Research Article

Extreme Points of the Unit Ball in the Dual Space of Some Real Subspaces of Banach Spaces of Lipschitz Functions

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Received 20 October 2011; Accepted 16 November 2011

Academic Editor: C. Zhu

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Let X be a compact Hausdorff space, τ be a continuous involution on X and $C(X, \tau)$ denote the uniformly closed real subalgebra of $C(X)$ consisting of all $f \in C(X)$ for which $f \circ \tau = \bar{f}$. Let (X, d) be a compact metric space and let $\text{Lip}(X, d^\alpha)$ denote the complex Banach space of complex-valued Lipschitz functions of order α on (X, d) under the norm $\|f\|_{X, p_\alpha} = \max\{\|f\|_X, p_\alpha(f)\}$, where $\alpha \in (0, 1]$. For $\alpha \in (0, 1)$, the closed subalgebra of $\text{Lip}(X, \alpha)$ consisting of all $f \in \text{Lip}(X, d^\alpha)$ for which $|f(x) - f(y)|/d^\alpha(x, y) \rightarrow 0$ as $d(x, y) \rightarrow 0$, denoted by $\text{lip}(X, d^\alpha)$. Let τ be a Lipschitz involution on (X, d) and define $\text{Lip}(X, \tau, d^\alpha) = \text{Lip}(X, d^\alpha) \cap C(X, \tau)$ for $\alpha \in (0, 1]$ and $\text{lip}(X, \tau, d^\alpha) = \text{lip}(X, d^\alpha) \cap C(X, \tau)$ for $\alpha \in (0, 1)$. In this paper, we give a characterization of extreme points of B_{A^*} , where A is a real linear subspace of $\text{Lip}(X, d^\alpha)$ or $\text{lip}(X, d^\alpha)$ which contains 1, in particular, $\text{Lip}(X, \tau, d^\alpha)$ or $\text{lip}(X, \tau, d^\alpha)$.

1. Introduction and Preliminaries

We let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote the field of real numbers, complex numbers, and the unit circle, respectively. The symbol $\mathbb{K}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. The elements of $\mathbb{K}$ are called scalars.

Let $\mathfrak{X}$ be a normed space over $\mathbb{K}$. We denote by $\mathfrak{X}^*$ and $B_{\mathfrak{X}}$ the dual space $\mathfrak{X}$ and the closed unit ball of $\mathfrak{X}$, respectively. If S is a subset $\mathfrak{X}$, let $\text{Ext}(S)$ denote the set of all extreme points of S . Let A be a subspace of $\mathfrak{X}$ and $\varphi \in A^*$. A Hahn-Banach extension of φ to $\mathfrak{X}$ is a continuous linear functional $\psi \in \mathfrak{X}^*$ such that $\psi|_A = \varphi$ and $\|\psi\| = \|\varphi\|$. The set of all Hahn-Banach extensions of φ to $\mathfrak{X}$ will be denoted by H_φ .

It is easy to see that if $\mathfrak{X}$ and $\mathfrak{Y}$ are normed spaces over $\mathbb{K}$ and $T : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a linear isometry from $\mathfrak{X}$ onto $\mathfrak{Y}$, then T is a bijection mapping between $\text{Ext}(B_{\mathfrak{X}})$ and $\text{Ext}(B_{\mathfrak{Y}})$.

For a complex normed space $\mathfrak{X}$, we assume that $\mathfrak{X}_r$ denotes $\mathfrak{X}$, regarded as a real normed space by restricting the scalar multiplication to real numbers.

Kulkarni and Limaye gave some conditions for $\varphi \in B_{A^*}$ to be an extreme point of B_{A^*} in terms of the Hahn-Banach extension of φ to $\mathfrak{X}$ and the extreme points of $B_{\mathfrak{X}^*}$ as the following.

Theorem 1.1 (see [1, Theorem 2]). *Let $\mathfrak{X}$ be a normed space over $\mathbb{K}$, A be a nonzero linear subspace of $\mathfrak{X}$ and $\varphi \in B_{A^*}$.*

(a) *Let $\varphi \in \text{Ext}(B_{A^*})$. Then,*

$$H_\varphi \cap \text{Ext}(B_{\mathfrak{X}^*}) = \text{Ext}(H_\varphi) \neq \emptyset. \quad (1.1)$$

In particular, φ has an extension to some $\psi \in \text{Ext}(B_{\mathfrak{X}^})$. Further, if such an extension is unique, then φ has a unique Hahn-Banach extension to $\mathfrak{X}$.*

(b) *Assume that whenever $\psi \in \text{Ext}(B_{\mathfrak{X}^*})$ and $\psi(f) = 1$ for all $f \in A$ with $\varphi(f) = 1 = \|f\|$, one has $\psi|_A = \varphi$, then $\varphi \in \text{Ext}(B_{A^*})$.*

(c) *If φ has a unique Hahn-Banach extension ψ to $\mathfrak{X}$ and if $\psi \in \text{Ext}(B_{\mathfrak{X}^*})$, then $\varphi \in \text{Ext}(B_{A^*})$.*

Let X be a compact Hausdorff space. We denote by $C(X)$ the complex Banach algebra of all continuous complex-valued functions on X under the uniform norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$. For $x \in X$, consider the evaluation functional e_x given by $e_x(f) = f(x)$, $f \in C(X)$. Clearly, $\lambda e_x \in B_{C(X)^*}$ for all $(x, \lambda) \in X \times \mathbb{T}$. It is well known [2, page 441] that

$$\text{Ext}(B_{C(X)^*}) = \{\lambda e_x : (x, \lambda) \in X \times \mathbb{T}\}. \quad (1.2)$$

For $x \in X$ and $\lambda \in \mathbb{T}$, we define the map $\varphi_{x,\lambda} : C(X)_r \rightarrow \mathbb{R}$ by $\varphi_{x,\lambda}(f) = \text{Re}(\lambda f(x))$ in fact, $\varphi_{x,\lambda} = \text{Re}(\lambda e_x)$. Clearly, $\varphi_{x,\lambda} \in B_{(C(X)_r)^*}$ for all $(x, \lambda) \in X \times \mathbb{T}$. Kulkarni and Limaye showed [1, Proposition 3] that

$$\text{Ext}(B_{(C(X)_r)^*}) = \{\varphi_{x,\lambda} : (x, \lambda) \in X \times \mathbb{T}\}, \quad (1.3)$$

and $\varphi_{y,\mu} = \varphi_{x,\lambda}$ if and only if $y = x$ and $\mu = \lambda$.

Let τ be a continuous involution on X ; that is, $\tau : X \rightarrow X$ is continuous and $\tau \circ \tau$ is the identity map on X . The map $\sigma : C(X) \rightarrow C(X)$ defined by $\sigma(f) = \bar{f} \circ \tau$, is an algebra involution on $C(X)$ which is called the algebra involution induced by τ on $C(X)$. Define $C(X, \tau) = \{f \in C(X) : \sigma(f) = f\}$. Then, $C(X, \tau)$ is a uniformly closed real subalgebra of $C(X)$ which contains 1. The real algebras $C(X, \tau)$ were first considered in [3]. For a detailed account of several properties of $C(X, \tau)$, we refer to [4].

