J is a Jordan ideal of R, then $2[R, R]J \subseteq J$ and $2J[R, R] \subseteq J$ [1, Lemma 1]. Moreover, From [2] we have $4jRj \subset J$, $4j^2R \subset J$ and $4Rj^2 \subset J$ for all $j \in J$.

A mapping $f : R \to R$ is called *strong commutativity preserving* on a subset S of R if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$. Recently, many authors have obtained commutativity theorems for $*$-prime (prime) rings admitting derivation, generalized derivation, and left multiplier (see [3–8]). In this paper, we will explore the commutativity of $*$-prime rings equipped with derivations satisfying certain differential identities on Jordan ideals.

2. Differential Identities with Commutator

We will make some use of the following well-known results.

Remarks 2.1. Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal.

(1) (see [6, Lemma 2]) If $aJb = a^*Jb = 0$, then $a = 0$ or $b = 0$.

(2) (see [6, Lemma 3]) If $[J, J] = 0$, then $J \subseteq Z(R)$.

(3) (see [7, Lemma 3]) If $J \subseteq Z(R)$, then R is commutative.

(4) (see [9], Lemma 3]) If d is a derivation such that $d(x^2) = 0$ for all $x \in J$, then $d = 0$.

We leave the proofs of the following two easy facts to the reader.

(5) If $aJa = 0$, then $a = 0$. In particular, if $aJ = 0$ or $Ja = 0$, then $a = 0$.

(6) If R admits a derivation d such that $d^2(J) = \{0\}$, then $d = 0$.

Lemma 2.2. *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal. If R admits a nonzero derivation d such that $[d(x), y^2] = 0$ for all $x, y \in J$, then R is commutative.*

Proof. First suppose that $Z(R) \cap J = 0$. From $[d(4xy^2), y^2] = 0$ it follows that

$$\left[x, y^2\right] d\left(y^2\right) = 0 \quad \forall x, y \in J. \tag{2.1}$$

Substituting $2[r, s]x$ for x in (2.1), where $r, s \in R$, we obtain $[[r, s], y^2]xd(y^2) = 0$ which leads to

$$\left[[r, s], y^2\right] Jd\left(y^2\right) = 0 \quad \forall y \in J,\ r, s \in R. \tag{2.2}$$

For $y \in Sa_*(R) \cap J$, (2.2) together with Remarks 2.1(1) forces $[[r, s], y^2] = 0$, in which case $y^2 \in Z(R)$, or $d(y^2) = 0$. Since $J \cap Z(R) = 0$, in both the cases we arrive at

$$d\left(y^2\right) = 0 \quad \forall y \in J \cap Sa_*(R). \tag{2.3}$$

Let $y \in J$; using the fact that $d((y+y^*)^2) = 0$ and $d((y-y^*)^2) = 0$, we obtain $d(y^2) = -d((y^*)^2)$. Replacing y by y^* in (2.2), we get $([[r, s], y^2])^* Jd(y^2) = 0$ which combined with (2.2) yields

either $d(y^2) = 0$ or $y^2 \in Z(R)$ so that $y^2 = 0$. This implies that

$$d\left(y^2\right) = 0 \quad \forall y \in J, \tag{2.4}$$

and hence by Remarks 2.1(4), $d = 0$ which contradicts our hypothesis and thus $Z(R) \cap J \neq 0$.

Now let $0 \neq u \in Z(R) \cap J$; from $[d(4xu^2), y^2] = 0$ it follows that $[x, y^2]d(u^2) = 0$ for all $x, y \in J$ and thus either $[x, y^2] = 0$ or $d(u^2) = 0$. If $[x, y^2] = 0$, then using similar arguments as used in [5, Proof of Theorem 3] we conclude that R is commutative. Assume that $d(u^2) = 0$; the fact that $[d(4u^2r), y^2] = 0$ yields $u^2[d(r), y^2] = 0$. Similarly in view of $[d(4(u^*)^2r), y^2] = 0$, we find that $(u^2)^*R[d(r), y^2] = 0$ and therefore

$$\left[d(r), y^2\right] = 0 \quad \forall y \in J,\ r \in R. \tag{2.5}$$

Replacing r by $d(r)s$ in (2.5) and using (2.5) we get $d^2(r)[s, y^2] = 0$. Again, replace s by st in the last equation, to get $d^2(r)s[t, y^2] = 0$ so that

$$d^2(r)R\left[t, y^2\right] = 0 \quad \forall y \in J,\ r, t \in R. \tag{2.6}$$

In view of $*$-primeness, (2.6) assures that either $d^2(r) = 0$ or $[t, y^2] = 0$.

