Research Article

Effect of Wind Stress on the Dynamics and Stability of Nonisothermal Power-Law Film down an Inclined Plane

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Dynamics and stability of a nonisothermal power-law liquid film down an inclined plane is considered in the presence of interfacial shear. Linear stability characteristics of the power-law liquid film using normal mode approach reveal that isothermal and evaporating films are unstable for any value of power-law index while there exists a critical value of power-law index for the case of condensate film above which condensate film flow system is always stable. This critical value of power-law index increases with the increase in shear stress at the interface. Weakly nonlinear stability analysis using method of multiple scales divulges the existence of zones due to supercritical stability and subcritical instability. The nonlinear evolution equation is solved numerically in a periodic domain. The results reveal that (1) for an isothermal dilatant (pseudoplastic) liquids, the maximum wave amplitude is always smaller (larger) than that for a Newtonian liquid and the amplitude of permanent wave increases with the increase in interfacial shear; (2) condensation of pseudoplastic film happens for the earlier instant of time when the phase change parameter increases and the effect of interfacial shear makes the film more corrugated; (3) dilatant (pseudoplastic) evaporating liquid film attains rupture faster (slower) than that of Newtonian liquid film, and the interfacial shear does not influence the time at which rupture occurs.

1. Introduction

Gravity-driven flow of a thin film down a vertical or an inclined plane has attracted much attention due to its importance in many industrial applications such as film coating and interface heat and mass transfer processes in chemical technology and energetics. Yih [1] and Benjamin [2] first studied the linear theory for the isothermal falling film. Yih [3] formulated the problem in terms of long-wave asymptotics and determined the critical Reynolds number

above which the instability would occur. Benney [4] extended the theory to nonlinear regime by deriving a nonlinear evolution equation. There have been number of extensions of this work as discussed by Chang and Demekhin [5] in their monograph.

Although the theory of laminar film condensation flow due to gravity has been analyzed by Nusselt [6], the stability analysis of condensate or evaporating liquid film down a vertical or an inclined plane has been studied only after 1970s [7–9]. Burelbach et al. [10] and Bankoff [11] formulated a one-sided model and studied nonlinear stability and breakdown of evaporating/condensing horizontal static liquid film. Long-wave instabilities of heated falling films have been investigated by Bankoff [12] and Joo et al. [13]. Effect of interfacial phase change on the linear stability of thin films has been considered by Spindler et al. [14], Ünsal and Thomas [15], and Spindler [16]. They have showed that vapor recoil effects are destabilizing for a volatile liquid. Using perturbation methods, Ünsal and Thomas [17] have investigated the nonlinear stability of vertical condensate film flow. Hwang and Weng [18] have examined the finite-amplitude stability analysis of liquid film down a vertical plane with and without interfacial phase change and have shown that both supercritical stability and subcritical instability are possible for condensate film flow system.

These results for a Newtonian film flows (isothermal, condensate, evaporating) cannot completely describe the dynamics of non-Newtonian film flows. For example, mud flows are pseudoplastic, saccharified honey is known to be a shear thickening fluid, whereas polymers exhibit generally a large elastic component. Non-Newtonian fluids generally exhibit a nonlinear relationship between shear stress and shear rate. These flows may be classified as inelastic and viscoelastic. The inelastic fluids may be subdivided as time-dependent fluids (thixotropic and rheopectic) and time-independent fluids (pseudoplastic, dilatant, Bingham plastic, pseudoplastic with yield stress). Apart from investigation on flow characteristics and stability analysis of viscoelastic fluid films along a vertical or on inclined plane [19–27], inelastic time-independent non-Newtonian fluids have received the greatest attention from rheologists [28–33] which has resulted in the development of a number of equations or models proposed to represent their flow behavior. The Ostwald de Waele power-law model represents several inelastic time-independent non-Newtonian fluids of practical interest and therefore has been used in this paper.

The investigations on the stability characteristics of Newtonian fluid film down an inclined/a vertical plane show that in the linear theory, the film flow system is unstable for any Reynolds number. However, the finite-amplitude stability analysis of liquid films down a vertical wall by Hwang and Weng [18] with interfacial phase change reveals that the isothermal and evaporating Newtonian films are unstable for any Reynolds number while there exists a finite critical Reynolds number for the case of condensate film below which condensate Newtonian film flow system is always stable. This shows that the effect of mass transfer at the interface of a Newtonian fluid film strongly modifies the stability characteristics of the film flow when the phase change is considered. Usha and Uma [34] have extended their study to condensate/evaporating power-law liquid film down an inclined plane with or without interfacial phase change to show that isothermal and evaporating films are unstable for any value of power-law index “ n ” while there exists a critical value of power-law index “ n ” for the case of condensate film above which condensate film flow system is always stable. van der Walls interactions are taken into account by Gorla [35] to study the rupture of thin power-law liquid film on a cylinder. He has shown that rupture time for the dilatant fluids are higher than that of Newtonian and pseudoplastic fluids. This shows that a detailed numerical investigation on nonlinear evolution of nonisothermal power-law liquid film down an inclined plane is required to shed more light on understanding the effect of

Figure 1: Schematic representation of a thin film down an inclined plane.

power-law index “ n ” on different kinds of permanent wave shapes (isothermal film), film condensation (condensate film), and film rupture (evaporating film).

Recently, Pascal and D’Alessio [36] have studied the generation and structure of roll waves on the surface of a isothermal power-law liquid film down an inclined plane. To have a more realistic description of flows taking place in an environment, they have considered the effect of wind stress acting on the surface of a power-law film down an incline. Effect of interfacial-induced shear on thin films has been first attempted by Sheintuch and Dukler [37]. Later, Pascal [38] studied the effect of superficial shear stress in connection with the study of wind-aided spreading of oil on the sea.

The chief motivation of the present study is to investigate the dynamics and stability of more realistic flows taking place in an environment. In this case, it becomes important to include the effects of superficial shear stress and phase change at the interface of a fluid film flowing down an inclined plane. The Ostwald de Waele power-law model is considered to represent the non-Newtonian fluid.

In this paper, dynamics and stability of condensate/evaporating power-law liquid film flowing down an inclined plane with the effect of wind stress and phase change at the interface are considered. Linear stability analysis using normal mode approach reveals that cocurrent superficial wind stress at the interface destabilizes the film flow system. Weakly nonlinear stability analysis using method of multiple scales divulges the existence of zones due to supercritical stability and subcritical instability. The nonlinear evolution equation describing the shape of the free surface has been solved numerically in a periodic domain using Lee’s three-time level method. Applications of the results indicate that the amplitude of permanent wave increases with the increase in prescribed shear stress at the interface for isothermal power-law film and pseudoplastic liquid film has the largest amplitude compared to Newtonian and dilatant film. Also, evaporating liquid film attains rupture faster than that of Newtonian and condensate liquid film, and the prescribed shear stress at the interface does not influence the time at which rupture occurs.

2. Mathematical Formulation

A thin power-law liquid film flowing down an inclined plane $\bar{y} = 0$ with the effect of prescribed superficial shear stress and phase change acting on the surface of a fluid layer $\bar{y} = \bar{h}(\bar{x}, \bar{t})$ (Figure 1) is considered.

Figure 2: Neutral stability curves for isothermal ($\xi = 0$), condensate ($\xi = 0.0872$), and evaporating ($\xi = -0.0872$) films with the effect of wind stress.

The governing equations [18, 24, 34] are the two-dimensional mass, momentum, and energy balance equations for the power-law model given by

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0,$$

$$\rho \left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} \right) = -\frac{\partial \bar{p}}{\partial \bar{x}} + \rho g \sin \theta (1 - \gamma) + \frac{\partial \bar{\tau}_{\bar{x}\bar{x}}}{\partial \bar{x}} + \frac{\partial \bar{\tau}_{\bar{x}\bar{y}}}{\partial \bar{y}},$$

$$\rho \left(\frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \right) = -\frac{\partial \bar{p}}{\partial \bar{y}} - \rho g \cos \theta (1 - \gamma) + \frac{\partial \bar{\tau}_{\bar{y}\bar{x}}}{\partial \bar{x}} + \frac{\partial \bar{\tau}_{\bar{y}\bar{y}}}{\partial \bar{y}},$$

$$\left(\frac{\partial \bar{T}}{\partial t} + \bar{u} \frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{T}}{\partial \bar{y}} \right) = \frac{K}{\rho c_p} \left(\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \right),$$

(2.1)

Figure 3: Growth rate for different values of phase change parameter and wind stress parameter.

where K is the thermal conductivity, ρ is the density, c_p is the liquid specific heat, g is the gravity, γ is the ratio of vapor density to liquid density, and $\bar{T}$ is the temperature.