Let $P = \{(x, \lambda) \in X \times \mathbb{T} : \tau(x) \neq x\} \cup \{(x, \lambda) \in X \times \mathbb{T} : \tau(x) = x, \lambda \in \{-1, 1\}\}$. For each $(x, \lambda) \in P$, let $\varphi_{x,\lambda}$ denote the restriction of $\varphi_{x,\lambda}$ to $C(X, \tau)$. Grzesiak obtained a characterization of the extreme points of $B_{(C(X, \tau))^*}$ in [5] and showed that $\varphi \in \text{Ext}(B_{(C(X, \tau))^*})$ if and only if $\varphi = \varphi_{x,\lambda}$ for some $(x, \lambda) \in P$. Further, if $(x, \lambda), (y, \mu) \in P$, then $\varphi_{y,\mu} = \varphi_{x,\lambda}$ if and only if $(y, \mu) = (x, \lambda)$ or $(y, \mu) = (\tau(x), \bar{\lambda})$.

Kulkarni and Limaye obtained [1, Theorem 4] a characterization of $\text{Ext}(B_{A^*})$, where A is a nonzero real linear subspace of $C(X, \tau)$.

Let (X, d) be a compact metric space. For $\alpha \in (0, 1]$, we denote by $\text{Lip}(X, d^\alpha)$ the set of all complex-valued functions f on X for which

$$p_\alpha(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in X, x \neq y \right\} \quad (1.4)$$

is finite. Then, $\text{Lip}(X, \alpha)$ is a complex subalgebra of $C(X)$ containing 1 and complex Banach space under the norm

$$\|f\|_{X, p_\alpha} = \max\{\|f\|_X, p_\alpha(f)\} \quad (f \in \text{Lip}(X, d^\alpha)). \quad (1.5)$$

For $\alpha \in (0, 1)$, the complex subalgebra of $\text{Lip}(X, d^\alpha)$ consisting of all $f \in \text{Lip}(X, d^\alpha)$ for which

$$\frac{|f(x) - f(y)|}{d^\alpha(x, y)} \longrightarrow 0 \quad \text{as } d(x, y) \longrightarrow 0, \quad (1.6)$$

is denoted by $\text{lip}(X, d^\alpha)$. Clearly, $\text{lip}(X, d^\alpha)$ is a closed linear subspace of $(\text{Lip}(X, d^\alpha), \|\cdot\|_{X, p_\alpha})$ and $1 \in \text{lip}(X, d^\alpha)$. These Banach spaces were first studied by Leeuw in [6].

Given a compact metric space (X, d) , let $\tilde{X} = \{(x, y) \in X \times X : x \neq y\}$, and let the compact Hausdorff space W be the disjoint union of X with $\beta\tilde{X}$, where $\beta\tilde{X}$ is the Stone-Cech compactification of $\tilde{X}$. For $\alpha \in (0, 1]$, consider the mapping $\Psi_\alpha : \text{Lip}(X, d^\alpha) \rightarrow C(W)$ defined for each $f \in \text{Lip}(X, d^\alpha)$ by

$$\Psi_\alpha(f)(w) = \begin{cases} f(w) & \text{if } w \in X, \\ (\beta\tilde{f})(w) & \text{if } w \in \beta\tilde{X}, \end{cases} \quad (1.7)$$

where

$$\tilde{f}(x, y) = \frac{f(x) - f(y)}{d^\alpha(x, y)}, \quad \forall (x, y) \in \tilde{X}, \quad (1.8)$$

and $\beta\tilde{f}$ is the norm-preserving extension of $\tilde{f}$ to $\beta\tilde{X}$. Clearly, Ψ_α is a linear isometry from $(\text{Lip}(X, d^\alpha), \|\cdot\|_{X, p_\alpha})$ into $(C(W), \|\cdot\|_W)$, which is called the Leeuw's linear isometry. Therefore, $\Psi_\alpha(\text{Lip}(X, d^\alpha))$ is a uniformly closed linear subspace of $C(W)$. It is well known (see [2, page 441]) that

$$\text{Ext}(B_{\Psi_\alpha(\text{Lip}(X, d^\alpha))^*}) \subseteq \left\{ \lambda e_w|_{\Psi_\alpha(\text{Lip}(X, d^\alpha))} : (w, \lambda) \in W \times \mathbb{T} \right\}, \quad (1.9)$$

where e_w is the evaluation functional at w on $C(W)$.

For each $x \in X$ and $w \in W$, define the linear functionals Δ_x and $\tilde{\Delta}_w$ in $\text{Lip}(X, d^\alpha)^*$ by $\Delta_x(f) = f(x)$ and $\tilde{\Delta}_w(f) = \Psi_\alpha(f)(w)$, respectively. Clearly, $|\Delta_x(f)| \leq \|f\|_{X, p_\alpha}$ and $|\tilde{\Delta}_w(f)| \leq \|f\|_{X, p_\alpha}$ for all $f \in \text{Lip}(X, d^\alpha)$. Therefore, $\Delta_x, \tilde{\Delta}_w \in B_{\text{Lip}(X, d^\alpha)^*}$. Moreover, $\tilde{\Delta}_x = \Delta_x$ for all $x \in X$ and $\tilde{\Delta}_w = e_w \circ \Psi_\alpha$ for all $w \in W$. Thus, we have the following result.

Theorem 1.2. For $\alpha \in (0, 1]$, every extreme point of $B_{\text{Lip}(X, d^\alpha)^*}$ must be either of the form $\lambda \Delta_x$ with $(x, \lambda) \in X \times \mathbb{T}$ or of the form $\lambda \tilde{\Delta}_w$ with $(w, \lambda) \in \beta \tilde{X} \times \mathbb{T}$.

Roy proved the following result by using a result of Leeuw [6, Lemma 1.2].

Theorem 1.3 (see [7, Lemma 1.2]). For each $(x, \lambda) \in X \times \mathbb{T}$, $\lambda \Delta_x$ is an extreme point of $B_{\text{Lip}(X, d^\alpha)^*}$.

Jimenez-Vargas and Villegas-Vallecillos used above results and obtained a characterization of linear isometries between $\text{Lip}(X, d_X)$ and $\text{Lip}(Y, d_Y)$ in [8].

A map $f : X \rightarrow Y$ is said to be Lipschitz map from the metric space (X, d_X) to the metric space (Y, d_Y) if there exists a constant $C > 0$ such that $d_Y(f(x), f(y)) \leq C d_X(x, y)$ for all $x, y \in X$.

Let (X, d) be a compact metric space. The mapping $\tau : X \rightarrow X$ is called a Lipschitz involution on (X, d) , if τ is a Lipschitz map from (X, d) to itself and an involution on X . Clearly, every Lipschitz involution on (X, d) is a continuous involution.

