Research Article

Uniqueness of the Infinite Component for Percolation on a Hierarchical Lattice

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We study a long-range percolation in the hierarchical lattice Ω_N of order N where probability of connection between two nodes separated by distance k is of the form $\min\{\alpha\beta^{-k}, 1\}$, $\alpha \geq 0$ and $\beta > 0$. We show the uniqueness of the infinite component for this model.

1. Introduction

Percolation theory in the Euclidean lattice $\mathbb{Z}^d$ started with the work of Broadbent and Hammersley in 1957. The infinity of the space of sites (or vertices) and its geometry are principal features of this model, see for example [1, 2]. Some questions of percolation in other non-Euclidean infinite systems are formulated in [3]. The study of long-range percolation on $\mathbb{Z}^d$ traces back to [4] and leads to a range of interesting results in probability theory and statistical physics [5–9]. On the other hand, hierarchical structures have been used in applications in the physics, genetics, and social sciences thanks to the multiscale organization of many natural objects [10–13].

Recently, long-range percolation is studied on the hierarchical lattice Ω_N of order N (to be defined below), where classical methods for the usual lattice break down. The asymptotic long-range percolation on Ω_N is addressed in [14] for $N \rightarrow \infty$. The works [15–17], analyze the phase transition of long-range percolation on Ω_N for finite N using different connection probabilities and methodologies. The contact process on Ω_N for fixed N has been investigated in [18]. In this paper, we investigate the question of uniqueness of infinite component in percolation on Ω_N for fixed N . The form of the connection probabilities used here follows from a prior work [17].

Figure 1: An illustration of hierarchical lattice Ω_2 of order 2. The distances between three nodes $\mathbf{0} = (0, 0, 0, \dots)$, $\mathbf{x} = (1, 0, 0, \dots)$ and $\mathbf{y} = (0, 1, 0, \dots)$ are $d(\mathbf{0}, \mathbf{x}) = 1$ and $d(\mathbf{0}, \mathbf{y}) = d(\mathbf{x}, \mathbf{y}) = 2$.

For an integer $N \geq 2$, we define the set

$$\Omega_N := \{\mathbf{x} = (x_1, x_2, \dots) : x_i \in \{0, 1, \dots, N-1\}, i = 1, 2, \dots, x_i \neq 0 \text{ only for finitely many } i\}, \quad (1.1)$$

and define a metric d on it:

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \mathbf{x} = \mathbf{y}, \\ \max\{i : x_i \neq y_i\}, & \mathbf{x} \neq \mathbf{y}. \end{cases} \quad (1.2)$$

The pair (Ω_N, d) is called the hierarchical lattice of order N , which may be thought of as the set of leaves at the bottom of an infinite regular tree without a root, where the distance between two nodes is the number of levels (generations) from the bottom to their most recent common ancestor. Figure 1 shows the lattice Ω_2 along with its metric generating tree.

Such a distance d satisfies the strong triangle inequality

$$d(\mathbf{x}, \mathbf{y}) \leq \max\{d(\mathbf{x}, \mathbf{z}), d(\mathbf{z}, \mathbf{y})\}, \quad (1.3)$$

for any triple $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Omega_N$. Hence, (Ω_N, d) is an ultrametric (or non-Archimedean) space [19]. From its ultrametricity, it is clear that for every $\mathbf{x} \in \Omega_N$ that there are $(N-1)N^{k-1}$ nodes at distance k from it.

Now consider a long-range percolation on Ω_N . For each $k \geq 1$, the probability of connection between $\mathbf{x}$ and $\mathbf{y}$ such that $d(\mathbf{x}, \mathbf{y}) = k$ is given by

$$p_k = \min\left\{\frac{\alpha}{\beta^k}, 1\right\}, \quad (1.4)$$

where $0 \leq \alpha < \infty$ and $0 < \beta < \infty$, all connections being independent. Two vertices $\mathbf{x}, \mathbf{y} \in \Omega_N$ are in the same component if there exists a finite sequence $\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n = \mathbf{y}$ of vertices such that each pair $(\mathbf{x}_{i-1}, \mathbf{x}_i)$, $i = 1, \dots, n$, of vertices presents an edge.

