Research Article

Some Blow-Up and Smooth Solutions about Landau-Lifshitz Equation and Their Spatial Curvature Behavior

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We construct the exact solution of $(2 + 1)$ -dimensional space-time Landau-Lifshitz equation (LLE) without the Gilbert term. Under suitable transformations, some exact solutions are obtained in the radially symmetric coordinates and nonsymmetric coordinates. The type of solutions cover the finite-time blow-up solution, smooth solution in time and vortex solution. At the end, some properties about these solutions and their spatial curvature are illustrated by the graphs.

1. Introduction

It is well known that Landau-Lifshitz equation (LLE) [1, 2] is one of the most important nonlinear equations in physics which bears a fundamental role in the understanding of nonequilibrium magnetism, just as the Navier-Stokes equation does in that of fluid dynamics. Under the principal assumption of the macroscopic theory of ferromagnetism, the state of a magnetic crystal is described by the magnetization vector $V(x, t)$, which is a function of position x and time t . For instance, a subcase of the LLE is as follows:

$$V_t = \alpha V \times H_{\text{eff}} - \beta V \times (V \times H_{\text{eff}}), \quad (1.1)$$

where $V = (v_1, v_2, v_3)$. α, β are constants. $\times$ denotes a vector cross-product. The effective magnetic field H_{eff} is given by the derivative of the magnetic crystal energy E with respect to the vector V :

$$H_{\text{eff}} = -\frac{\partial E}{\partial V}, \quad (1.2)$$

where E is given by the sum of the exchange energy, the anisotropy energy, and the magnetic field energy; that is,

$$E(V) = E_{ex}(V) + E_{an}(V) + E_{mf}(V). \quad (1.3)$$

Equation (1.1) exhibits a rich variety of dynamical properties of a spin vector in different backgrounds. In this letter, we discuss the following LLE which vanishes the Gilbert term (i.e., $\beta = 0$), anisotropy energy and the magnetic field energy (here, we set $\alpha = 1$):

$$V_t = V \times \Delta V_{R^2}, \quad \Delta V_{R^2} = \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2}. \quad (1.4)$$

In the 1-dimensional motion LLE, the soliton solutions have been studied by many physicists and mathematicians, see, for example, in 1976, Lakshmanan et al. constructed a class of solutions [3] of (1.4). Nakamura and Sasada constructed a solution [4] of the non-vanishing external magnetic field case in 1974. Tjon and Wright also found some solitons [5] for such case in 1977.

LLE is nonintegrable [6] in high dimensions ($n \geq 2$). Furthermore, as far as we know, many famous direct methods such as Hirota bilinear method and auxiliary function method [7] are difficult for constructing the exact solution of (1.4). So, we can only find out some particular exact solutions [8, 9] by various direct methods which base on some special ansatz about the solution. In 2000 and 2001, Guo et al. constructed some exact blow-up solutions [10, 11] for 2-dimension radially symmetric LLE. Similarly, for n -dimensions LLE, some exact blow-up solutions were constructed in [12, 13]. In this situation, a solution V blows up at time $T > 0$ if

$$\lim_{t \rightarrow T^-} \sup \|\nabla V(\cdot, t)\|_\infty = \infty. \quad (1.5)$$

The present work is organized as follows. We firstly deduce a blow-up solution of (1.4) in Section 2. Then, we construct a vortex solution and some other periodic solutions about LLE in Section 3. In Section 4, some other exact solutions which different from blow-up solution and vortex solution are presented. In Section 6, some solutions and their spatial curvature are illustrated in a graphic way.

2. Blow-Up Solution

We deduce an exact blow-up solution in this section. Under radially symmetric coordinates, (1.4) takes the form:

$$V_t = V \times V_{rr} + \frac{1}{r} V \times V_r, \quad (2.1)$$

where $r = \sqrt{x_1^2 + x_2^2}$, $V = (v_1(t, r), v_2(t, r), v_3(r))$.

We will find the explicit solution of (2.1) in the form of

$$\begin{aligned}v_1(t, r) &= \cos(m(t, r))f(r), \\v_2(t, r) &= \sin(m(t, r))f(r), \\v_3(t, r) &= \gamma f(r),\end{aligned}\tag{2.2}$$

where $m(t, r)$ and $f(r)$ are functions to be determined, γ is constant.

Example 2.1. If

$$m(t, r) = -\frac{1}{432} \frac{27C_1^3 r^4 + 108C_1^2 r^{8/3} C_2 + 144C_1 r^{4/3} C_2^2 + 64C_2^3}{t - T} + C_3, \tag{2.3}$$

$$f(r) = \frac{-81C_1^3 r^4 - 324C_1^2 r^{8/3} C_2 - 432C_1 r^{4/3} C_2^2 - 192C_2^3}{\gamma C_1^2 (16r^{1/3} C_2^2 + 24C_1 r^{5/3} C_2 + 9C_1^2 r^3)^2}, \tag{2.4}$$

where γ, C_1, C_2, C_3 are any constants, $T > 0$, then (2.2) is an exact blow-up solution of (2.1).