If $d^2(r) = 0$ for all $r \in R$, then $d = 0$ by Remarks 2.1(4) contradiction, thus $[t, y^2] = 0$ for all $t \in R$ and $y \in J$, then [5, Proof of Theorem 3] implies that R is commutative. □

Theorem 2.3. *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal. If R admits a nonzero derivation d, commuting with $*$, such that either d is strong commutativity preserving on J or $[d(x), d(y)] = 0$ for all $x, y \in J$, then R is commutative.*

Proof. (i) Assume that d is strong commutativity preserving on J. In this case the condition $d \neq 0$ is not necessary. Indeed, if $d = 0$, then $[J, J] = 0$ and Remarks 2.1(2) forces $J \subseteq Z(R)$. Thus R is commutative by Remarks 2.1(3).

We are given that

$$\left[d(x), d(y)\right] = \left[x, y\right] \quad \forall x, y \in J. \tag{2.7}$$

Replacing y by $4y^2z$ in the above expression, where $z \in J$, we get

$$d\left(y^2\right)[d(x), z] + \left[d(x), y^2\right]d(z) = 0 \quad \forall x, y, z \in J. \tag{2.8}$$

Again, replacing z by $2[a, uv]z$ in (2.8), where $u, v \in J$ and $a \in R$, and using (2.8) we obtain

$$d\left(y^2\right)[a, uv][d(x), z] + \left[d(x), y^2\right][a, uv]d(z) = 0 \quad \forall u, v, x, y, z \in J,\ a \in R. \tag{2.9}$$

Putting $4z^2t$ for z in (2.9), where $t \in R$ we find that

$$d\left(y^2\right)[a,uv]z^2[d(x),t] + \left[d(x),y^2\right][a,uv]z^2d(t) = 0 \tag{2.10}$$

for all $u,v,x,y,z \in J$ and $a,t \in R$. Substituting $td(x)$ for t in (2.10), then we have $[d(x),y^2][a,uv]z^2td^2(x) = 0$ and therefore

$$\left[d(x),y^2\right][a,uv]z^2Rd^2(x) = 0 \quad \forall u,v,x,y,z \in J,\ a \in R. \tag{2.11}$$

Let $x_0 \in J \cap Sa_*(R)$; from (2.11) it follows that $[d(x_0),y^2][a,uv]z^2R(d^2(x_0))^* = 0$, so either $d^2(x_0) = 0$ or $[d(x_0),y^2][a,uv]z^2 = 0$.

Suppose that

$$\left[d(x_0),y^2\right][a,uv]z^2 = 0 \quad \forall u,v,y,z \in J,\ a \in R. \tag{2.12}$$

Replacing a by aj^2 in (2.12), where $j \in J$, we find that $[d(x_0),y^2]a[j^2,uv]z^2 = 0$ and therefore

$$\left[d(x_0),y^2\right]R\left[j^2,uv\right]z^2 = 0 \quad \forall j,u,v,y,z \in J. \tag{2.13}$$

Since $x_0 \in Sa_*(R)$, then (2.13) assures that

$$\left(\left[d(x_0),y^2\right]\right)^*R\left[j^2,uv\right]z^2 = 0 \quad \forall j,u,v,y,z \in J. \tag{2.14}$$

In view of (2.13) and (2.14), the $*$-primeness of R forces $[d(x_0),y^2] = 0$ for all $y \in J$ or $[j^2,uv]z^2 = 0$ for all $j,u,v,z \in J$.

Suppose that $[j^2,uv]z^2 = 0$ thus

$$u\left[j^2,v\right]z^2 + \left[j^2,u\right]vz^2 = 0 \quad \forall z,u,v,j \in J. \tag{2.15}$$

Replacing u by $2[r,s]u$ in (2.15) we arrive at $[j^2,[r,s]]uvz^2 = 0$ and thus $[j^2,[r,s]]uJz^2 = 0$ which leads to $[j^2,[r,s]] = 0$ or $z^2 = 0$. Hence, in both cases we find that $z^2 \in Z(R)$ for all $z \in J$ and [5, Proof of Theorem 3] assures that R is commutative, thereby $[d(x_0),y^2] = 0$ for all $y \in J$.