The rest of the paper is organized as follows. In Section 2, we provide the uniqueness result and Section 3 is devoted to the proof.

2. Main results

Let $\mathbb{N}$ be the nonnegative integers including 0, and denote by $\ell := \min\{k \in \mathbb{N} : \alpha \leq \beta^{k+1}\}$. Let $|S|$ be the size of a set S . The connected component containing the node $\mathbf{x} \in \Omega_N$ is denoted by $C(\mathbf{x})$. Since, for every node $\mathbf{x}$, $|C(\mathbf{x})|$ has the same distribution, it suffices to consider only $|C(\mathbf{0})|$. The percolation probability is defined as

$$\theta(\alpha, \beta) := P(|C(\mathbf{0})| = \infty), \quad (2.1)$$

and the critical percolation value is defined as

$$\alpha_c(\beta) := \inf\{\alpha \geq 0 : \theta(\alpha, \beta) > 0\}. \quad (2.2)$$

The following theorem characterizes the phase transition for this model.

Theorem 2.1 (see [17]). (i) If $\beta \leq N$, then $\alpha_c(\beta) = 0$.

(ii) If $N < \beta < N^2$, then $0 < \alpha_c(\beta) < \infty$.

(iii) If $\beta \geq N^2$, then $\alpha_c(\beta) = \infty$.

The uniqueness of infinite component is established in the following result.

Theorem 2.2. For $0 \leq \alpha < \infty$ and $0 < \beta < \infty$, there is at most one infinite component almost surely.

3. Proof of Theorem 2.2

For any node $\mathbf{x} \in \Omega_N$, define $B_r(\mathbf{x})$ the ball of radius r around $\mathbf{x}$, that is, $B_r(\mathbf{x}) = \{\mathbf{y} : d(\mathbf{x}, \mathbf{y}) \leq r\}$. From this definition, we make the following observations. Firstly, for any $\mathbf{x} \in \Omega_N$, $B_r(\mathbf{x})$ contains N^r vertices. Secondly, $B_r(\mathbf{x}) = B_r(\mathbf{y})$ if $d(\mathbf{x}, \mathbf{y}) \leq r$. Finally, for any $\mathbf{x}, \mathbf{y}$, and r , we either have $B_r(\mathbf{x}) = B_r(\mathbf{y})$ or $B_r(\mathbf{x}) \cap B_r(\mathbf{y}) = \emptyset$.

The proof of Theorem 2.2 follows the idea in [16, Theorem 1.2] and is based on several lemmas.

Lemma 3.1 (see [20]). Consider long range percolation on $\mathbb{Z}^d$ with the properties

(i) the model is translation-invariant

(ii) the model satisfies the positive finite energy condition.

Then there can be at most one infinite component almost surely.

Lemma 3.2. The original metric generating tree (as shown in Figure 1) can be embedded into $\mathbb{Z}$ in a stationary way.

Figure 2: A realization of a new metric generating tree for $N = 2$. $B_1(0)$ corresponds to $\{0, 1\}$, $B_2(0)$ corresponds to $\{-2, -1, 0, 1\}$, and $B_3(0)$ corresponds to $\{-2, -1, 0, 1, 2, 3, 4, 5\}$.

Proof. We will prove this lemma in two steps.

- (i) Construct a new metric generating tree, which is isomorphic to the original metric generating tree.
- (ii) The new metric generating tree is stationary on $\mathbb{Z}$.

To show step (i), we first describe the construction roughly and then provide the formal construction. The new metric generating tree embeds into $\mathbb{Z}$ in such a way that for every $r \in \mathbb{N}$, (a) any ball of radius r will be represented by N^r consecutive integers and (b) the collection of balls of radius r partitions $\mathbb{Z}$.