Proof. Substituting (2.2) into (2.1), we get

$$\gamma r \sin(m) m_r^2 f - \gamma r \cos(m) m_{rr} f - 2\gamma r \cos(m) m_r f_r - f \gamma \cos(m) m_r + \sin(m) m_t r = 0, \tag{2.5}$$

$$\gamma r \cos(m) m_r^2 f + \gamma r \sin(m) m_{rr} f + 2\gamma r \sin(m) m_r f_r + f \gamma \sin(m) m_r + \cos(m) m_t r = 0, \tag{2.6}$$

$$r m_{rr} f + 2r m_r f_r + m_r f = 0. \tag{2.7}$$

Furthermore, if we insert (2.7) into (2.5) and (2.6), we have

$$\begin{aligned}r m_{rr} f + 2r m_r f_r + m_r f &= 0, \\ \gamma m_r^2 f + m_t &= 0.\end{aligned}\tag{2.8}$$

If we settle down on the following ansatzs:

$$\begin{aligned}m &= \frac{g(r)}{t - T} + C, \\ f &= g(r)j(r),\end{aligned}\tag{2.9}$$

where $g(r)$ and $j(r)$ are the functions about r to be determined, T and C are constants.

From (2.8) and (2.9), we have

$$\begin{aligned}\gamma g_r^2 g j - g &= 0, \\ r g_r^2 g j + 2r g_r (g_r j + g j_r) + g_r g j &= 0.\end{aligned}\tag{2.10}$$

To construct the exact solution of the ordinary differential equations (ODEs for short), we can resort to the mathematic software. Concretely, we find out the exact solution of (2.10) as follows:

$$\begin{aligned} g &= \frac{1}{64}C_1^3r^4 + \frac{1}{16}C_1^2r^{8/3}C_2 + \frac{1}{12}C_1r^{4/3}C_2^2 + \frac{1}{27}C_2^3, \\ j &= \frac{20736}{\gamma C_1^2(9C_1^2r^3 + 24C_1r^{5/3}C_2 + 16r^{1/3}C_2^2)^2}, \end{aligned} \quad (2.11)$$

where γ, C_1, C_2 are any constants.

This completes the proof of Example 2.1. $\square$

In the case of ODE, we mention here that deducing process of software tries to solve it using either classification methods or symmetry methods. In these process, the coefficient about r in (2.11) will be extend to a general form. It is why the forms of (2.3) and (2.4) are so specific.

For this finite-time blow-up solution (2.2), it is easy to verify the following.

Remark 2.2. (i) It holds that

$$|V(t, r)|^2 = \frac{9(144C_1r^{4/3}C_2^2 + 108C_1^2r^{8/3}C_2 + 27C_1^3r^4 + 64C_2^3)^2(1 + \gamma^2)}{\gamma^2C_1^4(16r^{1/3}C_2^2 + 24C_1r^{5/3}C_2 + 9C_1^2r^3)^4} \quad (2.12)$$

is independent of t , so $\int_{r_0}^{r_1} r|V|^2 dr (0 < r_0 \leq r_1 < \infty)$ is a conservation quantity.

(ii) When $r_0 \leq r \leq r_1, T > 0$, it is not difficult to verify $\lim_{t \rightarrow T} |V_r|^2 = +\infty$. For example, in the case of $T = 1$ and $\gamma = C_1 = C_3 = 1, C_2 = 0$, we can verify that

$$|V_r|^2 = \frac{1}{16} \frac{r^8 - 256t + 128 + 128t^2}{r^6(t - 1)^2} \quad (2.13)$$

is not bounded as $t \rightarrow 1$.

(iii) For any first orthogonal matrix B (the elements of B are constants.), VB is also a blow up solution of (2.1).

3. Vortex Solution and Periodic Solutions

In this section, we concentrate on the solution on the S^2 (here, we mean that $|V|^2 = 1$). We firstly consider the vortex solution in the radially symmetric coordinates. Here, these time periodic solutions, which under the radially symmetric coordinates are called magnetic vortices or vortex solutions are of particular importance in manifesting interesting physical phenomena and topological structures of LLE. After that, some exact solutions which under the nonradially symmetric coordinates are proposed.