In conclusion,

$$d^2(x_0) = 0 \quad \text{or} \quad \left(\left[d(x_0),y^2\right] = 0\ \forall y \in J\right) \quad \forall x_0 \in J \cap Sa_*(R). \tag{2.16}$$

Let $x \in J$; as $x - x^* \in J \cap Sa_*(R)$, from the above relation it follows that $d^2(x) = d^2(x^*) = d^2(x)^*$ or $[d(x),y^2] = [d(x^*),y^2]$ for all $y \in J$.

Assume that $d^2(x) = d^2(x)^*$. Hence (2.11) can be rewritten as

$$\left[d(x),y^2\right][a,uv]z^2Rd^2(x)^* = 0 \quad \forall u,v,y,z \in J,\ a \in R. \tag{2.17}$$

Combining the latter relation with (2.11), we get $d^2(x) = 0$ or $[d(x), y^2][a, uv]z^2 = 0$ which makes it possible to conclude, using similar arguments as above, that $[d(x), y^2] = 0$ or $d^2(x) = 0$.

Now suppose $[d(x), y^2] = [d(x^*), y^2]$. In relation to (2.11) let x be x^*; then we have

$$\left[d(x), y^2\right][a, uv]z^2 Rd^2(x^*) = 0 \quad \forall u, v, y, z \in J,\ a \in R, \tag{2.18}$$

which gives, because d commutes with $*$,

$$\left[d(x), y^2\right][a, uv]z^2 Rd^2(x)^* = 0 \quad \forall u, v, y, z \in J,\ a \in R. \tag{2.19}$$

Whence it follows, according to (2.11), that $d^2(x) = 0$ or $[d(x), y^2] = 0$. Thus in both cases we find that

$$d^2(x) = 0 \quad \text{or} \quad \left[d(x), y^2\right] = 0 \quad \forall x, y \in J. \tag{2.20}$$

Now let $J_1 = \{x \in J \mid d^2(x) = 0\}$ and $J_2 = \{x \in J \mid [d(x), y^2] = 0 \text{ for all } y \in J\}$. Clearly, J_1 and J_2 are additive subgroups of J whose union, because of (2.20), is Hence, by Brauer's trick, either $J = J_1$ or $J = J_2$. If $J = J_1$, then $d^2(J) = 0$ and hence, by Remarks 2.1(6), $d = 0$ thus $J = J_2$, so $[d(x), y^2] = 0$ for all $x, y \in J$, whence it follows, according to Lemma 2.2, that R is commutative.

(ii) Assume that

$$[d(x), d(y)] = 0 \quad \forall x, y \in J. \tag{2.21}$$

Substituting $4y^2z$ for y in (2.21), where $z \in J$, we get

$$d\left(y^2\right)[d(x), z] + \left[d(x), y^2\right]d(z) = 0 \quad \forall\ x, y, z \in J. \tag{2.22}$$

Since (2.22) is the same as (2.8), reasoning as in the first case, we conclude that R is commutative. □

In [10] Herstein proved that if R is a prime ring of characteristic not 2 equipped with a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. As an application of the above theorem, we get the following theorem which generalizes Herstein's result for Jordan ideals.

Theorem 2.4. *Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal. If R admits a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in J$, then R is commutative.*

Proof. Let $\mathfrak{D}$ be the additive mapping defined on $\mathcal{R} = R \times R^0$ by $\mathfrak{D}(x, y) = (d(x), d(y))$. Clearly, $\mathfrak{D}$ is a nonzero derivation of $\mathcal{R}$. Moreover, if we set $\mathcal{J} = J \times J$, then $\mathcal{J}$ is a $*_{\text{ex}}$-Jordan ideal of $\mathcal{R}$ such that $[\mathfrak{D}(x, y), \mathfrak{D}(u, v)] = (0, 0)$ for all $(x, y), (u, v) \in \mathcal{J}$. Since $\mathfrak{D}$ commutes with $*_{\text{ex}}$ and $\mathcal{R}$ is $*_{\text{ex}}$-prime, then Theorem 2.3 assures that $\mathcal{R}$ is commutative and thus so is R. □

An application of similar arguments yields the following.

Theorem 2.5. *Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal. If R admits a derivation d strong commutativity preserving on J, then R is commutative.*

In 2010, Oukhtite et al. [8, Theorem 2] established that if a 2-torsion free $*$-prime ring admits a nonzero derivation d such that $[d(x), d(y)] = d([x, y])$ for all x, y in a nonzero square closed Lie ideal U, then $d = 0$ or $U \subset Z(R)$. Motivated by this result, our aim in the following theorem is to explore the commutativity of $*$-prime rings admitting a nonzero derivation d satisfying the above condition on $*$-Jordan ideals.

