Research Article

Multiple Periodic Solutions to a Kind of Lotka-Volterra Food-Chain System with Delays and Impulses on Time Scales

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By using Mawhin's continuation theorem of coincidence degree theory and some skills of inequalities, we establish the existence of at least 2^n periodic solutions for a kind of n -species Lotka-Volterra food-chain system with delays and impulses on time scales. One example is given to illustrate the effectiveness of our results.

1. Introduction

The food-chain phenomenon is universal and interesting in ecosystem. It is one of important methods to investigate this phenomenon by establishing the model of differential equations. The importance of studying the dynamics of food chain was pointed by Rosenzweig. In his famous paper on the paradox of enrichment [1], he wrote "Man must be very careful in attempting to enrich an ecosystem in order to increase its food yield. There is a real chance that such activity may result in decimation of the food species that are wanted in greater abundance." Hereafter, many scholars investigated various kinds of food-chain systems (see [2-9]).

In [10], Li and Zhao have studied the existence of multiple positive periodic solutions for the following Lotka-Volterra food-chain system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)(a_1(t) - b_1(t)x_1(t) - c_{12}(t)x_2(t)) - h_1(t), \\ &\vdots\end{aligned}$$

$$\begin{aligned}
\dot{x}_i(t) &= x_i(t)(-d_i(t) - b_i(t)x_i(t) + c_{i,i-1}(t)x_{i-1}(t) - c_{i,i+1}(t)x_{i+1}(t)) \\
&\quad - h_i(t), \quad 2 \leq i \leq n-1, \\
&\vdots \\
\dot{x}_n(t) &= x_n(t)(-d_n(t) - b_n(t)x_n(t) + c_{n,n-1}(t)x_{n-1}(t)) - h_n(t).
\end{aligned} \tag{1.1}$$

As we know, in population dynamics, many evolutionary processes experience short-time rapid change after undergoing relatively long sooth variation. Examples include stocking of species and annual immigration. Incorporating these phenomena give us impulsive differential equations. For the theory of impulsive differential equations, we refer the reader to [11, 12]. In addition, there are few results on the existence of multiple periodic solutions for the delay food-chain system with impulsive effects in literatures. This motivates us to modify system (1.1) to the form

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)(a_1(t) - b_1(t)x_1(t - \tau_{11}(t)) - c_{12}(t)x_2(t - \tau_{12}(t))) - h_1(t), \quad t \neq t_k, \\
&\vdots \\
\dot{x}_i(t) &= x_i(t)(-d_i(t) - b_i(t)x_i(t - \tau_{ii}(t)) + c_{i,i-1}(t)x_{i-1}(t - \tau_{i,i-1}(t)) - c_{i,i+1}(t)x_{i+1}(t - \tau_{i,i+1}(t))) \\
&\quad - h_i(t), \quad 2 \leq i \leq n-1, \quad t \neq t_k, \\
&\vdots \\
\dot{x}_n(t) &= x_n(t)(-d_n(t) - b_n(t)x_n(t - \tau_{nn}(t)) + c_{n,n-1}(t)x_{n-1}(t - \tau_{n,n-1}(t))) - h_n(t), \quad t \neq t_k, \\
\Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) = g_{ik}x_i(t_k), \quad 1 \leq i \leq n,
\end{aligned} \tag{1.2}$$

where $g_{ik} \in (-1, +\infty)$ ($i = 1, 2, \dots, n$; $k \in \mathbb{N} = \{1, 2, \dots, n\}$). $\{t_k\}_{k \in \mathbb{N}}$ is a strictly increasing sequence with $t_1 > 0$ and $\lim_{t \rightarrow \infty} t_k = \infty$. $x_i(t)$ ($i = 1, 2, \dots, n$) is the i th species population density. $a_1(t) > 0$ is the growth rate of the only producer. $b_i(t) > 0$ ($i = 1, 2, \dots, n$) and $h_i(t) > 0$ ($i = 1, 2, \dots, n$) stand for the i th species intraspecific competition rate and harvesting rate, respectively. $d_i(t) > 0$ ($i = 2, 3, \dots, n$) is the i th species death rate. $c_{i,i+1}(t) > 0$ ($i = 1, 2, \dots, n-1$) represents the $(i+1)$ th species predation rate on the i th species. $c_{i,i-1}(t) > 0$ ($i = 2, 3, \dots, n$) stands for the transformation rate from the $(i-1)$ th species to the i th species. $\tau_{i,i+1}(t) \geq 0$ ($i = 1, 2, \dots, n-1$) stands for the time-lag in the process of the $(i+1)$ th species predation rate on the i th species. $\tau_{i,i-1}(t) \geq 0$ ($i = 2, 3, \dots, n$) stands for the time-lag in the process of transformation from the $(i-1)$ th species to the i th species. $\tau_{ii}(t) \geq 0$ ($i = 1, 2, \dots, n$) represents the time-lag in the process of intraspecific competition. In addition, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Therefore, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.), which leads us to assume that $a_i(t)$, $b_i(t)$, $c_{ij}(t)$, $\tau_{ij}(t)$, and $h_i(t)$ ($i, j = 1, 2, \dots, n$) are all