Under the radially symmetric coordinates, we set

$$\begin{aligned}v_1(t, r) &= f(t) \sin(q(r)), \\v_2(t, r) &= \sqrt{1 - (f(t))^2} \sin(q(r)), \\v_3(t, r) &= \cos(q(r)),\end{aligned}\tag{3.1}$$

Submitting (3.1) into (2.1), (2.1) can be transformed into

$$\sqrt{1 - (f(t))^2} r^2 \frac{d^2}{dr^2} q(r) + \sqrt{1 - (f(t))^2} \left(\frac{d}{dr} q(r) \right) r + \left(\frac{d}{dt} f(t) \right) \sin(q(r)) r^2 = 0. \tag{3.2}$$

So we have

$$\begin{aligned}\sqrt{1 - (f(t))^2} + \frac{1}{C} \frac{d}{dt} f(t) &= 0, \\r^2 \frac{d^2}{dr^2} q(r) + \left(\frac{d}{dr} q(r) \right) r - C \sin(q(r)) r^2 &= 0.\end{aligned}\tag{3.3}$$

Solving (3.3), we have

$$\begin{aligned}f(t) &= \sin(Ct + C_1) \quad \text{or} \quad f(t) = \cos(Ct + C_1), \\ \sqrt{1 - f^2(t)} &= \cos(Ct + C_1) \quad \text{or} \quad \sqrt{1 - f^2(t)} = \sin(Ct + C_1), \\ q(r) &= \pm 4, \arctan\left(\frac{r^{\pm\sqrt{C}}}{\lambda}\right),\end{aligned}\tag{3.4}$$

where $C > 0$, C_1 , λ are constants.

We point out there that (3.4) is vortex solution of (2.1). Similarly, if we set (C is a constant)

$$\begin{aligned}v_1(t, x, y) &= \cos(t + C) \sin(Q(x, y)), \\v_2(t, x, y) &= \sin(t + C) \sin(Q(x, y)), \\v_3(t, x, y) &= \cos(Q(x, y)),\end{aligned}\tag{3.5}$$

Equation (1.4) can be transformed into

$$\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} = \sin Q. \tag{3.6}$$

Here, we propose the following two subcase solutions of (3.6).

(I) *Functional separable solutions of (3.6)*

let

$$Q(x, y) = \pm 4 \arctan[f(\xi_1)g(\xi_2)], \quad (3.7)$$

where $\xi_1 = ax + b$, $\xi_2 = cy + d$.

Substituting (3.7) into (3.6), we can deduce the following equations:

$$\begin{aligned} 4a^2 f'' g + 4a^2 f'' f^2 g^3 - 8a^2 (f')^2 f g^3 + 4c^2 g'' f + 4c^2 g'' f^3 g^2 \\ - 8c^2 (g')^2 f^3 g - 4f g + 4f^3 g^3 = 0. \end{aligned} \quad (3.8)$$

Introducing two auxiliary function about f' , g' ,

$$(f')^2 = A_1 f^4 + B_1 f^3 + C_1 f^2 + D_1 f + E_1, \quad (3.9)$$

$$(g')^2 = A_2 g^4 + B_2 g^3 + C_2 g^2 + D_2 g + E_2. \quad (3.10)$$

According to (3.9) and (3.10), we have

$$f'' = \frac{1}{2} D_1 + 2A_1 f^3 + \frac{3}{2} B_1 f^2 + C_1 f, \quad (3.11)$$

$$g'' = \frac{1}{2} D_2 + 2A_2 g^3 + \frac{3}{2} B_2 g^2 + C_2 g. \quad (3.12)$$

Substituting (3.9)–(3.12) into (3.8), we have

$$\begin{aligned} -2B_1 a^2 f^4 g^4 - 2B_2 c^2 f^3 g^4 - 4(C_1 a^2 + C_2 c^2 - 1) f^3 g^3 - 6D_2 c^2 f^3 g^2 \\ + 8(A_1 a^2 - E_2 c^2) f^3 g - 6D_1 a^2 f^2 g^3 + 6B_1 a^2 f^2 g \\ + 8(A_2 c^2 - E_1 a^2) f g^3 + 6B_2 c^2 f g^2 \\ + 4(C_1 a^2 + C_2 c^2 - 1) f g + 2D_2 c^2 f + 2D_1 a^2 g = 0. \end{aligned} \quad (3.13)$$

Equating the coefficients of different powers of fg in (3.13), we obtain a set of algebraic equations:

$$\begin{aligned} B_1 = B_2 = D_1 = D_2 = 0, \\ C_1 a^2 + C_2 c^2 - 1 = 0, \\ A_1 a^2 - E_2 c^2 = 0, \\ A_2 c^2 - E_1 a^2 = 0. \end{aligned} \quad (3.14)$$

If we set

$$A = A_1 a^2, \quad B = C_1 a^2, \quad C = E_1 a^2, \quad (3.15)$$

according to (3.14), functions $f = f(\xi_1)$ and $g = g(\xi_2)$ are determined by the first-order autonomous ordinary differential equations as follows:

$$(f_{\xi_1})^2 = A f^4 + B f^2 + C, \quad (g_{\xi_2})^2 = C g^4 + (1 - B) g^2 + A, \quad (3.16)$$

Here, we omit some details about the deducing and present some exact solutions of (3.16).