Theorem 2.6. *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal. If R admits a nonzero derivation d which commutes with $*$ such that $[d(x), d(y)] = d([x, y])$ for all $x, y \in J$, then R is commutative.*

Proof. Suppose that

$$[d(x), d(y)] = d([x, y]) \quad \forall x, y \in J. \tag{2.23}$$

Replacing x by $2[a, uv]x$ in (2.23) where $u, v \in J$ and $a \in R$, in light of $2[a, uv] \in J$, we find that

$$d([a, uv])[x, d(y) - y] + [[a, uv], d(y) - y]d(x) = 0 \quad \forall u, v, x, y \in J,\ a \in R. \tag{2.24}$$

Again replacing x by $2[r, pq]x$ in (2.24) with $p, q \in J$ and $r \in R$, we get

$$d([a, uv])[r, pq][x, d(y) - y] + [[a, uv], d(y) - y][r, pq]d(x) = 0. \tag{2.25}$$

Substituting x by $4x^2t$ in (2.25) with $t \in R$ and employing (2.25), we obtain

$$d([a, uv])[r, pq]x^2[t, d(y) - y] + [[a, uv], d(y) - y][r, pq]x^2d(t) = 0. \tag{2.26}$$

Replacing t by ts in (2.26), where $s \in R$, then we have

$$d([a, uv])[r, pq]x^2t[s, d(y) - y] + [[a, uv], d(y) - y][r, pq]x^2td(s) = 0, \tag{2.27}$$

for all $p, q, u, v, x, y \in J$ and for all $a, r, t, s \in R$. Putting $s = d(y) - y$ in (2.27), we find that

$$[[a, uv], d(y) - y][r, pq]x^2td(d(y) - y) = 0 \tag{2.28}$$

and therefore

$$[[a, uv], d(y) - y][r, pq]x^2Rd(d(y) - y) = 0 \quad \forall p, q, u, v, x, y \in J,\ a, r \in R. \tag{2.29}$$

Let $y \in J \cap Sa_*(R)$; from (2.29) it follows, in light of $*$-primeness, that

$$[[a, uv], d(y) - y]\,[r, pq]\,x^2 = 0 \quad \text{or} \quad d(d(y) - y) = 0. \tag{2.30}$$

Assume that

$$[[a, uv], d(y) - y]\,[r, pq]\,x^2 = 0 \quad \forall p, q, u, v, x \in J,\ a, r \in R. \tag{2.31}$$

Replacing r by rj^2 in the above expression, we get $[[a, uv], d(y) - y]r[j^2, pq]x^2 = 0$ and thus

$$[[a, uv], d(y) - y]\,R\left[j^2, pq\right]x^2 = 0. \tag{2.32}$$

R being $*$-prime implies that either $[[a, uv], d(y) - y] = 0$ or $[j^2, pq]x^2 = 0$.

If $[j^2, pq]x^2 = 0$, then

$$p\left[j^2, q\right]x^2 + \left[j^2, p\right]qx^2 = 0 \quad \forall j, p, q, x \in J. \tag{2.33}$$

As (2.33) is the same as (2.15), then reasoning as in the proof of Theorem 2.3 we find that R is commutative and therefore $[[a, uv], d(y) - y] = 0$.

In conclusion,

$$[[a, uv], d(y) - y] = 0 \text{ or } d(d(y) - y) = 0 \quad \forall y \in J \cap Sa_*. \tag{2.34}$$

Let $y \in J$; as $y - y^* \in J \cap Sa_*(R)$, in view of (2.34) we obtain that either $d(d(y-y^*)-(y-y^*)) = 0$ or $[[a, uv], d(y - y^*) - (y - y^*)] = 0$.

If $d(d(y - y^*) - (y - y^*)) = 0$, then $d(d(y) - y) = d(d(y^*) - y^*)$ and hence, because d commutes with $*$, (2.29) reduces to

$$[[a, uv], d(y) - y]\,[r, pq]\,x^2 R d(d(y) - y)^* = 0. \tag{2.35}$$

In light of (2.29), the latter expression together with $*$-primeness of R shows that either $d(d(y)-y) = 0$ or $[[a, uv], d(y)-y][r, pq]x^2 = 0$ which, as above, forces $[[a, uv], d(y)-y] = 0$.