continuous ω -periodic functions. For impulsive effects, we further assume that there exists a $q \in \mathbb{N}$ such that $g_{i(k+q)} = g_{ik}$ ($i = 1, 2, \dots, n$) and $t_{k+q} = t_k + \omega$.

However, dynamics in each equally spaced time interval may vary continuously. So, it may be more realistic to assume that the population dynamics involves the hybrid discrete-continuous processes. For example, Gamarra and Solé pointed out that such hybrid processes appear in the population dynamics of certain species that feature nonoverlapping generations: the change in population from one generation to the next is discrete and so is modeled by a difference equation, while within-generation dynamics vary continuously (due to mortality rates, resource consumption, predation, interaction, etc.) and thus are described by a differential equation [13]. The theory of calculus on time scales (see [14, 15] and references cited therein) was initiated by Hilger in his Ph.D. thesis in 1988 [16] in order to unify continuous and discrete analysis, and it has become an effective approach to the study of mathematical models involving the hybrid discrete-continuous processes. This motivates us to consider Lotka-Volterra food-chain system with delays and impulses on time scales $\mathbb{T}$ of the following form:

$$\begin{aligned}
 u_1^\Delta(t) &= a_1(t) - b_1(t)e^{u_1(t-\tau_{11}(t))} - c_{12}(t)e^{u_2(t-\tau_{12}(t))} - h_1(t)e^{-u_1(t)}, \quad t \in \mathbb{T}, \quad t \neq t_k, \\
 &\vdots \\
 u_i^\Delta(t) &= -d_i(t) - b_i(t)e^{u_i(t-\tau_{ii}(t))} + c_{i,i-1}(t)e^{u_{i-1}(t-\tau_{i,i-1}(t))} - c_{i,i+1}(t)e^{u_{i+1}(t-\tau_{i,i+1}(t))} \\
 &\quad - h_i(t)e^{-u_i(t)}, \quad 2 \leq i \leq n-1, \quad t \in \mathbb{T}, \quad t \neq t_k, \\
 &\vdots \\
 u_n^\Delta(t) &= -d_n(t) - b_n(t)e^{u_n(t-\tau_{nn}(t))} + c_{n,n-1}(t)e^{u_{n-1}(t-\tau_{n,n-1}(t))} - h_n(t)e^{-u_n(t)}, \quad t \in \mathbb{T}, \quad t \neq t_k, \\
 \Delta u_i(t_k) &= u_i(t_k^+) - u_i(t_k^-) = \ln(1 + g_{ik}), \quad 1 \leq i \leq n,
 \end{aligned} \tag{1.3}$$

where $\mathbb{T}$ is a ω -periodic time scale. $g_{ik} \in (-1, +\infty)$ ($i = 1, 2, \dots, n$; $k \in \mathbb{N} = \{1, 2, \dots, n\}$) is a constant. $\{t_k\}_{k \in \mathbb{N}}$ is a strictly increasing sequence with $t_1 > 0$ and $\lim_{k \rightarrow \infty} t_k = \infty$. $a_1(t) > 0$, $b_i(t) > 0$, $h_i(t) > 0$, $d_i(t) > 0$, $c_{i,i+1}(t) > 0$, $c_{i,i-1}(t) > 0$, $\tau_{i,i+1}(t)$, $\tau_{i,i-1}(t)$, $\tau_{ii}(t) \geq 0$ are rd-continuous ω -periodic functions. For impulsive effects, we further assume that there exists a integer $q \in \mathbb{N}$ such that $g_{i(k+q)} = g_{ik}$ ($i = 1, 2, \dots, n$) and $t_{k+q} = t_k + \omega$. We denote by $\kappa = \min\{[0, +\infty) \cap \mathbb{T}\}$, $I_\omega = [\kappa, \kappa + \omega] \cap \mathbb{T}$. Without loss of generality, we also assume that $I_\omega \cap \{t_k : k \in \mathbb{N}\} = \{t_1, t_2, \dots, t_q\}$.