(A) If $A = C = 0, 0 < B < 1$, then

$$f = \exp(\pm \sqrt{B} \xi_1), \quad g = \exp(\pm \sqrt{1 - B} \xi_2). \quad (3.17)$$

(B) If $A = 0, B = 1, C > 0$, solving (3.16), we have

$$f = \sqrt{C} \sinh(\xi_1), \quad g = \pm \frac{1}{\sqrt{C} \xi_2}. \quad (3.18)$$

(C) If $A = 0, B > 1, C > 0$, then

$$f = \sqrt{\frac{C}{B}} \sinh(\sqrt{B} \xi_1), \quad g = \sqrt{\frac{B-1}{C}} \sec(\sqrt{B-1} \xi_2). \quad (3.19)$$

(D) According to Table 3.1 which is proposed in [7], we can deduce a solution easily. Defining

$$sc(\xi, R) = \frac{sn(\xi, R)}{cn(\xi, R)}, \quad ds(\xi, R) = \frac{dn(\xi, R)}{sn(\xi, R)}, \quad (3.20)$$

where R is the modulus of Jacobi elliptic function.

Suppose $R = \sqrt{3}/2, A = -3/16, B = 1/2, C = 1$, we have

$$f = sd\left(\xi_1, \frac{\sqrt{3}}{2}\right), \quad g = ds\left(\xi_2, \frac{\sqrt{3}}{2}\right). \quad (3.21)$$

(II) Another functional separable solution of (3.6)

$$\begin{aligned} Q(x, y) &= 4 \arctan \left(\cot C_1 \frac{\cosh f}{\cosh g} \right), \quad f = \frac{\cos C_1}{\sqrt{1 + C_2^2}} (x - C_2 y), \\ g &= \frac{\sin C_1}{\sqrt{1 + C_2^2}} (y - C_2 x), \end{aligned} \quad (3.22)$$

where C_1 and C_2 are arbitrary constants.

According to the above derivation, we conclude our main result in the following example.

Example 3.1. (i) (3.1) and (3.4) are a vortex solution of (2.1);
(ii) (3.5), (3.7) (3.17)–(3.19), and (3.21) are exact periodic solutions about (1.4);
(iii) (3.5) and (3.22) are a periodic solution about (1.4).

Remark 3.2. (i) For the solution (3.1) and (3.4),

$$|V_r(t, r)|^2 = 16 \frac{r^{2\sqrt{C}-2} C \lambda^2}{(\lambda^2 + r^{2\sqrt{C}})^2} \quad (3.23)$$

is independent of t . Furthermore, $\int_0^\infty |V_r|^2 r dr$ is a conservation quantity. For example, if we set $C = 4$ and $\lambda = 1$ in (3.23), we have $\int_0^\infty |V_r|^2 r dr = 16 \gg 0$.

(ii) For the solutions proposed in (ii)–(iii) in Example 3.1, it is a tedious expression for $|\nabla V(t, x, y)|^2$ (∇ is about the spacial directions) to each of them. Therefore, if we omit the details of $Q(t, x, y)$, obviously $|\nabla V(t, x, y)|^2$ of (ii) and (iii) will adopt the same form as follows:

$$|\nabla V(t, x, y)|^2 = \left(\frac{\partial}{\partial x} Q(x, y) \right)^2 + \left(\frac{\partial}{\partial y} Q(x, y) \right)^2. \quad (3.24)$$

The graphic demonstration will be provided in Section 6.

(iii) For any first orthogonal matrix B (the elements of B are constants.), VB is also a blow up solution of (1.4) or (2.1).

4. Non-Blow-Up Solutions on the Disc

In Section 1, we propose some blow-up solutions on the disc. However, whether or not there are non-blow-up solutions (here the non-blow-up solution contains discontinuous one) on the disc is not so clear as far as we know. By constructing some solutions, we give a positive answer about it.

According to results which provided in [13], some exact solutions can be constructed via an explicit transform between LLE with effective magnetic field and the one without

external magnetic field. So, if we meet some difficulties in searching the solutions about (1.4), we can firstly try to find out some solutions about (1.1).

Similar to Section 2, we can find out an exact solution for (2.1) as follows:

$$\begin{aligned} v_1(t, r) &= -\frac{\cos\left(C_3 t + (3/4)r(C_1\sqrt{r})^{2/3} + C_2\right)C_3}{(C_1\sqrt{r})^{4/3}}, \\ v_2(t, r) &= -\frac{\sin\left(C_3 t + (3/4)r(C_1\sqrt{r})^{2/3} + C_2\right)C_3}{(C_1\sqrt{r})^{4/3}}, \\ v_3(r) &= -\frac{C_3}{(C_1\sqrt{r})^{4/3}}, \end{aligned} \quad (4.1)$$

where C_1, C_2, C_3 are any constants.

Although (4.1) is an exact solution, we can find out some other exact solutions about (2.1) by adding an effective magnetic field. This fact will be demonstrated in the following illustration which concentrates on Proposition 4.4.