If $[[a, uv], d(y - y^*) - (y - y^*)] = 0$, then $[[a, uv], d(y) - y] = [[a, uv], d(y^*) - y^*]$. Replacing y by y^* in (2.29) and using similar arguments as above, we find that $[[a, uv], d(y) - y] = 0$ or $d(d(y) - y) = 0$.

Hence in both cases we find that

$$[[a, uv], d(y) - y] = 0 \quad \text{or} \quad d(d(y) - y) = 0 \quad \forall y, u, v \in J,\ a \in R. \tag{2.36}$$

The set of $y \in J$ for which these two properties hold are additive subgroups of J whose union is J; accordingly, we must have either $[[a, uv], d(y) - y] = 0$ for all $u, v, y \in J$, $a \in R$ or $d(d(y) - y) = 0$ for all $y \in J$.

Assume that

$$d^2(y) = d(y) \quad \forall y \in J. \tag{2.37}$$

Replacing y by $4y^2z$ with $z \in J$, the last expression becomes

$$d\left(y^2\right)d(z) = 0. \tag{2.38}$$

Writing $4zy^2$ instead of z we get $d(y^2)zd(y^2) = 0$, that is,

$$d\left(y^2\right)Jd\left(y^2\right) = 0 \quad \forall y \in J. \tag{2.39}$$

In view of Remarks 2.1(5), the above relation yields that $d(y^2) = 0$ for all $y \in J$. Hence, using Remarks 2.1(4), we conclude that $d = 0$ contradiction, thus

$$[[a, uv], d(y) - y] = 0 \quad \forall u, v, y \in J,\ a \in R. \tag{2.40}$$

Substitution $[r, s](d(y) - y)$ for a in the latter relation, we get

$$[[r, s], d(y) - y]u[d(y) - y, v] + [[r, s], d(y) - y]\,[d(y) - y, u]v = 0. \tag{2.41}$$

Replacing v by $2v[r, s]$ we obtain

$$[[r, s], d(y) - y]uv[d(y) - y, [r, s]] = 0 \tag{2.42}$$

so that

$$v[[r, s], d(y) - y]Jv[[r, s], d(y) - y] = 0 \quad \forall v, y \in J \text{ and } r, s \in R. \tag{2.43}$$

Now, an application of Remarks 2.1(5) yields that $[[r, s], d(y) - y] = 0$ for all $y \in J,\ r, s \in R$ which obviously leads to $d(y) - y \in Z(R)$ for all $y \in J$. Thus, d is centralizing on J and from [7, Theorem 1] we get the required result. □

As an application of Theorem 2.6, we get the following theorem for which the proof goes through in the same way as the proof of Theorem 2.4.

Theorem 2.7. *Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal of R. If R admits a nonzero derivation d such that $[d(x), d(y)] = d([x, y])$ for all $x, y \in J$, then R is commutative.*

3. Differential Identities with Anticommutator

This section is devoted to finding out if commutativity still holds when the commutator in the conditions of the preceding section is replaced by anticommutator.

Theorem 3.1. *Let R be a 2-torsion free $*$-prime ring and J be a nonzero $*$-Jordan ideal. Then R admits no nonzero derivation d which commutes with $*$ such that $d(x) \circ d(y) = 0$ for all $x, y \in J$.*

Proof. Assume that there exists a nonzero derivation d which commutes with $*$ and satisfying

$$d(x) \circ d(y) = 0 \quad \forall x, y \in J. \tag{3.1}$$

Replacing y by $4y^2z$ in (3.1), where $z \in J$, we get

$$-d\left(y^2\right)[d(x), z] + \left[d(x), y^2\right]d(z) = 0. \tag{3.2}$$

Substituting $2[a, uv]z$ for z in (3.2) with $u, v \in J$, $a \in R$ and using (3.2) again, we obtain

$$-d\left(y^2\right)[a, uv][d(x), z] + \left[d(x), y^2\right][a, uv]d(z) = 0. \tag{3.3}$$

Replacing z by $4z^2t$ in (3.3) with $t \in R$, and using (3.3) again, we get

$$-d\left(y^2\right)[a, uv]z^2[d(x), t] + \left[d(x), y^2\right][a, uv]z^2d(t) = 0. \tag{3.4}$$

Writing tw instead of t in (3.4) with $w \in R$, we find that

$$-d\left(y^2\right)[a, uv]z^2t[d(x), w] + \left[d(x), y^2\right][a, uv]z^2td(w) = 0. \tag{3.5}$$