Let τ be a Lipschitz involution on the compact metric space (X, d) and let σ be the algebra involution induced by τ on $C(X)$. Clearly,

$$\sigma(\text{Lip}(X, d^\alpha)) = \text{Lip}(X, d^\alpha), \quad \sigma(\text{lip}(X, d^\alpha)) = \text{lip}(X, d^\alpha). \quad (1.10)$$

We define

$$\begin{aligned} \text{Lip}(X, \tau, d^\alpha) &= \{h \in \text{Lip}(X, d^\alpha) : \sigma(h) = h\}, \\ \text{lip}(X, \tau, d^\alpha) &= \{h \in \text{lip}(X, d^\alpha) : \sigma(h) = h\}. \end{aligned} \quad (1.11)$$

Then, the following statements hold.

- (i) $\text{Lip}(X, \tau, d^\alpha)$ ($\text{lip}(X, \tau, d^\alpha)$, resp.) is a real subalgebra of $\text{Lip}(X, d^\alpha)$ ($\text{lip}(X, d^\alpha)$, resp.).
- (ii) $\text{Lip}(X, \tau, d^\alpha) = \text{Lip}(X, d^\alpha) \cap C(X, \tau)$ and $\text{lip}(X, \tau, d^\alpha) = \text{lip}(X, d^\alpha) \cap C(X, \tau)$.
- (iii) $\text{Lip}(X, d^\alpha) = \text{Lip}(X, \tau, d^\alpha) \oplus i \text{Lip}(X, \tau, d^\alpha)$ and $\text{lip}(X, d^\alpha) = \text{lip}(X, \tau, d^\alpha) \oplus i \text{lip}(X, \tau, d^\alpha)$.
- (iv) $\text{Lip}(X, \tau, d^\alpha)$ ($\text{lip}(X, \tau, d^\alpha)$, resp.) is a real subalgebra of $C(X, \tau)$ which contains 1 and separates the points of X .
- (v) $\text{Lip}(X, \tau, d^\alpha)$ ($\text{lip}(X, \tau, d^\alpha)$, resp.) is uniformly dense in $C(X, \tau)$ (use (iv) and the Stone-Weierstrass theorem for real subalgebra of $C(X, \tau)$ [3, Proposition 1.1]).
- (vi) For $0 < \alpha < \beta \leq 1$,

$$\text{Lip}(X, \tau, d^\beta) \subseteq \text{lip}(X, \tau, d^\alpha) \subseteq \text{Lip}(X, \tau, d^\alpha). \quad (1.12)$$

- (vii) There exists a constant $C \geq 1$ such that

$$\max\{\|f\|_{X, p_\alpha}, \|g\|_{X, p_\alpha}\} \leq C^\alpha \|f + g\|_{X, p_\alpha}, \quad (1.13)$$

for all $f, g \in \text{Lip}(X, \tau, d^\alpha)$.

- (viii) $\text{Lip}(X, \tau, d^\alpha), \|\cdot\|_{X, p_\alpha}$ is a real Banach space and $\text{lip}(X, \tau, d^\alpha)$ is its closed real subspace.

The real Banach spaces $\text{Lip}(X, \tau, d^\alpha)$ and $\text{lip}(X, \tau, d^\alpha)$ are called real Banach spaces of complex Lipschitz functions and first studied in [9].

We give a characterization of extreme points of the unit ball in the dual space of $\text{Lip}(X, d^\alpha)_r$, $\text{Lip}(X, \tau, d^\alpha)$ and some its real linear subspaces for $\alpha \in (0, 1]$ in Section 2. Next, we give a characterization of extreme points of the unit ball in the dual spaces of $\text{lip}(X, d^\alpha)$, $\text{lip}(X, d^\alpha)_r$, $\text{lip}(X, \tau, d^\alpha)$ and some its real linear subspaces for $\alpha \in (0, 1)$ in Section 3.

2. Real Linear Subspaces of $\text{Lip}(X, d^\alpha)$ Containing 1

In the remainder of this paper, we assume that $\alpha \in (0, 1]$, (X, d) is a compact metric space, $\tilde{X} = \{(x, y) \in X \times X, x \neq y\}$, $\beta\tilde{X}$ is the Stone-Cech compactification of $\tilde{X}$, W is the compact Hausdorff space $X \cup \beta\tilde{X}$, Ψ_α is the Leeuw's linear isometry from $(\text{Lip}(X, d^\alpha), \|\cdot\|_{X, p_\alpha})$ into $(C(W), \|\cdot\|_W)$, and τ is a Lipschitz involution on (X, d) .

For each $(w, \lambda) \in W \times \mathbb{T}$, we define the map $\tilde{\varphi}_{w, \lambda} : \text{Lip}(X, d^\alpha)_r \rightarrow \mathbb{R}$ by $\tilde{\varphi}_{w, \lambda}(f) = \text{Re}(\lambda \tilde{\Delta}_w(f))$ in fact, $\tilde{\varphi}_{w, \lambda} = \text{Re}(\lambda(e_w \circ \Psi_\alpha))$. Clearly, $\tilde{\varphi}_{w, \lambda} \in B_{(\text{Lip}(X, d^\alpha)_r)^*}$ for all $(w, \lambda) \in W \times \mathbb{T}$. Moreover, $\tilde{\varphi}_{x, \lambda} = \text{Re}(\lambda \Delta_x)$ for all $(x, \lambda) \in X \times \mathbb{T}$.

We first give a characterization of the extreme points of the unit ball in the $(\text{Lip}(X, d^\alpha)_r)^*$ as the following.

Proposition 2.1. *By above notations,*

$$\{\tilde{\varphi}_{x, \lambda} : (x, \lambda) \in X \times \mathbb{T}\} \subseteq \text{Ext}(B_{(\text{Lip}(X, d^\alpha)_r)^*}) \subseteq \{\tilde{\varphi}_{w, \lambda} : (w, \lambda) \in W \times \mathbb{T}\}. \quad (2.1)$$

Further, for $(x, \lambda), (y, \mu) \in X \times \mathbb{T}$ one has $\tilde{\varphi}_{x, \lambda} = \tilde{\varphi}_{y, \mu}$ if and only if $(x, \lambda) = (y, \mu)$.

Proof. We define the map $T : (\text{Lip}(X, d^\alpha)_r)^* \rightarrow (\text{Lip}(X, d^\alpha)_r)^*$ by $T(\varphi) = \text{Re } \varphi$. Clearly, T is a real-linear mapping. For each $u \in (\text{Lip}(X, d^\alpha)_r)^*$, defining the map $\varphi : \text{Lip}(X, d^\alpha)^* \rightarrow \mathbb{C}$ by $\varphi(f) = u(f) - iu(if)$. Clearly, $\varphi \in (\text{Lip}(X, d^\alpha)^*)^*$ and $u = \text{Re } \varphi$. It follows that $\varphi \in (\text{Lip}(X, d^\alpha)^*)_r$ and $T(\varphi) = u$. Thus, T is onto.