Imposing a special anisotropy energy on (1.4), we have the following system:

$$U_t = U \times (\Delta U + UJ), \quad (4.2)$$

where $J = \text{diag}\{a, a, b\}$, a, b are constants.

Under radially symmetric coordinates, (4.2) takes the form:

$$U_t = U \times U_{rr} + \frac{1}{r}U \times U_r + U \times UJ, \quad (4.3)$$

where $r = \sqrt{x_1^2 + x_2^2}$, $U = (u_1(t, r), u_2(t, r), u_3(t, r))$.

We deduce a solution of (4.3) in the following example.

Example 4.1. If $Z(r)$ is the root of

$$Z(r)^4 - C_1\sqrt{r}Z(r) - a + b = 0, \quad (4.4)$$

$$\begin{aligned} u_1(t, r) &= \frac{\cos\left(C_3 t + \int Z(r)^2 dr + C_2\right)C_3}{-Z(r)^4 - b + a}, \\ u_2(t, r) &= \frac{\sin\left(C_3 t + \int Z(r)^2 dr + C_2\right)C_3}{-Z(r)^4 - b + a}, \\ u_3(r) &= \frac{C_3}{-Z(r)^4 - b + a}, \end{aligned} \quad (4.5)$$

where C_1, C_2, C_3 are any constants, then (4.4)-(4.5) is a solution of (4.3).

Proof. We settle down on the following setting:

$$\begin{aligned}u_1(t, r) &= \cos(m(t, r))f(r), \\u_2(t, r) &= \sin(m(t, r))f(r), \\u_3(r) &= f(r),\end{aligned}\tag{4.6}$$

where $m(t, r)$ and $f(r)$ are functions to be determined.

Similar to the deducing of Example 2.1, if we insert (4.6) into (4.3), we have

$$\begin{aligned}rm_{rr}f + 2rm_rf_r + m_rf &= 0, \\ \gamma m_r^2 f + (b - a)f + m_t &= 0.\end{aligned}\tag{4.7}$$

Solving (4.7), we get

$$\begin{aligned}m(t, r) &= C_3 t + \int Z(r)^2 dr + C_2, \\ f(r) &= \frac{C_3}{-Z(r)^4 - b + a},\end{aligned}\tag{4.8}$$

where $Z(r)$ satisfies $Z(r)^4 - C_1\sqrt{r}Z(r) - a + b = 0$.

Therefore, we have completed the proof of Example 4.1. $\square$

Remark 4.2. (i) For (4.4), we can find out the exact solution of it. Let

$$\begin{aligned}G &= 108C_1^2r + 12\sqrt{768a^3 - 2304a^2b + 2304ab^2 - 768b^3 + 81C_1^4r^2}, \\ Q &= \sqrt{\frac{G^{2/3} - 48a + 48b}{G^{1/3}}},\end{aligned}\tag{4.9}$$

Then

$$Z(r) = \pm \frac{\sqrt{6}}{12}Q \pm \frac{1}{12}\sqrt{-\frac{6QG^{2/3} - 288Qa + 288Qb - 72\sqrt{6}C_1\sqrt{r}G^{1/3}}{G^{1/3}Q}}.\tag{4.10}$$

(ii) it holds that

$$|U(t, r)|^2 = \frac{2C_3^2}{C_1^2rZ(r)^2}\tag{4.11}$$

is independent of t , so $\int_{r_0}^{r_1} r|U|^2 dr$ ($0 < r_0 \leq r_1 < \infty$) is a conservation quantity.

Figure 1: (2.2)–(2.4) with $r \in [1, 3]$, $t \in [0, 9/10]$.

(iii) When $0 < r \leq r_1$, it will not a difficult work to verify $|V_r|^2 \leq C$ (C is a positive constant). For instant, in the case of $b = 0$ and $a = C_1 = C_3 = 1$, we can verify that

$$|U_r|^2 = \frac{(r^{1/2}Z(r) + 1)(8 + 16r^2 + 9r^3Z(r)^2 + 8r^{1/2}Z(r) + 24r^{5/2}Z(r))}{r^3(24r^{1/2}Z(r)^3 + 16Z(r)^2 + 9r^{3/2}Z(r) + 9r)} \quad (4.12)$$

is bounded as $0 < r \leq r_1$. In fact, $|U_r|^2$ decay quickly as $r \rightarrow +\infty$. We can see this property in Figure 6.

(iv) For any first orthogonal matrix B (the elements of B are constants.), UB is also a solution of (4.3).

In order to get the exact solution of (2.1), we firstly propose an important Lemma which borrow from [13] as follows.

Figure 2: Part shape of (2.13) with $T = 1$.

Figure 3: (3.23) as $\lambda = 0, 1, 2, 3, 4$, respectively.

Figure 4: Part shape of periodic function (3.24) and (6.1).