The boundary conditions at the wall are the no-slip condition of velocity and a constant wall temperature ($\bar{T}_w$) given by

$$\bar{u} = 0, \quad \bar{v} = 0, \quad \bar{T} = \bar{T}_w \quad \text{on } \bar{y} = 0. \quad (2.2)$$

The effect of wind stress is implemented into the model by prescribing the shear stress $\bar{\tau}_w$ at the free surface of the fluid layer [36]. The boundary conditions at the liquid-vapor interface [17, 18, 34, 36] are the balance of normal and tangential stresses, the relation of interfacial energy balances, and the equality of liquid and saturated vapor temperatures ($\bar{T}_s$) and are

Figure 4: Neutral stability curves in the κ - n plane for different values of phase change parameter ξ ($\xi = 0$ —isothermal film flow; $\xi = 0.0872$ —condensate film flow; $\xi = -0.0872$ —evaporating film flow) and wind stress parameters τ .

given by

$$\begin{aligned}
 & \bar{p} + \left(2 \frac{\partial \bar{h}}{\partial \bar{x}} \bar{\tau}_{\bar{x}\bar{y}} - \left(\frac{\partial \bar{h}}{\partial \bar{x}} \right)^2 \bar{\tau}_{\bar{x}\bar{x}} - \bar{\tau}_{\bar{y}\bar{y}} \right) \left(1 + \left(\frac{\partial \bar{h}}{\partial \bar{x}} \right)^2 \right)^{-1} + K^2 h_{fs}^{-2} \frac{(\gamma - 1)}{\rho \gamma} \left(\frac{\partial \bar{T}}{\partial \bar{y}} - \frac{\partial \bar{h}}{\partial \bar{x}} \frac{\partial \bar{T}}{\partial \bar{x}} \right)^2 \\
 & \times \left(1 + \left(\frac{\partial \bar{h}}{\partial \bar{x}} \right)^2 \right)^{-1} + \sigma \frac{\partial^2 \bar{h}}{\partial \bar{x}^2} \left[1 + \left(\frac{\partial \bar{h}}{\partial \bar{x}} \right)^2 \right]^{-3/2} = \bar{p}_a \quad \text{on } \bar{y} = \bar{h}, \\
 & \frac{\partial \bar{h}}{\partial \bar{x}} (\bar{\tau}_{\bar{y}\bar{y}} - \bar{\tau}_{\bar{x}\bar{x}}) + \left(1 - \left(\frac{\partial \bar{h}}{\partial \bar{x}} \right)^2 \right) \bar{\tau}_{\bar{x}\bar{y}} = \bar{\tau}_w \quad \text{on } \bar{y} = \bar{h}, \\
 & K \left(\frac{\partial \bar{T}}{\partial \bar{y}} - \frac{\partial \bar{h}}{\partial \bar{x}} \frac{\partial \bar{T}}{\partial \bar{x}} \right) - \rho h_{fs} \left(\frac{\partial \bar{h}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{h}}{\partial \bar{x}} - \bar{v} \right) = 0 \quad \text{on } \bar{y} = \bar{h}, \\
 & \bar{T} = \bar{T}_s \quad \text{on } \bar{y} = \bar{h},
 \end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
 \bar{\tau}_{\bar{x}\bar{x}} &= 2\mu_n \frac{\partial \bar{u}}{\partial \bar{x}} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^{n-1}, \\
 \bar{\tau}_{\bar{x}\bar{y}} &= \bar{\tau}_{\bar{y}\bar{x}} = \mu_n \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^n, \\
 \bar{\tau}_{\bar{y}\bar{y}} &= -2\mu_n \frac{\partial \bar{u}}{\partial \bar{x}} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^{n-1},
 \end{aligned} \tag{2.4}$$

μ_n is the consistency coefficient, and n is the flow index. When the power-law exponent n is equal to 1, then the model describes the Newtonian fluid; If $n < 1$, the fluid is said to be pseudoplastic or shear thinning, and, if $n > 1$, the fluid is called dilatant or shear thickening. Using the dimensionless quantities defined by

$$\begin{aligned}
 u &= \frac{\bar{u}}{\bar{u}_0}, \quad v = \frac{\bar{v}}{\kappa \bar{u}_0} \text{ (velocities in the } x \text{ and } y \text{ directions resp.)}, \quad x = \frac{\kappa \bar{x}}{\bar{h}_0}, \quad y = \frac{\bar{y}}{\bar{h}_0}, \\
 t &= \frac{\kappa \bar{u}_0 \bar{t}}{\bar{h}_0} \text{ (time)}, \quad H = \frac{\bar{h}}{\bar{h}_0} \text{ (film thickness)}, \quad \text{Fr} = \frac{\bar{u}_0^2}{g(1-\gamma)\bar{h}_0} \text{ (Froude number)}, \\
 \bar{u}_0 &= \frac{n}{n+1} \left(\frac{\rho g (1-\gamma) \sin \theta}{\mu_n} \right)^{1/n} \bar{h}_0^{(n+1)/n} \text{ (reference velocity)}, \quad \kappa = \frac{2\pi \bar{h}_0}{\lambda} \text{ (wave number)}, \\
 p &= \frac{\bar{p} - \bar{p}_a}{\rho \bar{u}_0^2} \text{ (pressure)}, \quad \Theta = \frac{\bar{T} - \bar{T}_w}{\bar{T}_s - \bar{T}_w} \text{ (temperature)}, \quad \text{Re}_n = \frac{\rho \bar{u}_0^{2-n} \bar{h}_0^n}{\mu_n} \text{ (Reynolds number)}, \\
 \text{Pr}_n &= \frac{\mu_n \bar{u}_0^{n-1} \bar{h}_0^{1-n} c_p}{K} \text{ (Prandtl number)}, \quad \text{Pe}_n = \text{Pr}_n \cdot \text{Re}_n \text{ (Peclet number)}, \quad \Delta \bar{T} = \bar{T}_s - \bar{T}_w, \\
 \xi &= \frac{c_p \Delta \bar{T}}{h_{fg}} (h_{fg} - \text{latent heat}), \quad \text{Fi}_n = \frac{\sigma}{\rho \bar{u}_0^2 \bar{h}_0} = \text{We}_n \text{Re}_n^{-(3n+2)/(n+2)} \left(\frac{1}{\sin \theta} \right)^{(3n-2)/(n+2)}, \\
 \text{Nd} &= \frac{\xi^2}{\gamma \text{Pr}_n^2}, \quad \tau = \frac{\bar{\tau}_w}{\rho g \bar{h}_0 \sin \theta} \text{ (Shear stress at the interface)}, \\
 \text{We}_n &= \sigma \rho^{-1} \left[\left(\frac{n+1}{n} \right)^{n(3n-2)} [(1-\gamma)g]^{- (3n-2)} \nu^{-4} \right]^{1/(n+2)} \text{ (Weber number)},
 \end{aligned} \tag{2.5}$$

the nondimensional governing equations and the boundary conditions are obtained as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2.6)$$

$$\kappa \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} \right) = \frac{\sin \theta}{\text{Fr}} + \frac{\kappa^2}{\text{Re}_n} \frac{\partial \tau_{xx}}{\partial x} + \frac{1}{\text{Re}_n} \frac{\partial \tau_{xy}}{\partial y}, \quad (2.7)$$

$$\kappa^2 \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} - \frac{\cos \theta}{\text{Fr}} + \frac{\kappa}{\text{Re}_n} \left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \right), \quad (2.8)$$

$$\frac{\partial^2 \Theta}{\partial y^2} = \kappa \text{Pe}_n \left(\frac{\partial \Theta}{\partial t} + u \frac{\partial \Theta}{\partial x} + v \frac{\partial \Theta}{\partial y} \right) - \kappa^2 \frac{\partial^2 \Theta}{\partial x^2}, \quad (2.9)$$

$$u = 0, \quad v = 0, \quad \Theta = 0 \quad \text{on } y = 0, \quad (2.10)$$

$$\begin{aligned} p + \frac{1}{\text{Re}_n} \left[2\kappa \frac{\partial H}{\partial x} \tau_{xy} - \kappa \tau_{yy} - \kappa^3 \left(\frac{\partial H}{\partial x} \right)^2 \tau_{xx} \right] & \left(1 + \kappa^2 \left(\frac{\partial H}{\partial x} \right)^2 \right)^{-1} + \kappa^2 \text{Fi}_n \frac{\partial^2 H}{\partial x^2} \\ & \times \left(1 + \kappa^2 \left(\frac{\partial H}{\partial x} \right)^2 \right)^{-3/2} + Nd \text{Re}_n^{-2} (\gamma - 1) \left(\frac{\partial \Theta}{\partial y} - \kappa^2 \frac{\partial H}{\partial x} \frac{\partial \Theta}{\partial x} \right)^2 \left(1 + \kappa^2 \left(\frac{\partial H}{\partial x} \right)^2 \right)^{-1} \\ & = 0 \quad \text{on } y = H, \end{aligned} \quad (2.11)$$