We claim that T is an isometric. Let $\varphi \in (\text{Lip}(X, d^\alpha)^*)_r$. Since

$$|T(\varphi)(f)| = |(\text{Re } \varphi)(f)| = |\text{Re}(\varphi(f))| \leq |\varphi(f)| \leq \|\varphi\| \|f\|_{X, p_\alpha}, \quad (2.2)$$

for each $f \in \text{Lip}(X, d^\alpha)_r$, we have

$$\|T(\varphi)\| \leq \|\varphi\|. \quad (2.3)$$

Let ε be an arbitrary positive number. There exists $f \in \text{Lip}(X, d^\alpha)_r$ with $\|f\|_{X, p_\alpha} \leq 1$ such that $\|\varphi\| < |\varphi(f)| + \varepsilon$. Choose $\gamma = 1$ if $\varphi(f) = 0$ and $\gamma = (1/\varphi(f))|\varphi(f)|$ if $\varphi(f) \neq 0$. Then, $\gamma \in \mathbb{C}$, $|\gamma| = 1$ and $\varphi(f) = \gamma|\varphi(f)|$. If $g = (1/\gamma)f$, then $g \in \text{Lip}(X, d^\alpha)_r$, $\|g\|_{X, p_\alpha} = \|f\|_{X, p_\alpha} \leq 1$ and so,

$$\|\varphi\| < |\text{Re}(\varphi(g))| + \varepsilon = |T(\varphi)(g)| + \varepsilon \leq \|T(\varphi)\| + \varepsilon. \quad (2.4)$$

It follows that

$$\|\varphi\| \leq \|T(\varphi)\|. \quad (2.5)$$

Thus, our claim is justified. The above arguments show that T is a real-linear isometry from $(\text{Lip}(X, d^\alpha)^*)_r$ onto $(\text{Lip}(X, d^\alpha)_r)^*$. Therefore,

$$\text{Ext}(B_{(\text{Lip}(X, d^\alpha)_r)^*}) = T(\text{Ext}(B_{\text{Lip}(X, d^\alpha)_r})). \quad (2.6)$$

Since

$$\{\lambda \tilde{\Delta}_x : (x, \lambda) \in X \times \mathbb{T}\} \subseteq \text{Ext}(B_{\text{Lip}(X, d^\alpha)_r}) \subseteq \{\lambda \tilde{\Delta}_w : (w, \lambda) \in W \times \mathbb{T}\}, \quad (2.7)$$

we conclude that

$$\{T(\lambda \tilde{\Delta}_x) : (x, \lambda) \in X \times \mathbb{T}\} \subseteq \text{Ext}(B_{(\text{Lip}(X, d^\alpha)_r)^*}) \subseteq \{T(\lambda \tilde{\Delta}_w) : (w, \lambda) \in W \times \mathbb{T}\}, \quad (2.8)$$

by Theorems 1.2 and 1.3.

Clearly, $T(\lambda \tilde{\Delta}_w) = \tilde{\Psi}_{\lambda, w}$ for all $(w, \lambda) \in W \times \mathbb{T}$. Therefore,

$$\{\tilde{\Psi}_{\lambda, x} : (x, \lambda) \in X \times \mathbb{T}\} \subseteq \text{Ext}(B_{(\text{Lip}(X, d^\alpha)_r)^*}) \subseteq \{\tilde{\Psi}_{\lambda, w} : (w, \lambda) \in W \times \mathbb{T}\}. \quad (2.9)$$

It is obvious that if $(x, \lambda), (y, \mu) \in X \times \mathbb{T}$ and $(x, \lambda) = (y, \mu)$, then $\tilde{\Psi}_{\lambda, x} = \tilde{\Psi}_{\mu, y}$. We now assume that $\tilde{\Psi}_{\lambda, x} = \tilde{\Psi}_{\mu, y}$, where $(x, \lambda), (y, \mu) \in X \times \mathbb{T}$. Letting $f = 1$ and $f = i$, we see that $\text{Re } \lambda = \text{Re } \mu$ and $\text{Re}(\lambda i) = \text{Re}(\mu i)$; that is, $\lambda = \mu$. If $x \neq y$, there exists $f \in \text{Lip}(X, d^\alpha)_r$ such that $f(y) = \bar{\mu}$, but $f(x) = 0$ (define $f : X \rightarrow \mathbb{C}$ by $f(z) = (\bar{\mu}/d(y, x))d(z, x)$, $z \in X$); so that

$$|\mu|^2 = \text{Re}(\mu \bar{\mu}) = \text{Re}(\mu f(y)) = \tilde{\Psi}_{y, \mu}(f) = \tilde{\Psi}_{x, \lambda}(f) = \text{Re}(\lambda f(x)) = 0. \quad (2.10)$$

But this is not possible since $|\mu| = 1$. Thus, $x = y$. □

The next purpose is giving conditions for $\varphi \in B_{A^*}$ to be an extreme point of B_{A^*} , where A is a real subspace of $\text{Lip}(X, \tau, d^\alpha)$.

Theorem 2.2. *Let A be a real linear subspace of $\text{Lip}(X, \tau, d^\alpha)$ containing 1. For $(w, \lambda) \in W \times \mathbb{T}$, let $\tilde{\eta}_{w, \lambda} = \tilde{\Psi}_{w, \lambda}|_A$. Let $P = \{(x, \lambda) \in X \times \mathbb{T} : \tau(x) \neq x\} \cup \{(x, \lambda) \in X \times \{-1, 1\} : \tau(x) = x\} \cup \{(w, \lambda) : w \in \beta \tilde{X}, \lambda \in \mathbb{T}\}$.*

Let Q_A denote the set of $(x, \lambda) \in X \times \{-1, 1\}$ such that

- (i) *there is $g \in A$ with $\|g\|_{X, p_\alpha} \leq 1$ and $g(x) = \lambda$,*
- (ii) *for every $y \in X \setminus \{x, \tau(x)\}$, there is some $h \in A$ with $\|h\|_{X, p_\alpha} \leq 1$, $h(x) = 1$ and $|h(y)| < 1$.*

Then,

$$\{\tilde{\eta}_{x, \lambda} : (x, \lambda) \in Q_A\} \subseteq \text{Ext}(B_{A^*}) \subseteq \{\tilde{\eta}_{w, \lambda} : (w, \lambda) \in P\}. \quad (2.11)$$

Further, if $(x, \lambda) \in Q_A$ and $(y, \mu) \in P \cap (X \times \mathbb{T})$, then $\tilde{\eta}_{y, \mu} = \tilde{\eta}_{x, \lambda}$ if and only if either $(y, \mu) = (x, \lambda)$ or $(y, \mu) = (\tau(x), \lambda)$.

Proof. Let $\varphi \in \text{Ext}(B_{A^*})$. Letting $\mathfrak{X} = \text{Lip}(X, d^a)_r$ in part (a) of Theorem 1.1, and using Proposition 2.1, we see that $\varphi = \tilde{\eta}_{w, \lambda}$ for some $(w, \lambda) \in W \times \mathbb{T}$. To prove that $(w, \lambda) \in P$, we consider $w = x \in X$ with $\tau(x) = x$ and show that $\lambda \in \{-1, 1\}$. For every $f \in A$, we have $\overline{f(x)} = f(\tau(x)) = f(x)$ that is, $f(x)$ is a real number. Hence,

$$|\varphi(f)| = |\tilde{\eta}_{x, \lambda}(f)| = |\tilde{\psi}_{x, \lambda}(f)| = |\text{Re}(\lambda f(x))| = |\text{Re } \lambda| |f(x)| \leq |\text{Re } \lambda| \|f\|_{X, p_\alpha}. \quad (2.12)$$

This shows that $\|\varphi\| \leq |\text{Re } \lambda|$. But since φ is an extreme point of B_{A^*} , we must have $\|\varphi\| = 1$. Thus, $1 \leq |\text{Re } \lambda| \leq |\lambda| = 1$ so that $\lambda \in \{-1, 1\}$.