Figure 5: Part shape of periodic function (3.24) and (6.2).

Lemma 4.3. *Let*

$$B = \begin{bmatrix} \frac{\sqrt{b_2^2 + b_3^2}}{\sqrt{b_1^2 + b_2^2 + b_3^2}} & -\frac{b_1 b_2}{\sqrt{b_2^2 + b_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}} & -\frac{b_1 b_3}{\sqrt{b_2^2 + b_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}} \\ 0 & \frac{b_3}{\sqrt{b_2^2 + b_3^2}} & -\frac{b_2}{\sqrt{b_2^2 + b_3^2}} \\ \frac{b_1}{\sqrt{b_1^2 + b_2^2 + b_3^2}} & \frac{b_2}{\sqrt{b_1^2 + b_2^2 + b_3^2}} & \frac{b_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}} \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} \sqrt{b_2^2 + b_3^2} & -\frac{b_1 b_2}{\sqrt{b_2^2 + b_3^2}} & -\frac{b_1 b_3}{\sqrt{b_2^2 + b_3^2}} \\ 0 & \frac{b_3}{\sqrt{b_2^2 + b_3^2}} & -\frac{b_2}{\sqrt{b_2^2 + b_3^2}} \\ b_1 & b_2 & b_3 \end{bmatrix}, \quad (4.13)$$

$$\tilde{A} = \begin{bmatrix} \frac{\sqrt{\tilde{h}_2^2 + \tilde{h}_3^2} \cos(\tilde{\rho})}{|\tilde{H}|} & -\frac{\tilde{h}_1 \tilde{h}_2 \cos(\tilde{\rho})}{|\tilde{H}| \sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} & -\frac{\tilde{h}_3 \sin(\tilde{\rho})}{\sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} & -\frac{\tilde{h}_1 \tilde{h}_3 \cos(\tilde{\rho})}{|\tilde{H}| \sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} & +\frac{\tilde{h}_2 \sin(\tilde{\rho})}{\sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} \\ \frac{\sqrt{\tilde{h}_2^2 + \tilde{h}_3^2} \sin(\tilde{\rho})}{|\tilde{H}|} & -\frac{\tilde{h}_1 \tilde{h}_2 \sin(\tilde{\rho})}{|\tilde{H}| \sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} & +\frac{\tilde{h}_3 \cos(\tilde{\rho})}{\sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} & -\frac{\tilde{h}_1 \tilde{h}_3 \sin(\tilde{\rho})}{|\tilde{H}| \sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} & -\frac{\tilde{h}_2 \cos(\tilde{\rho})}{\sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} \\ \frac{\tilde{h}_1}{|\tilde{H}|} & \frac{\tilde{h}_2}{|\tilde{H}|} & \frac{\tilde{h}_3}{|\tilde{H}|} & \frac{\tilde{h}_3}{|\tilde{H}|} & \frac{\tilde{h}_2}{|\tilde{H}|} \end{bmatrix},$$

where $\tilde{H} = (\tilde{h}_1, \tilde{h}_2, \tilde{h}_3) = (h_1, h_2, h_3)B^T$, $|\tilde{H}| = \sqrt{\tilde{h}_1^2 + \tilde{h}_2^2 + \tilde{h}_3^2}$, and $\tilde{\rho} = |\tilde{H}| \int_0^t \mu(s) ds$.
Considering the following two LLEs $(\mathbb{P}_1)$ and $(\mathbb{P}_2)$:

$$(\mathbb{P}_1) \begin{cases} \frac{\partial V}{\partial t} = V \times (\Delta V + \mu(t)H + H(V)), & \text{in } R^n \times (0, \infty), \\ V|_{t=0} = V_0, & \text{in } R^n, \end{cases} \quad (4.14)$$

where $\int_0^t \mu(s) ds \in C[0, \infty)$ and $\mu(t) \in L^2(0, \infty)$, $H(V) = \sum_{i,j=1}^3 b_{ij} u_i(b_1, b_2, b_3)$, $b_3 \neq 0$,

$$b_{ij} = \begin{cases} \alpha + (\beta - \alpha)b_i b_j, & i = j, \\ (\beta - \alpha)b_i b_j, & i \neq j. \end{cases} \quad (4.15)$$

$$(\mathbb{P}_2) \begin{cases} \frac{\partial U}{\partial t} = U \times (\Delta U + H_1(U)), & \text{in } R^n \times (0, \infty), \\ U|_{t=0} = U_0, & \text{in } R^n, \end{cases} \quad (4.16)$$

where $H_1(U) = (0, 0, (\beta - \alpha)(b_1^2 + b_2^2 + b_3^2)u_3)$.

By the transform $V = U\tilde{B}\tilde{A}B$, (4.14) is equivalent to (4.16).

Figure 6: (4.1) with $r \in [1/2, 25]$ near $t = 0$.