Taking $w = d(x)$ in (3.5), we get $[d(x), y^2][a, uv]z^2td^2(x) = 0$ and thus

$$\left[d(x), y^2\right][a, uv]z^2Rd^2(x) = 0. \tag{3.6}$$

Since (3.6) is the same as (2.11), reasoning as in the proof of Theorem 2.3 we conclude that R is commutative and (3.1) becomes

$$2d(x)d(y) = 0 \quad \forall x, y \in J \tag{3.7}$$

hence

$$d(x)Rd(y) = 0 \quad \forall x, y \in J \tag{3.8}$$

so that $d(J) = 0$ and $d = 0$, a contradiction. □

Using the same arguments as used in the proof of Theorem 2.4, an application of Theorem 3.1 yields the following result.

Theorem 3.2. *Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal. Then R admits no nonzero derivation d such that $d(x) \circ d(y) = 0$ for all $x, y \in J$.*

Theorem 3.3. *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal. Then R admits no nonzero derivation d such that $d(x) \circ d(y) = d(x \circ y)$ for all $x, y \in J$.*

Proof. Suppose that there exists a derivation d such that

$$d(x) \circ d(y) = d(x \circ y) \quad \forall x, y \in J. \tag{3.9}$$

Replacing y by $2y[r,s]$ in (3.9), we find that

$$-d(y)[d(x)-x,[r,s]] + ((d(x)-x)\circ y)d([r,s]) - y[d(x),d([r,s])] + yd([x,[r,s]]) = 0. \tag{3.10}$$

Substituting $2y[a,uv]$ for y in (3.10), where $u, v \in J$ and $a \in R$, we get

$$-d(y)[a,uv][d(x)-x,[r,s]] + [d(x)-x,y][a,uv]d([r,s]) = 0. \tag{3.11}$$

Replacing y by $4ty^2$ in (3.11), where $t \in R$, and using (3.11) again, we find that

$$-d(t)y^2[a,uv][d(x)-x,[r,s]] + [d(x)-x,t]y^2[a,uv]d([r,s]) = 0. \tag{3.12}$$

Writing wt instead of t in (3.12), where $w \in R$, and using (3.12) we get

$$-d(w)ty^2[a,uv][d(x)-x,[r,s]] + [d(x)-x,w]ty^2[a,uv]d([r,s]) = 0. \tag{3.13}$$

Taking $w = d(x) - x$ in (3.13), we obtain $d(d(x)-x)ty^2[a,uv][d(x)-x,[r,s]] = 0$, that is,

$$d(d(x)-x)Ry^2[a,uv][d(x)-x,[r,s]] = 0. \tag{3.14}$$

Since (3.14) is the same as (2.29), reasoning as in the proof of Theorem 2.6, it follows that R is commutative and thus (3.9) becomes

$$2d(x)d(y) - 2d(x)y - 2xd(y) = 0. \tag{3.15}$$

Replacing y by $2yz$ in (3.15) where $z \in J$ we get

$$d(x)yd(z) - xyd(z) = 0. \tag{3.16}$$

Replacing x by $2xt$ in (3.16) where $t \in J$ we obtain

$$d(t)xyd(z) = 0 \tag{3.17}$$

which leads to $d(J) = 0$ and therefore $d = 0$. □

Using the same arguments as used in the proof of Theorem 2.4, an application of Theorem 3.3 yields the following result.

Theorem 3.4. *Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal. Then R admits no nonzero derivation d such that $d(x) \circ d(y) = d(x \circ y)$ for all $x, y \in J$.*

To end this paper, we give examples proving that our results cannot be extended to semiprime rings.

Example 3.5. Let $(R_1, *)$ be a noncommutative semiprime ring, with involution, which admits a nonzero derivation d and let $R = R_1 \times R_1$. Consider $J = 0 \times R_1$ and define a derivation D on R by setting $D(x, y) = (d(x), 0)$. Obviously, J is a nonzero τ-Jordan ideal of R, where τ is the involution defined on R by $\tau(x, y) = (x^*, y^*)$. Furthermore,

$$\begin{aligned} [D(u), D(v)] = 0, \quad [D(u), D(v)] = D([u, v]), \\ D(u) \circ D(v) = 0, \quad D(u) \circ D(v) = D(u \circ v) \end{aligned} \tag{3.18}$$

for all $u, v \in J$; but R is noncommutative. Hence Theorems 2.3, 2.6, 3.1, and 3.3 cannot be extended to a semiprime ring.

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