Next, let $(x, \lambda) \in Q_A$. We claim that the following statement hold.

For $(y, \mu) \in X \times \mathbb{T}$ with $(y, \mu) \neq (x, \lambda)$ and $(y, \mu) \neq (\tau(x), \lambda)$, there is $f \in A$ such that $\|f\|_{X, p_\alpha} = 1$, $\text{Re}(\lambda f(x)) = 1$, but $\text{Re}(\mu f(y)) \neq 1$.

By condition (i), there is $g \in A$ such that $\|g\|_{X, p_\alpha} \leq 1$ and $g(x) = \lambda$.

Case 1 ($y \in \{x, \tau(x)\}$). Let $f = g$. Then, $\|f\|_{X, p_\alpha} \leq 1$ and $|f(x)| = |\bar{\lambda}| = 1$ so that $\|f\|_{X, p_\alpha} = 1$. Also,

$$\text{Re}(\lambda f(x)) = \text{Re}(\lambda g(x)) = \text{Re}(\lambda^2) = 1. \quad (2.13)$$

If $y = x$, then $(y, \mu) \neq (x, \lambda)$ implies that $\mu \neq \lambda$ that is, $\mu\lambda \neq 1$. Since $|\mu\lambda| = 1$, this shows that

$$\text{Re}(\mu f(x)) = \text{Re}(\mu g(x)) = \text{Re}(\mu\lambda) \neq 1. \quad (2.14)$$

If $y = \tau(x)$, then $(y, \mu) \neq (\tau(x), \lambda)$ implies that $\mu \neq \lambda$, that is, $\mu\lambda \neq 1$. Since $|\mu\lambda| = 1$, this shows that

$$\text{Re}(\mu f(y)) = \text{Re}(\mu g(\tau(x))) = \text{Re}(\mu\bar{\lambda}) = \text{Re}(\mu\lambda) \neq 1. \quad (2.15)$$

Case 2 ($y \notin \{x, \tau(x)\}$). By condition (ii), there is $h \in A$ such that $\|h\|_{X, p_\alpha} \leq 1$, $h(x) = 1$ and $|h(y)| < 1$. Let $f = gh$. Now, $\|f\|_{X, p_\alpha} \leq 1$ and $|f(x)| = |f(x)||g(x)| = |g(x)| = |\lambda| = 1$, so that $\|f\|_{X, p_\alpha} = 1$. Also,

$$\begin{aligned} \text{Re}(\lambda f(x)) &= \text{Re}(\lambda g(x)h(x)) = \text{Re}(\lambda^2) = 1, \\ \text{Re}(\mu f(y)) &= \text{Re}(\mu g(y)h(y)) \leq |\mu g(y)h(y)| \\ &= |g(y)||h(y)| \leq |h(y)| < 1. \end{aligned} \quad (2.16)$$

Thus, our claim is justified. Let $\mathfrak{X} = \text{Lip}(X, d^a)_r$ and $\varphi = \tilde{\eta}_{x, \lambda}$ in part (b) of Theorem 1.1. Consider $\psi \in \text{Ext}(B_{\mathfrak{X}^*})$ such that $\psi(f) = 1$ for all $f \in A$ with $\varphi(f) = 1 = \|f\|_{X, p_\alpha}$. By Proposition 2.1, $\psi = \tilde{\psi}_{w, \mu}$ for some $(w, \mu) \in W \times \mathbb{T}$. Thus, $\text{Re}(\mu \tilde{\Delta}_w(f)) = 1$ for all $f \in A$

with $\operatorname{Re}(\lambda f(x)) = 1 = \|f\|_{X, p_\alpha}$. Since $\lambda 1 \in A$, $\|\lambda 1\|_{X, p_\alpha} = 1$ and $\operatorname{Re}(\lambda(\lambda 1)(x)) = \operatorname{Re}(\lambda^2) = 1$, we have $\operatorname{Re}(\mu \psi_\alpha(\lambda 1)(w)) = 1$. If $w' \in \beta \tilde{X}$, then

$$\begin{aligned} \operatorname{Re}(\mu \psi_\alpha(\lambda 1)(w')) &= \operatorname{Re}(\mu \lambda \psi_\alpha(1)(w')) = \operatorname{Re}(\mu \lambda (\beta \tilde{1})(w')) \\ &\leq \left| (\mu \lambda (\beta \tilde{1})(w')) \right| = \left| (\beta \tilde{1})(w') \right| \\ &\leq \left\| \beta \tilde{1} \right\|_{\beta \tilde{X}} = \left\| \tilde{1} \right\|_{\tilde{X}} \\ &= p_\alpha(1) = 0. \end{aligned} \tag{2.17}$$

Thus, it must be $w \in X$. Choose $w = y$. Then, $(y, \mu) \in X \times \mathbb{T}$ and $\operatorname{Re}(\mu f(y)) = 1$ for all $f \in A$ with $\|f\|_{X, p_\alpha} = 1 = \operatorname{Re}(\lambda f(x))$. By our claim, we must have $(y, \mu) = (x, \lambda)$ or $(y, \mu) = (\tau(x), \lambda)$.

If $(y, \mu) = (x, \lambda)$, then clearly

$$\psi|_A = \tilde{\psi}_{y, \mu}|_A = \tilde{\psi}_{x, \lambda}|_A = \tilde{\eta}_{x, \lambda} = \varphi. \tag{2.18}$$

Now, let $(y, \mu) = (\tau(x), \lambda)$. Since $\lambda \in \mathbb{R}$, $f \in A$ implies that

$$\begin{aligned} \varphi(f) &= \operatorname{Re}(\mu f(y)) = \operatorname{Re}(\lambda f(\tau(x))) \\ &= \operatorname{Re}(\lambda \overline{f(x)}) = \lambda \operatorname{Re}(\overline{f(x)}) = \lambda \operatorname{Re}(f(x)) \\ &= \operatorname{Re}(\lambda f(x)) = \tilde{\psi}_{x, \lambda}(f) \\ &= \tilde{\eta}_{x, \lambda}(f) = \varphi(f). \end{aligned} \tag{2.19}$$

Therefore, $\varphi|_A = \varphi$, again. Hence, we see that $\tilde{\eta}_{(x, \lambda)} = \varphi \in \operatorname{Ext}(B_{A^*})$. This also shows that $\tilde{\eta}_{\tau(x), \lambda} = \tilde{\eta}_{x, \lambda}$ for all $(x, \lambda) \in X \times \{-1, 1\}$.