$$\kappa^2 \frac{\partial H}{\partial x} (\tau_{yy} - \tau_{xx}) + \left(1 - \kappa^2 \left(\frac{\partial H}{\partial x} \right)^2 \right) \tau_{xy} = \frac{\text{Re}_n \sin \theta}{\text{Fr}} \tau \quad \text{on } y = H, \quad (2.12)$$

$$\xi \left(\frac{\partial \Theta}{\partial y} - \kappa^2 \frac{\partial H}{\partial x} \frac{\partial \Theta}{\partial x} \right) - \kappa \text{Pe}_n \left(\frac{\partial H}{\partial t} + u \frac{\partial H}{\partial x} - v \right) = 0 \quad \text{on } y = H, \quad (2.13)$$

$$\Theta = 1 \quad \text{on } y = H. \quad (2.14)$$

In (2.13), $\xi > 0$ corresponds to condensate film flow (saturated vapor temperature is more than the wall temperature; $(\bar{T}_s > \bar{T}_w)$), $\xi < 0$ corresponds to evaporating film flow (saturated vapor temperature is less than the wall temperature; $(\bar{T}_s < \bar{T}_w)$), and $\xi = 0$ corresponds to isothermal film (saturated vapor temperature is equal to the wall temperature; $(\bar{T}_s = \bar{T}_w)$).

It is to be noted that, when $n = 1, \tau = 0, \theta = 90^\circ$, the above equations and boundary conditions reduce to evaporating or condensating Newtonian flow down a vertical wall investigated by Hwang and Weng [18] and, for $\tau = 0$, they reduce to the equations obtained for nonisothermal power-law fluid film flow investigated by Usha and Uma [34]. Since the long wavelength modes are the most unstable ones for the film flow, the physical quantities u, v, p , and Θ are expanded in powers of small wave number κ . Substituting these in (2.6)–(2.14) and collecting the coefficients of like powers of κ , the zeroth- and the first-order equations are obtained. Noting that We_n is large in practical applications, $\kappa^2 \text{We}_n$ is taken to be of order one. Also, since the effect of Nd has been found to be negligible on stability [15], $Nd \text{Re}_n^{-2}$ is taken to be of order κ^2 . Further, in the analysis, $\text{Re}_n \simeq O(1)$ and $\text{Pe}_n \simeq O(1)$. The

first-order solutions, u_1 , v_1 , and Θ_1 , will have the time derivative H_t in the right hand side, which is replaced by the space derivative from the leading order representation [18] given by

$$H_t = \frac{\xi}{\kappa \text{Pe}_n} \Theta_{0y} - u_0 H_x + v_0. \quad (2.15)$$

The zeroth- and the first-order (after replacing all the time derivatives with (2.15)) solutions are presented in the appendix.

The solutions (A.1) and (A.2) are substituted into the expanded kinematic boundary condition (2.13) yields:

$$H_t = \frac{\xi}{\kappa \text{Pe}_n} \left[\Theta_{0y} + \kappa \Theta_{1y} - \kappa^2 H_x (\Theta_{0x} + \kappa \Theta_{1x}) \right] - \{ (u_0 + \kappa u_1) H_x - v_0 - \kappa v_1 \}. \quad (2.16)$$

The simplification of (2.16) now gives the generalized kinematic equation as

$$H_t + X(H) + A(H)H_x + B(H)H_{xx} + C(H)H_{xxx} + D(H)H_x^2 + E(H)H_x H_{xxx} + O(\kappa^2) = 0, \quad (2.17)$$

where

$$\begin{aligned} X(H) &= -\frac{\xi}{\kappa \text{Pe}_n} \left(1 - \frac{\xi}{3} \right) \frac{1}{H}, \\ A(H) &= \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{1/n} \\ &\quad \times \left\{ (H + \tau)^{1/n} H - \frac{\xi n}{n+1} \right. \\ &\quad \times \left[\frac{1}{6} (H + \tau)^{(n+1)/n} - \frac{n^2}{(2n+1)(3n+1)} \left(\frac{(H + \tau)^{(3n+1)/n} + \tau^{(3n+1)/n}}{H^2} \right) \right. \\ &\quad \left. \left. + \frac{2n^3}{(2n+1)(3n+1)(4n+1)} \left(\frac{(H + \tau)^{(4n+1)/n} - \tau^{(4n+1)/n}}{H^3} \right) \right] \right\} \\ &\quad + \frac{\text{Re}_n \xi}{(n+1) \text{Pe}_n} \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(2-n)/n} \end{aligned}$$

$$\begin{aligned}
& \times \left\{ -\frac{2}{n}(H+\tau)^{(2-n)/n}(H-n\tau) + \left(\frac{(H+\tau)^{(1-n)/n}}{(n+1)H} - \frac{n(H+\tau)^{1/n}}{(n+1)H^2} \right) \right. \\
& \quad \times \left[\left((H+\tau)^{(n+1)/n}(H-n\tau) + n\tau^{(2n+1)/n} \right) - \frac{n}{2n+1} \left((H+\tau)^{(2n+1)/n} - \tau^{(2n+1)/n} \right) \right] \\
& \quad + \frac{n^2}{2(n+1)(n+2)} \frac{1}{H^2} \left[(H+\tau)^{2(n+1)/n} - \tau^{2(n+1)/n} \right] \\
& \quad - \frac{n}{(n+2)} \frac{1}{H} (H+\tau)^{(n+2)/n} + \frac{(H+\tau)^{2/n}(H-n\tau)}{H} \\
& \quad \left. - \frac{n^2}{n+1} \frac{\tau^{(n+1)/n}}{H^2} \left[(H+\tau)^{(n+1)/n} - \tau^{(n+1)/n} \right] + \frac{\tau^{(n+1)/n}}{H} n(H+\tau)^{1/n} \right\},
\end{aligned}$$

$$\begin{aligned}
B(H) &= \kappa \text{Re}_n \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(3-n)/n} (H+\tau)^{1/n} \\
& \times \left\{ \frac{3n^2}{(n+1)(n+2)(2n+1)} (H+\tau)^{2(n+1)/n} H - \frac{n}{(n+1)(n+2)} (H+\tau)^{(n+2)/n} H^2 \right. \\
& \quad + (H+\tau)^{2/n} H^3 - \frac{3n^3}{(n+1)(n+2)(2n+1)(3n+2)} \\
& \quad \times \left[(H+\tau)^{(3n+2)/n} - \tau^{(3n+2)/n} \right] + \frac{2n^2}{(n+1)(2n+1)} \tau^{(2n+1)/n} \\
& \quad \times \left[\frac{n}{n+1} \left((H+\tau)^{(n+1)/n} - \tau^{(n+1)/n} \right) - (H+\tau)^{1/n} H \right] \\
& \quad + \frac{2n^2}{(n+1)(2n+1)} (H+\tau)^{1/n} H \left[(H+\tau)^{(2n+1)/n} - \tau^{(2n+1)/n} \right] \\
& \quad - \frac{2n^3}{(n+1)^2(2n+1)} (H+\tau)^{(n+1)/n} \left[(H+\tau)^{(2n+1)/n} - \tau^{(2n+1)/n} \right] \\
& \quad \left. + \frac{2n^2}{(n+1)^2} (H+\tau)^{2(n+1)/n} H - \frac{3n}{n+1} (H+\tau)^{(n+2)/n} H^2 \right\} \\
& - \frac{\kappa \text{Re}_n}{(n+1)} \frac{\cos \theta}{\text{Fr}} \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(1-n)/n} \\
& \times \left\{ (H+\tau)^{1/n} (H-n\tau) H - \frac{n}{n+1} \left[(H+\tau)^{(n+1)/n} (H-n\tau) + n\tau^{(2n+1)/n} \right] \right. \\
& \quad \left. + \frac{n^2}{(n+1)(2n+1)} \left[(H+\tau)^{(2n+1)/n} - \tau^{(2n+1)/n} \right] \right\},
\end{aligned}$$