Combining Example 4.1 with Lemma 4.3, one has the following Proposition, in which one also adopts the notations of Lemma 4.3.

Proposition 4.4. *Let*

$$V = U \tilde{B} \tilde{A} B \quad (4.17)$$

be a transform between the initial problem of the isotropic Landau-Lifshitz equation:

$$(\mathbb{Q}_1) \begin{cases} \frac{\partial V}{\partial t} = V \times (\Delta V), & \text{in } R^n \times (0, \infty), \\ V|_{t=0} = V_0, & \text{in } R^n, \end{cases} \quad (4.18)$$

and the initial problem of the LLE with an anisotropic magnetic field:

$$(\mathbb{Q}_2) \begin{cases} \frac{\partial U}{\partial t} = U \times (\Delta U + H_1(U)), & \text{in } R^n \times (0, \infty), \\ U|_{t=0} = U_0, & \text{in } R^n, \end{cases} \quad (4.19)$$

where $H_1(\mathcal{U}) = (0, 0, (\beta - \alpha)(b_1^2 + b_2^2 + b_3^2)u_3)$,

$$\tilde{A} = \begin{bmatrix} \frac{\sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}}{|\tilde{H}|} & -\frac{\tilde{h}_1\tilde{h}_2}{|\tilde{H}|\sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} & -\frac{\tilde{h}_1\tilde{h}_3}{|\tilde{H}|\sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} \\ 0 & \frac{\tilde{h}_3}{\sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} & -\frac{\tilde{h}_2}{\sqrt{\tilde{h}_2^2 + \tilde{h}_3^2}} \\ \frac{\tilde{h}_1}{|\tilde{H}|} & \frac{\tilde{h}_2}{|\tilde{H}|} & \frac{\tilde{h}_3}{|\tilde{H}|} \end{bmatrix}. \quad (4.20)$$

If

$$\left\{ \beta = \frac{\alpha(-1 + b_3^2)}{b_3^2}, \quad b_1 = b_2 = b_3 \right\} \quad \text{or} \quad \{ \alpha = 0, \quad b_1 = -b_2 - b_3 \}, \quad (4.21)$$

then V is a solution of $(\mathbb{Q}_1)$ if and only if $\mathcal{U}$ is a solution of $(\mathbb{Q}_2)$.

Proof. According to Lemma 4.3, to prove Proposition 4.4, we just need to set $\mu(t)H = 0$ and $H(V) = \sum_{i,j=1}^3 b_{ij}u_i(b_1, b_2, b_3) = 0$ in (4.19).

$H(V) = 0$ if and only if

$$\begin{aligned} 3\alpha + (\beta - \alpha)(b_1^2 + b_1b_2 + b_1b_3) &= 0, \\ 3\alpha + (\beta - \alpha)(b_1b_2 + b_2^2 + b_2b_3) &= 0, \\ 3\alpha + (\beta - \alpha)(b_1b_3 + b_2b_3 + b_3^2) &= 0. \end{aligned} \quad (4.22)$$

Solving (4.22), we have $\{\beta = \alpha(-1 + b_3^2)/b_3^2, b_1 = b_2 = b_3\}$ or $\{\alpha = 0, b_1 = -b_2 - b_3\}$.

On the other hand, if $\mu(t) = 0$, $\tilde{A}$ will degenerate as $\tilde{A}$. □

Remark 4.5. According to Proposition 4.4, we can construct a smooth solution about (2.1) on the disc from (4.17), (4.21), and (4.4)-(4.5).

5. Discussions

In Sections 2, and 4, we construct some solutions in which $|V|$ is not a constant. In the original physical model [1, 2] about LLE, LLE is under the constrain of $V \in S^2$ (here, we means that $|V| = C > 0$). However, many papers [10–13] do some investigations about the case that $|V|$ is not a constant. If $|V|$ is not a constant, is (1.4) still a physical significance model? To answer

this question, we firstly mention another LLE. Exactly speaking, under radially symmetric coordinates, $(2 + 1)$ -dimensional inhomogeneous LLE [14] takes the following form:

$$U_t = q(r)U \times \left[U_{rr} + \frac{1}{r}U_r \right] + q(r)_r[U \times U_r], \quad (5.1)$$

where $q(r)$ is the inhomogeneous term.

Suppose $V = (v_1(t, r), v_2(t, r), v_3(t, r))$ is a solution of (2.1), $Z = (z_1(t, r), z_2(t, r), z_3(t, r))$, $z_1^2(t, r) + z_2^2(t, r) + z_3^2(t, r) = C > 0$. Furthermore, the relationship between V and Z is in the form of

$$\begin{aligned} v_1(t, r) &= z_1(t, r)G(\alpha r), \\ v_2(t, r) &= z_2(t, r)G(\alpha r), \\ v_3(t, r) &= z_3(t, r)G(\alpha r), \end{aligned} \quad (5.2)$$

where $G(\alpha r)$ is a functions, α is constant.