Conversely, our claim implies that if $(x, \lambda) \in Q_A$, $(y, \mu) \in X \times \mathbb{T}$ and $\tilde{\eta}_{y, \mu} = \tilde{\eta}_{x, \lambda}$, then $(y, \mu) = (x, \lambda)$ or $(y, \mu) = (\tau(x), \lambda)$. Thus, for $(x, \lambda) \in Q_A$ and $(y, \mu) \in P \cap (X \times \mathbb{T})$, we have $\tilde{\eta}_{y, \mu} = \tilde{\eta}_{x, \lambda}$ if and only if $(y, \mu) = (x, \lambda)$ or $(y, \mu) = (\tau(x), \lambda)$. $\square$

By using the above theorem, we give a characterization of extreme points of unit ball in the dual space of $\operatorname{Lip}(X, \tau, d^\alpha)$.

Corollary 2.3. *Let P be as in Theorem 2.2. For $(w, \lambda) \in P$, let $\tilde{\eta}_{w, \lambda}$ denote the restriction of $\tilde{\psi}_{w, \lambda}$ to $\operatorname{Lip}(X, \tau, d^\alpha)$. Then,*

$$\{\tilde{\eta}_{x, \lambda} : (x, \lambda) \in X \times \{-1, 1\}\} \subseteq \operatorname{Ext}(B_{\operatorname{Lip}(X, \tau, d^\alpha)^*}) \subseteq \{\tilde{\eta}_{w, \lambda} : (w, \lambda) \in P\}. \tag{2.20}$$

Further, if $(x, \lambda) \in X \times \{-1, 1\}$ and $(y, \mu) \in P \cap (X \times \mathbb{T})$, then $\eta_{y, \mu} = \tilde{\eta}_{x, \lambda}$ if and only if $(y, \mu) = (x, \lambda)$ or $(y, \mu) = (\tau(x), \lambda)$.

Proof. Let $A = \text{Lip}(X, \tau, d^\alpha)$. By Theorem 2.2, we have

$$\text{Ext}(B_{A^*}) \subseteq \{\tilde{\eta}_{w,\lambda} : (w, \lambda) \in P\}. \quad (2.21)$$

To prove $\{\tilde{\eta}_{x,\lambda} : (x, \lambda) \in X \times \{-1, 1\}\} \subseteq \text{Ext}(B_{\text{Lip}(X, \tau, d^\alpha)^*})$, it is enough to show that for every $(x, \lambda) \in X \times \{-1, 1\}$,

- (i) there is $g \in \text{Lip}(X, \tau, d^\alpha)$ with $\|g\|_{X, p_\alpha} \leq 1$ and $g(x) = \lambda$,
- (ii) for every $y \in X \setminus \{x, \tau(x)\}$, there is $h \in \text{Lip}(X, \tau, d^\alpha)$ with

$$\|h\|_{X, p_\alpha} \leq 1, \quad h(x) = 1, \quad |h(y)| < 1. \quad (2.22)$$

Let $(x, \lambda) \in X \times \{-1, 1\}$. We first define the function $g : X \rightarrow \mathbb{C}$ by

$$g(z) = \lambda \left(1 - \frac{d(z, x)d(\tau(z), x)}{1 + \rho^2 + \rho^{2-2\alpha}} \right), \quad (2.23)$$

where $\rho = \text{diam}(X) = \sup\{d(y, z) : y, z \in X\}$. Clearly, $g \in C(X, \tau)$, $g(x) = \lambda$, $\|g\|_X \leq 1$ and $g \in \text{Lip}(X, d^1)$. Let $y, z \in X$ with $y \neq z$. Then,

$$\begin{aligned} \frac{|g(y) - g(z)|}{d^\alpha(y, z)} &= \frac{|d(z, x)d(\tau(z), x) - d(y, x)d(\tau(y), x)|}{(1 + \rho^2 + \rho^{2-2\alpha})d^\alpha(y, z)} \\ &\leq \frac{d^{1-\alpha}(y, z)}{(1 + \rho^2 + \rho^{2-2\alpha})} (d(z, x) + d(\tau(y), x)) \\ &\leq \frac{2\rho\rho^{1-\alpha}}{1 + \rho^2 + \rho^{2-2\alpha}} \\ &\leq 1. \end{aligned} \quad (2.24)$$

Therefore, $g \in \text{Lip}(X, d^\alpha)$ and $p_\alpha(g) \leq 1$. Consequently,

$$g \in \text{Lip}(X, \tau, d^\alpha), \quad \|g\|_{X, p_\alpha} \leq 1, \quad g(x) = \lambda. \quad (2.25)$$

Hence, (i) holds.

We now assume that $y \in X \setminus \{x, \tau(x)\}$ and define the function $h : X \rightarrow \mathbb{C}$ by

$$h(z) = 1 - \frac{\lambda g(z)d(x, z)d(x, \tau(z))}{2(1 + \rho^2 + \rho^{2-2\alpha})}. \quad (2.26)$$

Clearly, $h \in C(X, \tau)$, $h(x) = 1$, $\|h\|_X \leq 1$ and there exists $\gamma \in (0, 1]$ such that $h(y) = 1 - \gamma$ so that $|h(y)| < 1$. We define the complex-valued function g_1 on X by $g_1 = (1/2)(1 - \lambda g)$. Then,

$g_1 \in C(X, \tau)$, $\|g_1\|_X \leq 1/2$, $g_1 \in \text{Lip}(X, d^1)$, $p_\alpha(g_1) \leq 1/2$ and $h = 1 - g_1$. Thus, we have $h \in \text{Lip}(X, d^1)$, $h \in \text{Lip}(X, \tau, d^\alpha)$ and

$$p_\alpha(h) \leq p_\alpha(g) \|g_1\|_X + \|g\|_X p_\alpha(g_1) \leq \frac{1}{2} + \frac{1}{2} = 1, \quad (2.27)$$

so that $\|h\|_{X, p_\alpha} \leq 1$. Since $y \in X \setminus \{x, \tau(x)\}$, $|h(y)| = h(y) < 1$. Thus, (ii) holds. $\square$

3. Real Subspaces of $\text{lip}(X, d^\alpha)$ Containing 1

Throughout this section, we assume that $\alpha \in (0, 1)$. Consider the mapping $\Phi_\alpha : \text{lip}(X, d^\alpha) \rightarrow C(W)$ by $\Phi_\alpha = \Psi_\alpha|_{\text{lip}(X, d^\alpha)}$. Then, Φ_α is a linear isometric from $(\text{lip}(X, d^\alpha), \|\cdot\|_{X, p_\alpha})$ into $C(W)$.

For each $x \in X$ and $w \in W$, define the functionals δ_x and $\tilde{\delta}_w$ in $\text{lip}(X, d^\alpha)^*$ by $\delta_x(f) = f(x)$ and $\tilde{\delta}_w(f) = \Phi_\alpha(f)(w)$, respectively. Clearly, $|\delta_x(f)| \leq \|f\|_{X, p_\alpha}$ for all $f \in \text{lip}(X, d^\alpha)$ and $|\tilde{\delta}_w(f)| \leq \|f\|_{X, p_\alpha}$, and therefore, $\delta_x, \tilde{\delta}_w \in B_{\text{lip}(X, d^\alpha)^*}$. Moreover, $\tilde{\delta}_x = \delta_x$ for all $x \in X$ and $\tilde{\delta}_w = e_w \circ \Phi_\alpha$ for all $w \in W$.