$$\begin{aligned}
C(H) &= \frac{\kappa^3 \text{Re}_n}{(n+1)} \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(1-n)/n} \\
&\quad \times \text{Fi}_n \left\{ (H + \tau)^{1/n} (H - n\tau) H - \frac{n}{n+1} \left[(H + \tau)^{(n+1)/n} (H - n\tau) + n\tau^{(2n+1)/n} \right] \right. \\
&\quad \left. + \frac{n^2}{(n+1)(2n+1)} \left[(H + \tau)^{(2n+1)/n} - \tau^{(2n+1)/n} \right] \right\}, \\
D(H) &= \kappa \text{Re}_n \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(3-n)/n} \\
&\quad \times \left\{ \frac{n(6n^2 + 19n + 11)}{(n+1)^2(n+2)(2n+1)} (H + \tau)^{(n+3)/n} H - \frac{(7n+15)}{(n+1)(n+2)} (H + \tau)^{3/n} H^2 \right. \\
&\quad + \frac{3}{n} (H + \tau)^{(3-n)/n} H^3 - \frac{4n}{(n+1)(2n+1)} \tau^{(2n+1)/n} (H + \tau)^{(2-n)/n} H \\
&\quad + \frac{4n}{(n+1)(2n+1)} (H + \tau)^{(2-n)/n} H \left[(H + \tau)^{(2n+1)/n} - \tau^{(2n+1)/n} \right] \\
&\quad - \frac{3n^2 (H + \tau)^{(1-n)/n}}{(n+1)(n+2)(2n+1)(3n+2)} \left[(H + \tau)^{(3n+2)/n} - \tau^{(3n+2)/n} \right] \\
&\quad + \frac{2n^2}{(n+1)^2(2n+1)} \tau^{(2n+1)/n} (H + \tau)^{(1-n)/n} \left[(H + \tau)^{(n+1)/n} - \tau^{(n+1)/n} \right] \\
&\quad \left. - \frac{2n^2}{(n+1)^2(2n+1)} (H + \tau)^{2/n} \left[(H + \tau)^{(2n+1)/n} - \tau^{(2n+1)/n} \right] \right\} \\
&\quad - \frac{\kappa \text{Re}_n}{(n+1)} \frac{\cos \theta}{\text{Fr}} \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(1-n)/n} \left\{ \frac{1}{n} (H + \tau)^{(1-n)/n} (H - n\tau) H + (H + \tau)^{1/n} H \right\} \\
&\quad - \frac{\xi \kappa}{\text{Pe}_n} \left(1 - \frac{\xi}{3} \right) \frac{1}{H}, \\
E(H) &= \frac{\kappa^3 \text{Re}_n}{(n+1)} \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(1-n)/n} \text{Fi}_n \left\{ \frac{1}{n} (H + \tau)^{(1-n)/n} (H - n\tau) H + (H + \tau)^{1/n} H \right\}.
\end{aligned} \tag{2.18}$$

3. Stability Analysis

As the variation of the film thickness of the base flow is found to be very small for $|\kappa H_x| \ll 1$ using an analysis based on Nusselt assumption, the dimensionless film thickness is expressed as

$$H = 1 + \eta(x, t), \tag{3.1}$$

where $\eta(x, t)$ is the perturbation of the stationary film thickness. The approximation $|\kappa H_x| \ll 1$ gives qualitative results for the constant film thickness assumption at the zeroth order. It is important to note that this constant film thickness approximation with long-wave

perturbations are reasonable approximations only for certain segments of weakly condensing and evaporating flows. Substituting for $H(x, t)$ in (2.18) and retaining terms up to the order of η^3 , the evolution equation for η is obtained as

$$\begin{aligned} \eta_t + X'\eta + A\eta_x + B\eta_{xx} + C\eta_{xxx} \\ = - \left[\frac{X''\eta^2}{2} + \frac{X'''\eta^3}{6} + \left(A'\eta + A''\frac{\eta^2}{2} \right) \eta_x + \left(B'\eta + B''\frac{\eta^2}{2} \right) \eta_{xx} \right. \\ \left. + \left(C'\eta + C''\frac{\eta^2}{2} \right) \eta_{xxx} + (D + D'\eta)\eta_x^2 + (E + E'\eta)\eta_x\eta_{xx} \right] + O(\eta^4), \end{aligned} \quad (3.2)$$

where the values of X, A, B, C, D, E , and their derivatives are evaluated at the dimensionless height of the film $H = 1$. It is worth mentioning here that the analysis based on locally valid unsteady equation has been the subject of several investigations of laminar film condensation [7, 8, 15, 17, 18, 34, 39–42]. Equation (3.2) describes the behavior of finite-amplitude disturbances on the power-law film, and it is used to predict the time-wise behavior of an initially sinusoidal disturbance on the power-law film. It is important to remark that the simplified unsteady equation (3.2) is only locally valid.

In order to understand the flow characteristics and the associated time-dependent properties of power-law liquid film down an inclined plane, typical values of the physical parameters considered by Lin and Hwang [24], Usha and Uma [34], and Pascal and D'Alessio [36] have been used for numerical evaluation and are given by

$$\begin{aligned} \rho = 998 \text{ Kg/m}^3, \quad \mu_n = 1.002 \times 10^{-3} \text{ Pas}^n, \\ h_0 = 10^{-4} \text{ m}, \quad \sigma = 0.0727 \text{ N/m}, \quad \gamma = 0.000611, \quad \theta = 60^\circ. \end{aligned} \quad (3.3)$$

The temperature at the interface is taken as $\bar{T}_s = 373 \text{ K}$ and the temperature difference between the wall and the interface as $\bar{T}_s - \bar{T}_w = \pm 47 \text{ K}$. Under such temperature conditions, the phase change parameter ξ takes values $|\xi| = 0.0872$ (with phase change) and $|\xi| = 0$ (without phase change). Shear stress parameter τ varies between 0.0 and 2.0.

3.1. Linear Stability Analysis

For the linear stability analysis, the nonlinear terms of (3.2) are neglected and the linearized equation

$$\eta_t + X'\eta + A\eta_x + B\eta_{xx} + C\eta_{xxx} = 0 \quad (3.4)$$

is obtained. Assuming the normal mode solution as

$$\eta = ae^{i(x-dt)} + \bar{a}e^{-i(x-dt)}, \quad (3.5)$$

Figure 5: Threshold amplitude in the supercritical stable region.

the complex wave celerity corresponding to linear stability problem is given by

$$d = d_r + id_i = A + i(B - C - X'), \quad (3.6)$$

where d_r is the linear wave speed and d_i is the linear growth rate of the amplitudes. The flow is in a linearly unstable supercritical condition for $d_i > 0$ and in a linearly stable subcritical condition for $d_i < 0$. For $d_i = 0$, the flow is neutrally stable. Zero linear growth rate of the perturbation ($d_i = 0$) gives rise to a cut-off wave number $\kappa = \kappa_c(n, \text{Re}_n, \text{We}_n, \theta, \xi, \text{Pr}_n, \tau)$.

Figure 2 shows the neutral stability curve for isothermal, condensate, and evaporating power-law films with the effect of wind stress at the interface. It is observed from Figure 2 that the neutral stability curve plotted from the linear stability analysis separates $\kappa - n$ plane into two regions depending on the value of phase change parameter ξ and shear stress parameter τ . Region of linear instability ($d_i > 0$) increases with the increase in the shear stress τ at the interface and decreases with the increase in the phase change parameter ξ . The neutral stability curve for condensate film shows that there exists a critical value for n , above which condensate power-law fluid film is always stable, and this critical value increases with the increase in shear stress τ at the interface (Figure 2(d)). It is also observed that condensate

Figure 6: Nonlinear wave speed in the supercritical stable region.

power-law fluid film is more stable than the corresponding isothermal and evaporating power-law fluid film in the absence of wind stress.

Figure 3 shows the temporal growth rate of power-law fluid given by (3.6). Temporal growth rate of the power-law liquid film increases with the increase in shear stress parameter τ and decrease in power-law index n . It is observed that temporal growth rate is lower for a dilatant ($n = 1.05$) condensate ($\xi > 0$) film when $\tau = 0$ and higher for a pseudoplastic ($n = 0.95$) evaporating ($\xi < 0$) film when $\tau > 0$. The results of the linear stability analysis are in good agreement with those of Lin and Hwang [24] (isothermal power-law liquid film for zero shear stress at the interface) and Usha and Uma [34] (power-law condensate/evaporating film for zero shear stress at the interface).