Substituting (5.2) into (2.1), we get

$$Z_t = G(\alpha r)Z \times \left[Z_{rr} + \frac{1}{r}Z_r \right] + 2\alpha G(\alpha r)'[Z \times Z_r], \quad (5.3)$$

where “ r ” is a derivative about αr .

Furthermore, suppose $\alpha = 1/2$, (5.3) can be rearranged to give (for simplicity, we write G for $G(\cdot)$ here.

$$Z_t = GZ \times \left[Z_{rr} + \frac{1}{r}Z_r \right] + G'[Z \times Z_r]. \quad (5.4)$$

So we can regard (5.4) as a modification version of (5.1). According to the above deducing, the following theorem holds

Theorem 5.1. *Under the constrain (5.2), $V = (v_1(t, r), v_2(t, r), v_3(t, r))$ is a solution of (2.1) if and only if $Z = (z_1(t, r), z_2(t, r), z_3(t, r))$ ($G(\cdot) = G(r/2)$) is a solution of (5.4) where $z_1^2(t, r) + z_2^2(t, r) + z_3^2(t, r) = C > 0$.*

One mentions here that if $\alpha = 1$, $G(\alpha r) = G(r)$, (5.3) will adopt the following form:

$$U_t = G(r)Z \times \left[Z_{rr} + \frac{1}{r}Z_r \right] + 2G(r)_r[U \times U_r]. \quad (5.5)$$

So, (5.5) will not equal to (5.1) because of the coefficient 2 before the $G(r)_r[Z \times Z_r]$. And one cannot deduce a the solution of (5.1) from (2.1) according to Theorem 5.1. One has to choose another way to construct the exact solution of (5.1) which is now beyond the topic of this paper.

6. Geometric Significance

In this section, we investigate the evolution of solutions and the spacial gradient about them. Five different kinds of solutions are demonstrated in the following notations.

- (1) Figure 1 display part shape of solutions on solution (2.2)–(2.4) where $r \in [1, 3]$, $t \in [0, 9/10]$, $T = 1$, $\gamma = 1$, $C_1 = 1$, $C_2 = 0$, $C_3 = 1$ and $C_4 = -1$. Accordingly, from Figure 2, we can descry the blow up behavior of (2.13) near $T = 1$. Different from Figure 1, here we set $r \in [3/5, 10]$ in Figure 2.
- (2) For vortex solution (3.1) and (3.4), as shown in Figure 3, we can see the decaying behavior of square spatial curvature (3.23). Here, we set $C = 4$ and $\lambda = 0, 1, 2, 3, 4$, respectively. In Figure 3, begin from $\lambda = 1$, we can see that the shape of (3.23) has only a declining shift of the peak when λ increases.
- (3) From Figures 4-5 we can apperceive the periodic behavior of (3.24) which is determined by the periodic function:

$$Q(x, y) = 4 \arctan \left(sd \left(x + 1, \frac{\sqrt{3}}{2} \right) ds \left(y + 1, \frac{\sqrt{3}}{2} \right) \right), \quad (6.1)$$

$$Q(x, y) = 4 \arctan \left(\frac{\cot(1) \cosh \left(\left(\sqrt{2}/2 \right) \cos(1)(x - y) \right)}{\cosh \left(\left(\sqrt{2}/2 \right) \sin(1)(y - x) \right)} \right), \quad (6.2)$$

respectively. A singularity behavior of (3.24) can be seen in Figure 4.

- (4) Considering the solution (4.1), and setting $C_1 = 1$, $C_2 = 1$ and $C_3 = 1$, we draw the picture of $|U_r|^2 = (1/9)(9r^{8/3} + 8/r^{10/3})$ where $r \in [1/2, 25]$ in Figure 6. From Figure 6, we can find out that the energy cannot be bounded near $t = 0$. Setting $t \in [0, 3]$ and $r \in [1/10, 3]$, we draw the picture of (4.1) in Figure 7. Amazedly, we point out here that (2.2)–(2.4) will perform a blow-up behavior while (4.1) will not under the infinite energy initial condition.

7. Conclusions

In the present work, we have obtained the exact solutions of the LLE by using some ansatz about the solutions. By using the functional separable technique and general Jacobi elliptic-function method, we have obtained the blow-up solution, vortex solution, various periodic solutions, and non-blow-up solutions in the radially symmetric coordinates or not. For a better understanding about the property of the solutions, some solutions and their spatial curvature which cover five different types of solutions have been also illustrated in a graphic way. From Sections 2 and 4, we can see that the solution of Section 2 will perform a blow-up behavior while the solutions in Section 4 will not under the infinite energy initial condition. Similarly, from Section 3, we can find out the same conclusion that the vortex solution will preserve the finite energy while the corresponding periodic solutions will not.

Figure 7: Part shape of (4.1) as $t \in [0, 3]$ and $r \in [1/10, 3]$.

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