We give a characterization of extreme points of the unit ball in the dual space $\text{lip}(X, d^\alpha)$ as the following.

Theorem 3.1. *Every extreme point of $B_{\text{lip}(X, d^\alpha)^*}$ must be either the form $\lambda \delta_x$ with $(x, \lambda) \in X \times \mathbb{T}$ or of the form $\lambda \tilde{\delta}_w$ with $(\lambda, w) \in \beta \tilde{X} \times \mathbb{T}$. Moreover, $\lambda \delta_x$ is an extreme point of $B_{\text{lip}(X, d^\alpha)^*}$ for all $(x, \lambda) \in X \times \mathbb{T}$.*

Proof. Since $(\text{lip}(X, d^\alpha), \|\cdot\|_{X, p_\alpha})$ is a Banach space and Φ_α is a linear isometry from $(\text{lip}(X, d^\alpha), \|\cdot\|_{X, p_\alpha})$ into $(C(W), \|\cdot\|_W)$, we conclude that $\Phi_\alpha(\text{lip}(X, d^\alpha))$ is a uniformly closed subspace of $C(W)$. It is well known [2, page 441] that

$$\text{Ext}(B_{(\Phi_\alpha(\text{lip}(X, d^\alpha)))^*}) \subseteq \left\{ \lambda e_w|_{\Phi_\alpha(\text{lip}(X, d^\alpha))} : (w, \lambda) \in W \times \mathbb{T} \right\}. \quad (3.1)$$

Let $A = \Phi_\alpha(\text{lip}(X, d^\alpha))$ and define $\Lambda_\alpha : \text{lip}(X, d^\alpha) \rightarrow A$ by $\Lambda_\alpha(f) = \Phi_\alpha(f)$. Then, Λ_α is a linear isometry from $(\text{lip}(X, d^\alpha), \|\cdot\|_{p_\alpha})$ onto $(A, \|\cdot\|_W)$, so Λ_α^* , the adjoint of Λ_α , is a linear isometry from $(A, \|\cdot\|_W)^*$ onto $(\text{lip}(X, d^\alpha))^*$. It is easily to show that $\Lambda_\alpha^*(e_w|_A) = \tilde{\delta}_w$ for all $w \in W$. Let $\varphi \in \text{Ext}(B_{\text{lip}(X, d^\alpha)^*})$. Then, $(\Lambda_\alpha^*)^{-1}(\varphi) \in \text{Ext}(B_{A^*})$. Thus, there exists $(w, \lambda) \in W \times \mathbb{T}$ such that $(\Lambda_\alpha^*)^{-1}(\varphi) = \lambda e_w|_A$ so that $\varphi = \lambda_\alpha^*(e_w|_A) = \lambda \tilde{\delta}_w$. It follows that

$$\text{Ext}(B_{\text{lip}(X, d^\alpha)^*}) \subseteq \left\{ \lambda \tilde{\delta}_w : (w, \lambda) \in W \times \mathbb{T} \right\}. \quad (3.2)$$

Now, let $(x, \lambda) \in X \times \mathbb{T}$. Clearly, $\lambda \delta_x \in B_{\text{lip}(X, d^\alpha)^*}$. Assume that $\varphi \in \text{Ext}(B_{\text{lip}(X, d^\alpha)^*})$ and $\varphi(f) = 1$ for all $f \in \text{lip}(X, d^\alpha)$ with $(\lambda \delta_x)(f) = 1 = \|f\|_{X, p_\alpha}$. Since $\text{lip}(X, d^\alpha)$ is a nonzero linear subspace of $\text{Lip}(X, d^\alpha)$, we conclude that $\varphi = \mu \Delta_y$ for some $(y, \mu) \in X \times \mathbb{T}$ or $\varphi = \mu \tilde{\Delta}_w$ for

some $(w, \mu) \in (\beta\tilde{X}) \times \mathbb{T}$ by Theorem 1.1. Clearly, $\bar{\lambda}1 \in \text{lip}(X, d^\alpha)$ and $(\lambda\delta_x)(\bar{\lambda}1) = 1 = \|\bar{\lambda}1\|_{X, p_\alpha}$. If $\psi = \mu\tilde{\Delta}_w$ for some $(w, \mu) \in (\beta\tilde{X}) \times \mathbb{T}$, then we have

$$\begin{aligned} 1 &= |\psi(\bar{\lambda}1)| = |\mu\tilde{\Delta}_w(\bar{\lambda}1)| \\ &= |\tilde{\Delta}_w(1)| = |(\beta\tilde{1})(w)| \\ &\leq \|\beta\tilde{1}\|_{\beta\tilde{X}} = p_\alpha(1) \\ &= 0. \end{aligned} \tag{3.3}$$

Thus, it must be $\psi = \mu\Delta_y$ for some $(y, \mu) \in X \times \mathbb{T}$. Since $\psi(\bar{\lambda}1) = 1$, we have

$$1 = (\mu\Delta_y)(\bar{\lambda}1) = \mu\bar{\lambda}. \tag{3.4}$$

It follows that $\mu = \lambda$. We claim that $y = x$. Let $y \in X \setminus \{x\}$. We define the function $g : X \rightarrow \mathbb{C}$ by

$$g(z) = 1 - \frac{d(z, x)}{1 + \rho + \rho^{1-\alpha}}, \tag{3.5}$$

where $\rho = \text{diam } X$. It is easy to show that $g \in \text{Lip}(X, d^1)$, $g(x) = 1$, $\|g\|_X = 1$ and $p_\alpha(g) \leq 1$. Now, we define the function $f : X \rightarrow \mathbb{C}$ by

$$f(z) = \bar{\lambda} \left(1 - \frac{g(z)d(z, x)}{2(1 + \rho + \rho^{1-\alpha})} \right), \tag{3.6}$$

and the function $g_1 : X \rightarrow \mathbb{C}$ by $g_1 = (1/2)(1 - g)$. It is easy to see that $g_1 \in \text{Lip}(X, d^1)$, $\|g_1\|_X \leq 1/2$, $p_\alpha(g_1) \leq 1/2$, and $f = \bar{\lambda}(1 - gg_1)$. Therefore, we have $f \in \text{Lip}(X, d^1) \subseteq \text{lip}(X, d^\alpha)$, $f(x) = \bar{\lambda}$, $\|f\|_X \leq 1$, and

$$p_\alpha(f) \leq p_\alpha(g)\|g_1\|_X + \|g\|_X p_\alpha(g_1) \leq \frac{1}{2} + \frac{1}{2} = 1, \tag{3.7}$$

so that $\|f\|_{X, p} = 1 = (\lambda\delta_x)(f)$. It must be that $\psi(f) = 1$; that is, $(\lambda\Delta_y)(f) = 1$. But

$$\begin{aligned} |(\lambda\Delta_y)(f)| &= |\lambda f(y)| = |f(y)| \\ &= 1 - \frac{g(y)d(y, x)}{2(1 + \rho + \rho^{1-\alpha})} < 1. \end{aligned} \tag{3.8}$$