3.2. Weakly Nonlinear Stability Analysis

As the perturbed wave grows to a finite amplitude, linear stability theory cannot be used to predict the flow behavior accurately. Therefore, in order to examine whether the finite-amplitude disturbance in the linearly stable region causes instability (subcritical instability) and to investigate whether the subsequent nonlinear evolution of disturbances in the linearly

Figure 7: Evolution of free surface for an isothermal ($\xi = 0$) pseudoplastic ($n = 0.95$) film when $\kappa = 0.06$ and $\tau = 0, 0.2, 0.4$.

unstable region develops into a new equilibrium state with a finite-amplitude (supercritical stability) or grows to be unstable, the nonlinear stability analysis is employed. The nonlinear stability analysis of (3.2) by the method of multiple scales [43] yields

$$\left(L_0 + \epsilon L_1 + \epsilon^2 L_2\right) \left(\epsilon \eta_1 + \epsilon^2 \eta_2 + \epsilon^3 \eta_3\right) = -\epsilon^2 N_2 - \epsilon^3 N_3 + o(\epsilon^4), \quad (3.7)$$

where

$$\begin{aligned} \eta(\epsilon, x, x_1, t, t_1, t_2) &= \epsilon \eta_1 + \epsilon^2 \eta_2 + \epsilon^3 \eta_3, \quad t_1 = \epsilon t, \quad t_2 = \epsilon^2 t, \quad x_1 = \epsilon x, \\ \frac{\partial}{\partial t} &\longrightarrow \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2}; \quad \frac{\partial}{\partial x} \longrightarrow \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial x_1}, \\ L_0 &= \frac{\partial}{\partial t} + X' + A \frac{\partial}{\partial x} + B \frac{\partial^2}{\partial x^2} + C \frac{\partial^4}{\partial x^4}, \end{aligned}$$

Figure 8: Evolution of free surface for an isothermal ($\xi = 0$) Newtonian ($n = 1.0$) film when $\kappa = 0.05$ and $\tau = 0, 0.2, 0.4$.

$$\begin{aligned}
 L_1 &= \frac{\partial}{\partial t_1} + A \frac{\partial}{\partial x_1} + 2B \frac{\partial^2}{\partial x \partial x_1} + 4C \frac{\partial^4}{\partial x^3 \partial x_1}, \\
 L_2 &= \frac{\partial}{\partial t_2} + B \frac{\partial^2}{\partial x_1^2} + 6C \frac{\partial^4}{\partial x^2 \partial x_1^2}, \\
 N_2 &= \frac{X''}{2} \eta_1^2 + A' \eta_1 \eta_{1x} + B' \eta_1 \eta_{1xx} + C' \eta_1 \eta_{1xxx} + D \eta_{1x}^2 + E \eta_{1x} \eta_{1xxx}, \\
 N_3 &= X'' \eta_1 \eta_2 + \frac{X'''}{6} \eta_1^3 + A' (\eta_1 \eta_{2x} + \eta_{1x} \eta_2 + \eta_1 \eta_{1x_1}) \\
 &\quad + B' (\eta_1 \eta_{2xx} + 2\eta_1 \eta_{1xx_1} + \eta_{1xx} \eta_2) \\
 &\quad + C' (\eta_1 \eta_{2xxx} + 4\eta_1 \eta_{1xxx_1} + \eta_{1xxx} \eta_2) + D (2\eta_{1x} \eta_{2x} + 2\eta_{1x} \eta_{1x_1}) \\
 &\quad + E (\eta_{1x} \eta_{2xxx} + 3\eta_{1x} \eta_{1xxx_1} + \eta_{1xxx} \eta_{2x} + \eta_{1xxx} \eta_{1x_1}) \\
 &\quad + \frac{A''}{2} \eta_1^2 \eta_{1x} + \frac{B''}{2} \eta_1^2 \eta_{1xx} + \frac{C''}{2} \eta_1^2 \eta_{1xxx} + D' \eta_1 \eta_{1x}^2 + E' \eta_1 \eta_{1x} \eta_{1xxx}.
 \end{aligned}$$

(3.8)

Figure 9: Evolution of free surface for isothermal ($\xi = 0$) dilatant ($n = 1.05$) film when $\kappa = 0.03$ and $\tau = 0, 0.2, 0.4$.

The solution of (3.7) at the order $O(\epsilon)$ is obtained by solving $L_0\eta_1 = 0$ and is in the form

$$\eta_1 = ae^{i(x-d_r t)} + \bar{a}e^{-i(x-d_r t)}, \quad (3.9)$$

where $a(x_1, t_1, t_2)$ is the nonlinear amplitude function and $\bar{a}(x_1, t_1, t_2)$ is its complex conjugate. The solution of the equation $L_0\eta_2 + L_1\eta_1 = -N_2$ at the $O(\epsilon^2)$ is in the form

$$\eta_2 = ea^2e^{2i(x-d_r t)} + \bar{e}\bar{a}^2e^{-2i(x-d_r t)}. \quad (3.10)$$

Figure 10: Maximum and minimum amplitude of the waves down an inclined plane for an isothermal power-law film.

Using the solutions for η_1 and η_2 in the $O(\epsilon^3)$ equation given by $L_0\eta_3 + L_1\eta_2 + L_2\eta_1 = -N_3$, the equation for the perturbation amplitude $a(x_1, t_1, t_2)$ is obtained as

$$\frac{\partial a}{\partial t_2} + D_1 \frac{\partial^2 a}{\partial x_1^2} - \epsilon^{-2} d_i a + (E_1 + iF_1) a^2 \bar{a} = 0 \quad (3.11)$$

from the secular condition for $O(\epsilon^3)$, where

$$\begin{aligned} D_1 &= B - 6C, \quad F_1 = (X'' - 5B' + 17C' + 4D - 10E)e_i + A'e_r + \frac{A''}{2}, \\ E_1 &= (X'' - 5B' + 17C' + 4D - 10E)e_r - A'e_i + \left(\frac{X'''}{2} - \frac{3}{2}B'' + \frac{3}{2}C'' + D' - E' \right), \\ e_r + ie_i &= \frac{(-(X''/2) + B' - C' + D - E) - iA'}{(16C - 4B + X')}. \end{aligned} \quad (3.12)$$

Figure 11: Evolution of free surface for an isothermal film ($\xi = 0$) when $\tau = 2.0$.

The weakly nonlinear behavior of the fluid film can be investigated using (3.12). It is important to note that such an expansion is only valid for wave numbers close to neutral and not near critical when κ approaches zero. The solution of (3.12) for a filtered wave in which spatial modulation does not exist and the diffusion terms in (3.12) vanishes is obtained by taking $a = a_0 e^{-ib(t_2)t_2}$. This leads to the Ginzburg-Landau equation given by

$$\frac{\partial a_0}{\partial t_2} = (\epsilon^{-2} d_i - E_1 a_0^2) a_0, \quad (3.13)$$

$$\frac{\partial(b(t_2)t_2)}{\partial t_2} = F_1 a_0^2. \quad (3.14)$$

The second term in (3.13) induced by the effect of nonlinearity can either accelerate or decelerate the exponential growth of the linear disturbance depending upon the signs of d_i

and E_1 . The perturbed wave speed caused by the infinitesimal disturbances appearing in the nonlinear system can be modified using (3.13). The threshold amplitude ϵa_0 is given by

$$\epsilon a_0 = \sqrt{\frac{d_i}{E_1}}, \quad (3.15)$$

and the nonlinear wave speed is given as

$$Nc_r = \epsilon^2 b = d_r + d_i \frac{F_1}{E_1}. \quad (3.16)$$

It is observed from (3.15) that in the linearly unstable region ($d_i > 0$), the condition for existence of a supercritical stable region is $E_1 > 0$ and ϵa_0 is the threshold amplitude. In the linearly stable region ($d_i < 0$), if $E_1 < 0$, then the flow has the behavior of subcritical instability and ϵa_0 is the threshold amplitude. The condition for the existence of a subcritical stable region is $E_1 > 0$, and $E_1 = 0$ gives the condition of existence of a neutral stability curve.

The neutral stability curves are obtained from (3.6) and (3.12) by equating to zero, the linear amplification rate d_i , and the nonlinear amplification rate E_1 . Figure 4 shows the regions of subcritical stability ($d_i < 0, E_1 > 0$) and subcritical instability ($d_i < 0, E_1 < 0$) in the linearly stable region and supercritical stability ($d_i > 0, E_1 > 0$) and supercritical explosive state ($d_i > 0, E_1 < 0$) in the linearly unstable region. It is observed that subcritical stable region ($d_i < 0, E_1 > 0$) decreases, and supercritical stability ($d_i > 0, E_1 > 0$) increases with the increase in shear stress parameter τ . However, for the condensate film ($\xi > 0$), region of supercritical explosive state ($d_i > 0, E_1 < 0$) and supercritical stability ($d_i > 0, E_1 > 0$) exists only for the pseudoplastic fluids ($n < 1$) which increases with the increase in τ .