We choose $B_1(0)$ uniformly at random among all N possible collections of N consecutive integers containing the origin 0 in $\mathbb{Z}$. Given the choice of $B_1(0)$, all other balls of radius 1 can be defined, although not specified at this point, by the above criteria (a) and (b). Next, the ball $B_2(0)$ is a union of N balls of radius 1 and contains $B_1(0)$. Since any ball of radius 2 is a collection of N^2 consecutive integers, there are N possible ways to achieve this. We choose one of the N possible ways to do this with probability $1/N$ each. Once we have chosen $B_2(0)$, all other balls of radius 2 are determined for the same reason as above. We continue this procedure to obtain the new metric generating tree, which is embedded in $\mathbb{Z}$. To get a picture of this, we illustrate in Figure 2 a possible implementation for $N = 2$.

Now we formalize the above construction. We choose the probability space as the unit interval $[0, 1]$ with Borel sigma field and Lebesgue measure. For $\eta \in [0, 1]$, denote by $\eta = 0 \cdot \eta_1 \eta_2 \dots$ the N -adic expansion. In other words,

$$\eta = \sum_{k=1}^{\infty} \eta_k N^{-k}, \quad (3.1)$$

where we assume that the expansion for η is unique without loss of generality. In the above construction, for each r , $B_{r-1}(0)$ is one of the balls of radius $r - 1$ among the balls making up $B_r(0)$. The new metric generating tree corresponding to η is obtained as follows. We let $B_r(0)$ be such that $B_{r-1}(0)$ is the $(\eta_r + 1)$ -st ball in $B_r(0)$ from left to right. In Figure 2, we can see, for example, $\eta_1 = \eta_3 = 0$ and $\eta_2 = 1$. It is clear that this construction formalize the informal description given earlier. By first identifying the $\mathbf{0}$ in Figure 1 and $\mathbf{0}$ in Figure 2, and then building up the balls $B_r(0)$ for $r = 1, 2, \dots$, in that order, we can see that the new metric generating tree is isomorphic to the original one.

Next, we move to step (ii). Let f be the map that assigns to each η a new metric generating tree as before. The map f is invertible on a set of complete Lebesgue measure. Denote by l the left-shift transformation, which translates the edges over one unit to the left

Figure 3: The commutative diagram.

on the new metric generating trees. Let the transformation $g : [0, 1] \rightarrow [0, 1]$ correspond to the left-shift transformation l on the space of new metric generating trees in the sense that $f \circ g = l \circ f$ (see Figure 3), hence $l = f g f^{-1}$. Let $A(\eta) := \min\{k : \eta_k \neq N - 1\}$, and then we can see that the i th digit in $g(\eta)$, $g(\eta)_i$, is given by

$$g(\eta)_i = \begin{cases} 0, & i < A(\eta), \\ \eta_i + 1, & i = A(\eta), \\ \eta_i, & i > A(\eta). \end{cases} \quad (3.2)$$

Furthermore, Lebesgue measure is invariant under the action of g , which implies that the construction of new random metric generating tree is stationary on $\mathbb{Z}$. $\square$

Now we are at the stage to prove Theorem 2.2.

Proof of Theorem 2.2. We assign a uniformly-[0, 1] distributed random variable δ_e to each edge e in such a way that the collection is independent. Given a new metric generating tree, an edge e is said to be open if $\delta_e \leq \min\{\alpha\beta^{-|e|}, 1\}$, where $|e|$ denotes the length of e . This gives a realization of the percolation process with the correct distribution and shows that the whole long-rang percolation process on the hierarchical lattice can be embedded as a stationary percolation process on $\mathbb{Z}$ involving Lemma 3.2.

For any $0 < \alpha < \infty$ and $0 < \beta < \infty$, every pair of vertices are connected by an edge with positive probability, irrespective of the presence or absence of other edges. Therefore, the positive finite energy condition is satisfied. The result then follows from Lemma 3.1. As for $\alpha = 0$, the result is immediate. $\square$

To conclude the paper, we mention that the uniqueness of infinite component has also been proved in [15, 16] for connection probabilities $p_k = c_k N^{-k(1+\delta)}$ with $\delta > -1$ and $p_k = 1 - \exp(-\beta^{-k}\alpha)$, respectively.

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