This contradiction implies that $y = x$ and so our claim is justified. Therefore, $\psi = \lambda \Delta_x$. It follows that $\psi|_{\text{lip}(X, d^\alpha)} = \lambda \delta_x$. Therefore, $\lambda \delta_x \in \text{Ext}(B_{\text{lip}(X, d^\alpha)^*})$ by part (b) of Theorem 1.1. Consequently,

$$\{\lambda \delta_x : (x, \lambda) \in X \times \mathbb{T}\} \subseteq \text{Ext}(B_{\text{lip}(X, d^\alpha)^*}) \subseteq \{\lambda \tilde{\delta}_w : (w, \lambda) \in W \times \mathbb{T}\}. \quad (3.9)$$

Thus, the proof is complete. $\square$

For each $(w, \lambda) \in W \times \mathbb{T}$, we define the map $\tilde{\varphi}_{w, \lambda} : \text{lip}(X, d^\alpha)_r \rightarrow \mathbb{R}$ by $\tilde{\varphi}_{w, \lambda}(f) = \text{Re}(\lambda \tilde{\delta}_w(f))$; in fact, $\tilde{\varphi}_{w, \lambda} = \text{Re}(\lambda(e_w \circ \Phi_\alpha))$. Clearly, $\tilde{\varphi}_{w, \lambda} \in B_{(\text{lip}(X, d^\alpha)_r)^*}$ for all $(w, \lambda) \in W \times \mathbb{T}$. Moreover, $\tilde{\varphi}_{x, \lambda} = \text{Re}(\lambda \delta_x)$ for all $(x, \lambda) \in X \times \mathbb{T}$.

We now give a characterization of extreme points of the unit ball in the $(\text{lip}(X, d^\alpha)_r)^*$ as the following.

Proposition 3.2. *By above notations,*

$$\{\tilde{\varphi}_{x, \lambda} : (x, \lambda) \in X \times \mathbb{T}\} \subseteq \text{Ext}(B_{(\text{lip}(X, d^\alpha)_r)^*}) \subseteq \{\tilde{\varphi}_{w, \lambda} : (w, \lambda) \in W \times \mathbb{T}\}. \quad (3.10)$$

Further, for $(x, \lambda), (y, \mu) \in X \times \mathbb{T}$, one has $\tilde{\varphi}_{x, \lambda} = \tilde{\varphi}_{y, \mu}$ if and only if $(x, \lambda) = (y, \mu)$.

Proof. The proof is similar to that of Proposition 2.1 by replacing $\text{lip}(X, d^\alpha)$, $\tilde{\delta}_x$, $\tilde{\delta}_w$, $\tilde{\varphi}_{x, \lambda}$, $\tilde{\varphi}_{w, \lambda}$ and Theorems 1.2 and 1.3 by $\text{Lip}(X, d^\alpha)$, Δ_x , $\tilde{\Delta}_w$, $\tilde{\psi}_{x, \lambda}$, $\tilde{\psi}_{w, \lambda}$ and Theorem 3.1, respectively. $\square$

Theorem 3.3. *Let A be a real linear subspace of $\text{lip}(X, \tau, d^\alpha)$ containing 1. For $(w, \lambda) \in W \times \mathbb{T}$, let $\tilde{\theta}_{w, \lambda} = \tilde{\varphi}_{w, \lambda}|_A$. Let $P = \{(x, \lambda) \in X \times \mathbb{T} : \tau(x) \neq x\} \cup \{(x, \lambda) \in X \times \mathbb{T} : \tau(x) = x, \lambda \in \{-1, 1\}\} \cup \{(w, \lambda) \in W \times \mathbb{T} : w \in \beta \tilde{X}\}$. Let Q_A denote the set of all $(x, \lambda) \in X \times \{-1, 1\}$ such that*

- (i) *there is $g \in A$ with $\|g\|_{X, p_\alpha} \leq 1$ and $g(x) = \bar{\lambda}$,*
- (ii) *for every $y \in X \setminus \{x, \tau(x)\}$, there is $h \in A$ with $\|h\|_{X, p_\alpha} \leq 1$, $h(x) = 1$ and $|h(y)| < 1$.*

Then,

$$\{\tilde{\theta}_{x, \lambda} : (x, \lambda) \in Q_A\} \subseteq \text{Ext}(B_{A^*}) \subseteq \{\tilde{\theta}_{w, \lambda} : (w, \lambda) \in P\}. \quad (3.11)$$

Further, if $(x, \lambda) \in Q_A$ and $(y, \mu) \in P \cap (X \times \mathbb{T})$, then $\tilde{\theta}_{y, \mu} = \tilde{\theta}_{x, \lambda}$ if and only if $(y, \mu) = (x, \lambda)$ or $(y, \mu) = (\tau(x), \lambda)$.

Proof. The proof is similar to that of Theorem 2.2 by replacing $\text{lip}(X, d^\alpha)$, $\tilde{\theta}_{w, \lambda}$, $\tilde{\theta}_{x, \lambda}$, $\tilde{\varphi}_{x, \lambda}$, Proposition 3.2 and Φ_α by $\text{Lip}(X, d^\alpha)$, $\tilde{\eta}_{w, \lambda}$, $\tilde{\eta}_{x, \lambda}$, $\tilde{\psi}_{x, \lambda}$, Proposition 2.1 and Ψ_α , respectively. $\square$

Corollary 3.4. *Let P be as in Theorem 3.3. For $(w, \lambda) \in P$, let $\tilde{\theta}_{w, \lambda}$ denote the restriction of $\tilde{\varphi}_{w, \lambda}$ to $\text{lip}(X, \tau, d^\alpha)$. Then,*

$$\{\tilde{\theta}_{x, \lambda} : (x, \lambda) \in X \times \{-1, 1\}\} \subseteq \text{Ext}(B_{\text{lip}(X, \tau, d^\alpha)^*}) \subseteq \{\tilde{\theta}_{w, \lambda} : (w, \lambda) \in P\}. \quad (3.12)$$

Further, if $(x, \lambda) \in X \times \{-1, 1\}$ and $(y, \mu) \in P \cap (X \times \mathbb{T})$, then $\theta_{y, \mu} = \tilde{\theta}_{x, \lambda}$ if and only if $(y, \mu) = (x, \lambda)$ or $(y, \mu) = (\tau(x), \lambda)$.

Proof. The proof is similar to that Corollary 2.3 by replacing $\text{lip}(X, \tau, d^\alpha)$, $\tilde{\theta}_{w, \lambda}$, $\tilde{\theta}_{x, \lambda}$ and $\text{lip}(X, d^\alpha)$ by $\text{Lip}(X, \tau, d^\alpha)$, $\tilde{\eta}_{w, \lambda}$, $\tilde{\eta}_{x, \lambda}$ and $\text{Lip}(X, d^\alpha)$, respectively. $\square$

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