Figures 5 and 6 show the threshold amplitude and nonlinear wave speed in the supercritical stable region for various values of phase change parameter ξ and shear stress parameter τ . The flow system is stable if the finite amplitude of the disturbance is more than the threshold amplitude and explosively unstable otherwise. The results show that the threshold amplitude and nonlinear wave speed increases with increase in τ and decreases with increase in ξ and n .

The above weakly nonlinear analysis shows the existence of both a supercritical stable region and a subcritical unstable region for isothermal and nonisothermal power-law liquid film down an inclined plane in the presence of shear stress at the interface. It is observed that the evolution of two-dimensional waves depends strongly on the initial disturbance wave number. There exists a value of κ_s (obtained from equating denominator of $E_1 = 0$) such that, when $\kappa_s < \kappa < \kappa_c$ (obtained from $d_i = 0$; cut-off wave number), the flow is supercritically stable and nonlinear equilibration occurs after the initial instability. The weakly nonlinear stability analysis shows that $\kappa_s = \kappa_c/2$, and the curve $\kappa = \kappa_s$ separates the linearly unstable region into two portions where the nonlinear waves attain a finite equilibrium amplitude ($d_i > 0, E_1 > 0$) or reach an explosive state ($d_i > 0, E_1 < 0$), and the flow is supercritically equilibrated.

3.3. Nonlinear Analysis

In order to understand the mechanism responsible for the transfer of energy from the basic state to the disturbance, the evolution of finite-amplitude perturbations is considered. On

Figure 12: Evolution of free surface for condensating ($\xi > 0$) pseudoplastic ($n = 0.95$) film when $\kappa = 0.06$ and $\tau = 0$.

the evolution of these perturbations, the influence of wind stress and mass transfer at the interface of a thin power-law liquid film down an inclined plane is examined by numerically solving the nonlinear evolution equation. The initial disturbance is taken to be a sinusoidal wave with small amplitude given by

$$H(x, 0) = 1 - 0.1 \cos(\kappa x). \quad (3.17)$$

The evolution of waves with time is obtained by solving the evolution equation (2.18) by using Lee's three-time level [44] finite-difference method in a periodic domain $[-\pi/\kappa, \pi/\kappa]$. The finite-difference scheme achieves linearity in the unknown $H_{i,j+1}$ by evaluating all coefficients of $H_{i,j+1}$ at a time level of known solution values, preserves stability by averaging $H_{i,j}$ over three time levels, and maintains accuracy by using central difference approximations, where j denotes the number of time steps and i is a spatial grid point. Special attention is given to the discretization of the wave propagation term so as to overcome the solution blowup for very small initial wave numbers. The scheme possesses both conservative and transportive properties and allows to follow the solution over larger periods

Figure 13: Evolution of free surface for condensating ($\xi = 0.0872$) pseudoplastic ($n = 0.95$) film when $\kappa = 0.06$ and $\tau = 0, 0.2, 0.4$.

of time. The resulting algebraic equations are solved using MATLAB, and the iterations are continued until the local minimum layer thickness becomes smaller than the maximum error bound of 10^{-10} . The computations are performed with $\Delta t = 0.01$ and with the number of nodes along the spatial direction as $N = 100$ and $\Delta x = 2\pi/N\kappa$. It is to be noted that the solution obtained with a decrease in Δt or an increase in N does not show any deviation from the values obtained for $\Delta t = 0.01$, $\Delta x = 2\pi/N\kappa$, $N = 100$. Further, it is confirmed that the wave that emerges on the film surface does not change its shape for a relatively long period of time.

The weakly nonlinear analysis predicts the occurrence of a wave number κ_s that separates the regimes of supercritical $\kappa_s < \kappa < \kappa_c$ and subcritical $\kappa < \kappa_s$ domains (Figure 4). For a wave number κ close to κ_c , the solution may evolve into a stable, almost sinusoidal wave of small finite amplitude. For κ close to $\kappa_m = \kappa_c/\sqrt{2}$, the surface wave approaches the form of a solitary wave as observed in experimental results by Alekseenko et al. [45]

Figure 14: Evolution of free surface for an evaporating film ($\xi < 0$) when $\kappa = 0.06$ and $\tau = 0$.

and Liu et al. [46]. The evolution of the film in the supercritical stable region is analyzed by performing numerical simulations for initial disturbance wave number $\kappa = \kappa_c$, $\kappa_m (= \kappa_c / \sqrt{2})$, and $\kappa_s (= \kappa_c / 2)$. The evolution of the film is examined for values of ξ in the range -0.0872 to 0.0872 and τ between 0 and 2.0 .

Figures 7–9 show the evolution of the free surface configuration of isothermal ($\xi = 0$) pseudoplastic ($n = 0.95$), Newtonian ($n = 1.0$), and dilatant ($n = 1.05$) fluids at various instants of time for different values of shear stress parameter at the interface. Wave number is chosen in the range $\kappa_s (= \kappa_c / 2) < \kappa < \kappa_c$ (Figure 7— $\kappa = 0.06$, $\kappa_c = 0.0868$ when $\tau = 0$, $\kappa_c = 0.0971$ when $\tau = 0.2$, and $\kappa_c = 0.1064$ when $\tau = 0.4$; Figure 8— $\kappa = 0.05$, $\kappa_c = 0.0569$ when $\tau = 0$, $\kappa_c = 0.0635$ when $\tau = 0.2$, and $\kappa_c = 0.0694$ when $\tau = 0.4$; Figure 9— $\kappa = 0.03$, $\kappa_c = 0.0365$ when $\tau = 0$, $\kappa_c = 0.0409$ when $\tau = 0.2$, and $\kappa_c = 0.0449$ when $\tau = 0.4$). It is observed that the growth rate is much more important and the distortion of the free surface is conspicuous with increase in time. The wave amplitude decays after reaching a maximum, and the rear becomes longer and longer to attain a one-hump solitary-like wave at different instant of time for various values of τ . The large amplitude wave is then followed by a small amplitude capillary wave. In all the cases, it is observed that finite-amplitude permanent wave emerges. For a given n , amplitude of the wave increases with the increase in shear stress parameter τ . The corresponding evolution of maximum ($H_{\max}$) and minimum ($H_{\min}$) thickness is presented in Figure 10. The disturbance amplitude $H_{\max}$ remains constant for initial times $t \sim 0$ – 5 for pseudoplastic fluids ($n = 0.95$), $t \sim 0$ – 10 for Newtonian fluids ($n = 1.0$), and $t \sim 0$ – 20 for dilatant fluids ($n = 1.05$) and then increases monotonically to attain a maximum amplitude.

Figure 15: Evolution of free surface for an evaporating film ($\xi = -0.00872$) when $\kappa = 0.06$ and $\tau = 0.2, 2.0$.

Isothermal pseudoplastic fluid attains the maximum amplitude in short time compared to Newtonian and dilatant fluid, and this amplitude further increases with the presence of shear stress at the interface. The distortion of the free surface is more significant when the effect of shear stress increases (Figure 11).

Weakly nonlinear stability analysis predicted that for the condensate film ($\xi > 0$), region of supercritical stability ($d_i > 0, E_1 > 0$) exists only for the pseudoplastic fluids ($n = 0.95$). This is the case when the saturated vapor temperature is more than the wall temperature. When condensation is considered, the static layer grows thicker with time. Evolution of condensate pseudoplastic liquid film down an inclined plane is depicted in Figure 12 for $\xi = 0.01, 0.02, 0.04$, and 0.06 and $\tau = 0$. The result shows that pseudoplastic film down an inclined plane thickens faster as the phase change parameter ξ increases gradually in the absence of shear stress parameter τ . The effect of wind stress on the interface of the pseudoplastic film takes little longer time to thicken (Figure 13) and makes the film more corrugated as τ increases (Figures 13(a)(ii), 13(b)(ii), 13(c)(ii)).

Figure 14 shows the evolution of the free-surface shape for evaporating pseudoplastic, Newtonian, and dilatant fluids when $\xi = -0.01$ and -0.0872 with a zero shear stress at the interface. This is the case when the saturated vapor temperature is less than the wall temperature. It is observed that as n increases, depth of the liquid layer disappears in a finite time $t = t_r$ decreases leading to rupture instability. The rupture instability is thus augmented by the evaporative effect. Plot describes that the dilatant film evaporates faster than the pseudoplastic and Newtonian film. For a given n (given pseudoplastic, Newtonian,

or dilatant fluid), evaporation of the film happens at earlier instant of time when the phase change parameter ξ decreases (Figure 14). Presence of wind stress on the interface does not influence the time of rupture t_r for any n whereas it has little influence on the position where the rupture takes place (Figure 15).

4. Conclusion

The influence of prescribed cocurrent superficial shear stress on the dynamics and stability of a condensate or evaporating power-law liquid film falling down an inclined plane has been analyzed by the method of long-wave perturbation. The interfacial boundary conditions include the effects of phase change across the interface. The evolution equation of the Benney type incorporating the effect of superficial shear stress on the surface of a nonisothermal power-law film has been derived. As the power-law exponent " n " decreases, the effective viscosity decreases, and, hence, " n " influences the Reynolds number Re_n , Weber number We_n , and Prandtl number Pr_n . The results of the linear stability analysis reveal that the effect of decreasing the phase change parameter ξ or increasing the shear stress parameter τ is to destabilize the film flow system. Further, the dimensional quantities used to discuss the stability characteristics of the power-law model in terms of the power-law exponent " n " show that there exists a critical value of " n " for the condensate film, above which the film flow system is always stable (Figure 2), and this critical value of " n " increases with increase in shear stress parameter τ (Figure 2(d)), while isothermal and evaporating power-law fluid films are unstable for any value of power-law index " n ". The weakly nonlinear stability analysis of the power-law film flow system using long-wave theory is even more qualitative than the linearized stability results, and it reveals that both subcritical instability ($d_i < 0, E_1 < 0$) and supercritical stability ($d_i > 0, E_1 > 0$) are possible for isothermal and evaporating power-law fluid films and supercritical stability ($d_i > 0, E_1 > 0$) and supercritical explosive state ($d_i > 0, E_1 < 0$) are not possible for condensate dilatant fluid films (Figure 4).

The nonlinear evolution of film thickness is found by numerically solving the evolution equation in a periodic domain. The numerical simulations confirm the results found on the basis of linear and weakly nonlinear stability analysis. For an isothermal power-law liquid film, the finite amplitude waves are stable solutions of the evolution equation when the wave numbers are less than the cut-off wave number κ_c . The permanent waves are nearly sinusoidal for initial wave numbers close to the cut-off wave number, and, for values much smaller than the cut-off value, the permanent waves are of solitary type. It is interesting to note that the range of supercritical stability broadens with an increase in the shear stress parameter τ for fixed values of other nondimensional parameters governing the film flow system. The nonlinear interactions in falling isothermal power-law films down an inclined plane exhibit a tendency toward permanent two-dimensional waves for very thin films at small wave numbers close to the cut-off wave number. The nonzero shear stress at the interface promotes the growth rate of the wave amplitude (Figure 10).

The results reveal that (i) for an isothermal dilatant (pseudoplastic) liquids, the maximum wave amplitude is always smaller (larger) than that for a Newtonian liquid and the amplitude of permanent wave increases with the increase in prescribed shear stress τ at the interface in cocurrent direction; (ii) condensation of pseudoplastic film happens for the earlier instant of time when the phase change parameter increases and the effect of shear stress at the interface makes the film more corrugated; (iii) dilatant (pseudoplastic) evaporating liquid film attains rupture faster (slower) than that of Newtonian liquid film,

and the prescribed shear stress τ at the interface does not influence the time of rupture and has little influence on the position where the rupture takes place.

Appendix

The zeroth- and the first-order solutions of the physical quantities are given by

$$\begin{aligned}
 u_0 &= \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{1/n} \left(\frac{n}{n+1} \right) \left[(H + \tau)^{(n+1)/n} - (H - y + \tau)^{(n+1)/n} \right], \\
 v_0 &= - \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{1/n} H_x \left[(H + \tau)^{1/n} y + \frac{n}{n+1} (H - y + \tau)^{(n+1)/n} - \frac{n}{n+1} (H + \tau)^{(n+1)/n} \right], \\
 p_0 &= \frac{\cos \theta}{\text{Fr}} (H - y) - \kappa^2 \text{Fi}_n H_{xx}, \\
 \Theta_0 &= \frac{y}{H},
 \end{aligned} \tag{A.1}$$

$$\begin{aligned}
 u_1 &= - \frac{\text{Re}_n \xi}{(n+1) \kappa \text{Pe}_n} \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(2-n)/n} \frac{1}{H} \\
 &\times \left\{ (H + \tau)^{1/n} \left[(H + \tau)^{1/n} (H - n\tau) - (H - y + \tau)^{1/n} (H - y - n\tau) \right] \right. \\
 &\quad + n\tau^{(n+1)/n} \left[(H + \tau)^{1/n} - (H - y + \tau)^{1/n} \right] \\
 &\quad \left. - \frac{n}{n+2} \left[(H + \tau)^{(n+2)/n} - (H - y + \tau)^{(n+2)/n} \right] \right\} \\
 &+ \text{Re}_n \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(3-n)/n} H_x (H + \tau)^{1/n} \\
 &\times \left\{ - \frac{n}{(n+1)(n+2)} \left[(H + \tau)^{(n+2)/n} H - (H - y + \tau)^{(n+2)/n} (H - y) \right] \right. \\
 &\quad + \frac{n^2 (4n+5)}{2(n+1)^2 (n+2)(2n+1)} \left[(H + \tau)^{2(n+1)/n} - (H - y + \tau)^{2(n+1)/n} \right] \\
 &\quad - \frac{2n^2}{(n+1)(2n+1)} \tau^{(2n+1)/n} \left[(H + \tau)^{1/n} - (H - y + \tau)^{1/n} \right] \\
 &\quad - (H + \tau)^{1/n} (H - y + \tau)^{1/n} (H - y) H + (H + \tau)^{2/n} H^2 \\
 &\quad - \frac{n}{n+1} \left[(H + \tau)^{(n+2)/n} H - (H + \tau)^{1/n} (H - y + \tau)^{(n+1)/n} H \right] \\
 &\quad - \frac{n}{n+1} \left[(H + \tau)^{(n+2)/n} H - (H + \tau)^{(n+1)/n} (H - y + \tau)^{1/n} (H - y) \right] \\
 &\quad \left. + \left(\frac{n}{n+1} \right)^2 \left[(H + \tau)^{2(n+1)/n} - (H + \tau)^{(n+1)/n} (H - y + \tau)^{(n+1)/n} \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\text{Re}_n}{(n+1)} \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(1-n)/n} \left(-\frac{\cos \theta}{\text{Fr}} H_x + \kappa^2 \text{Fi}_n H_{xxx} \right) \\
& \times \left[(H + \tau)^{1/n} (H - n\tau) - (H - y + \tau)^{1/n} (H - y - n\tau) \right], \\
v_1 = & - \frac{\text{Re}_n \xi}{(n+1) \kappa \text{Pe}_n} \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(2-n)/n} \frac{H_x}{H^2} \\
& \times \left\{ \frac{n}{n+2} \left[\frac{n}{2(n+1)} \left((H + \tau)^{2(n+1)/n} - (H - y + \tau)^{2(n+1)/n} \right) - (H + \tau)^{(n+2)/n} y \right] \right. \\
& + (H + \tau)^{1/n} \left[-\frac{n}{n+1} \left((H + \tau)^{(n+1)/n} (H - n\tau) - (H - y + \tau)^{(n+1)/n} (H - y - n\tau) \right) \right. \\
& \quad \left. + (H + \tau)^{1/n} (H - n\tau) y \right. \\
& \quad \left. + \frac{n^2}{(n+1)(2n+1)} \left((H + \tau)^{(2n+1)/n} - (H - y + \tau)^{(2n+1)/n} \right) \right] \\
& \left. - n\tau^{(n+1)/n} \left[\frac{n}{n+1} \left((H + \tau)^{(n+1)/n} - (H - y + \tau)^{(n+1)/n} \right) - (H + \tau)^{1/n} y \right] \right\} \\
& - \frac{\text{Re}_n \xi}{(n+1) \kappa \text{Pe}_n} \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(2-n)/n} \frac{H_x}{H} \\
& \times \left\{ -\frac{n}{n+2} \left[(H + \tau)^{(n+2)/n} - (H - y + \tau)^{(n+2)/n} \right] - \frac{2}{n} (H + \tau)^{(2-n)/n} (H - n\tau) y \right. \\
& + (H + \tau)^{1/n} \left[(H + \tau)^{1/n} (H - n\tau) - (H - y + \tau)^{1/n} (H - y - n\tau) \right] \\
& + \tau^{(n+1)/n} \left[n \left((H + \tau)^{1/n} - (H - y + \tau)^{1/n} \right) - (H + \tau)^{(1-n)/n} y \right] \\
& + \frac{(H + \tau)^{(1-n)/n}}{n+1} \left[\left((H + \tau)^{(n+1)/n} (H - n\tau) - (H - y + \tau)^{(n+1)/n} (H - y - n\tau) \right) \right. \\
& \quad \left. - \frac{n}{2n+1} \left((H + \tau)^{(2n+1)/n} - (H - y + \tau)^{(2n+1)/n} \right) \right] \Big\} \\
& - \text{Re}_n \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(3-n)/n} \left(H_{xx} (H + \tau)^{1/n} + \frac{H_x^2}{n} (H + \tau)^{(1-n)/n} \right) \\
& \times \left\{ \frac{n}{(n+1)(n+2)} \left[\frac{n}{2(n+1)} \left((H + \tau)^{2(n+1)/n} H - (H - y + \tau)^{2(n+1)/n} (H - y) \right) \right. \right. \\
& \quad \left. - (H + \tau)^{(n+2)/n} H y \right. \\
& \quad \left. - \frac{n^2}{2(n+1)(3n+2)} \left((H + \tau)^{(3n+2)/n} - (H - y + \tau)^{(3n+2)/n} \right) \right] \Big\}
\end{aligned}$$

$$\begin{aligned}
& - \frac{n^2(4n+5)}{2(n+1)^2(n+2)(2n+1)} \left[\frac{n}{3n+2} \left((H+\tau)^{(3n+2)/n} - (H-y+\tau)^{(3n+2)/n} \right) \right. \\
& \quad \left. - (H+\tau)^{2(n+1)/n} y \right] \\
& + \frac{2n^2}{(n+1)(2n+1)} \tau^{(2n+1)/n} \left[\frac{n}{n+1} \left((H+\tau)^{(n+1)/n} - (H-y+\tau)^{(n+1)/n} \right) - (H+\tau)^{1/n} y \right] \\
& + (H+\tau)^{1/n} H \left[- \frac{n}{n+1} \left((H+\tau)^{(n+1)/n} H - (H-y+\tau)^{(n+1)/n} (H-y) \right) \right. \\
& \quad \left. + \frac{2n^2}{(n+1)(2n+1)} \left((H+\tau)^{(2n+1)/n} - (H-y+\tau)^{(2n+1)/n} \right) \right] \\
& + (H+\tau)^{2/n} H^2 y - \frac{2n}{n+1} (H+\tau)^{(n+2)/n} H y + \left(\frac{n}{n+1} \right)^2 (H+\tau)^{2(n+1)/n} y \\
& + \frac{n}{n+1} (H+\tau)^{(n+1)/n} \left[\frac{n}{n+1} \left((H+\tau)^{(n+1)/n} H - (H-y+\tau)^{(n+1)/n} (H-y) \right) \right. \\
& \quad \left. - \frac{n^2}{(n+1)(2n+1)} \left((H+\tau)^{(2n+1)/n} - (H-y+\tau)^{(2n+1)/n} \right) \right] \\
& - \left(\frac{n}{n+1} \right)^2 \frac{n}{2n+1} (H+\tau)^{(n+1)/n} \left[(H+\tau)^{(2n+1)/n} - (H-y+\tau)^{(n+1)/n} \right] \Big\} \\
& - \operatorname{Re}_n \left(\frac{\operatorname{Re}_n \sin \theta}{\operatorname{Fr}} \right)^{(3-n)/n} H_x^2 (H+\tau)^{1/n} \\
& \times \left\{ \frac{2n}{(n+1)(2n+1)} \left[- \frac{n}{2(n+1)} \left((H+\tau)^{2(n+1)/n} - (H-y+\tau)^{2(n+1)/n} \right) \right. \right. \\
& \quad \left. \left. + (H+\tau)^{(n+2)/n} y \right] \right. \\
& + \frac{n}{n+1} \left[\frac{n}{n+2} \left((H+\tau)^{(n+2)/n} H - (H-y+\tau)^{(n+2)/n} (H-y) \right) - (H+\tau)^{2/n} y \right. \\
& \quad \left. - \frac{n^2}{2(n+1)(n+2)} \left((H+\tau)^{2(n+1)/n} - (H-y+\tau)^{2(n+1)/n} \right) \right] \\
& + \frac{2n}{(n+1)(2n+1)} \tau^{(2n+1)/n} \left[n \left((H+\tau)^{1/n} - (H-y+\tau)^{1/n} \right) - (H+\tau)^{(1-n)/n} y \right] \\
& + \frac{1}{n} (H+\tau)^{(1-n)/n} H \left[- \frac{n}{n+1} \left((H+\tau)^{(n+1)/n} H - (H-y+\tau)^{(n+1)/n} (H-y) \right) \right. \\
& \quad \left. + \frac{n^2}{(n+1)(2n+1)} \left((H+\tau)^{(2n+1)/n} - (H-y+\tau)^{(2n+1)/n} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{n}(H + \tau)^{1/n}H \left[-n \left((H + \tau)^{1/n}H - (H - y + \tau)^{1/n}(H - y) \right) \right. \\
& \quad \left. + \frac{n^2}{n+1} \left((H + \tau)^{(n+1)/n} - (H - y + \tau)^{(n+1)/n} \right) \right] \\
& + \frac{n}{(n+1)(2n+1)}(H + \tau)^{(1-n)/n}H \left[(H + \tau)^{(2n+1)/n} - (H - y + \tau)^{(2n+1)/n} \right] \\
& + \frac{1}{n+1}(H + \tau)^{(n+1)/n} \left[n \left((H + \tau)^{1/n}H - (H - y + \tau)^{1/n}(H - y) \right) \right. \\
& \quad \left. - \frac{n^2}{n+1} \left((H + \tau)^{(n+1)/n} - (H - y + \tau)^{(n+1)/n} \right) \right] \\
& + \frac{2}{n}(H + \tau)^{(2-n)/n}H^2y - \frac{2}{n+1}(H + \tau)^{2/n}Hy \Big\} \\
& + \frac{\text{Re}_n}{(n+1)} \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(1-n)/n} \left(\frac{\cos \theta}{\text{Fr}} H_{xx} - \kappa^2 \text{Fi}_n H_{xxx} \right) \\
& \times \left\{ (H + \tau)^{1/n}(H - n\tau)y \right. \\
& \quad - \frac{n}{n+1} \left[(H + \tau)^{(n+1)/n}(H - n\tau) - (H - y + \tau)^{(n+1)/n}(H - y - n\tau) \right] \\
& \quad \left. + \frac{n^2}{(n+1)(2n+1)} \left[(H + \tau)^{(2n+1)/n} - (H - y + \tau)^{(2n+1)/n} \right] \right\} \\
& + \frac{\text{Re}_n}{(n+1)} \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{(1-n)/n} \left(\frac{\cos \theta}{\text{Fr}} H_x^2 - \kappa^2 \text{Fi}_n H_x H_{xxx} \right) \\
& \times \left\{ \frac{1}{n}(H + \tau)^{(1-n)/n}(H - n\tau)y + (H + \tau)^{1/n}y \right. \\
& \quad \left. - \left[(H + \tau)^{1/n}(H - n\tau) - (H - y + \tau)^{1/n}(H - y - n\tau) \right] \right\}, \\
\Theta_1 = \text{Pe}_n & \left[\frac{\xi}{6\kappa\text{Pe}_n H^3} (H^3 - y^3) - \frac{\xi}{6\kappa\text{Pe}_n H} (H - y) + \left(\frac{\text{Re}_n \sin \theta}{\text{Fr}} \right)^{1/n} \frac{n}{n+1} \frac{1}{H^2} H_x \right. \\
& \times \left\{ (H + \tau)^{(n+1)/n} \left[\frac{(H - y)^3}{6} - \frac{H^2(H - y)}{6} \right] \right. \\
& \quad - \frac{n^2}{(2n+1)(3n+1)} \left[(H - y + \tau)^{(3n+1)/n} - (H + \tau)^{(3n+1)/n} \right] (H - y) \\
& \quad + \frac{2n^3}{(2n+1)(3n+1)(4n+1)} \\
& \quad \left. \times \left[(H - y + \tau)^{(4n+1)/n} + \left(\frac{y}{H} - 1 \right) (H + \tau)^{(4n+1)/n} - \frac{y}{H} \tau^{(4n+1)/n} \right] \right\} \Big].
\end{aligned}
\tag{A.2}$$

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