Remark 1.1. In (1.3), let $x_i(t) = e^{u_i(t)}$, $i = 1, 2, \dots, n$. If $\mathbb{T} = \mathbb{R}$ (the set of all real numbers), then (1.3) reduces to (1.2).

To the best of our knowledge, few papers have been published on the existence of multiple periodic solutions for this model. Our main purpose of this paper is by using Mawhin's continuation theorem of coincidence degree theory [17], to establish the existence of at least 2^n periodic solutions for system (1.3). For the work concerning the multiple existence of periodic solutions of periodic population models which was done using coincidence degree theory, we refer to [18–25].

The organization of the rest of this paper is as follows. In the Section 2, we will introduce some basic notations and lemmas which are used in what follows. In Section 3, by employing the continuation theorem of coincidence degree theory and the skills of inequalities, we establish the existence of at least 2^n periodic solutions of system (1.3). In the final Section, one example is given to illustrate the effectiveness of our results.

2. Preliminaries on Time Scales

In this section, we briefly recall some basic definitions and lemmas on time scales which are used in what follows. For more details, one can see [14–16].

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ and the graininess $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t. \quad (2.1)$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m , then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

Let $\omega > 0$. Throughout this paper, the time scale $\mathbb{T}$ is assumed to be ω -periodic, that is, $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$. In particular, the time scale $\mathbb{T}$ under consideration is unbounded above and below.

Definition 2.1. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called regulated provided its right-side limits exist (finite) at all right-side points in $\mathbb{T}$ and its left-side limits exist (finite) at all left-side points in $\mathbb{T}$.

Definition 2.2. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by $C_{\text{rd}} = C_{\text{rd}}(\mathbb{T}) = C_{\text{rd}}(\mathbb{T}, \mathbb{R})$.

Definition 2.3. Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$. Then we define $f^\Delta(t)$ to be the number (if it exists) with the property that given any $\varepsilon > 0$ there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] \right| < \varepsilon |\sigma(t) - s| \quad (2.2)$$

for all $s \in U$. we call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t . The set of functions $f : \mathbb{T} \rightarrow \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C_{\text{rd}}^1 = C_{\text{rd}}^1(\mathbb{T}) = C_{\text{rd}}^1(\mathbb{T}, \mathbb{R})$.

If f is continuous, then f is rd-continuous. If f is rd-continuous, the f is regulated. If f is delta differentiable at t , then f is continuous at t .

Lemma 2.4. *Let f be regulated, then there exists a function F which is delta differentiable with region of differentiation D such that*

$$F^\Delta(t) = f(t) \quad \forall t \in D. \quad (2.3)$$

Definition 2.5. Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a regulated function. Any function F as in Lemma 2.4 is called a Δ -antiderivative of f . We define the indefinite integral of a regulated function f by

$$\int f(t) \Delta t = F(t) + C, \quad (2.4)$$

where C is an arbitrary constant and F is a Δ -antiderivative of f . We define the Cauchy integral by

$$\int_a^b f(s) \Delta s = F(b) - F(a) \quad \forall a, b \in \mathbb{T}. \quad (2.5)$$

A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided

$$F^\Delta(t) = f(t) \quad \forall t \in \mathbb{T}^k. \quad (2.6)$$

Lemma 2.6. *If $a, b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $f, g \in C(\mathbb{T}, \mathbb{R})$, then*

- (i) $\int_a^b [\alpha f(t) + \beta g(t)] \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t$;
- (ii) if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t) \Delta t \geq 0$;
- (iii) if $|f(t)| \leq g(t)$ on $[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$, then $|\int_a^b f(t) \Delta t| \leq \int_a^b g(t) \Delta t$.

For convenience, we now introduce some notation to be used throughout this paper. Let

$$\begin{aligned} \kappa &= \min\{[0, +\infty) \cap \mathbb{T}\}, & I_\omega &= [\kappa, \kappa + \omega] \cap \mathbb{T}, & g^M &= \sup_{t \in I_\omega} g(t), \\ g^l &= \inf_{t \in I_\omega} g(t), & \bar{g} &= \frac{1}{\omega} \int_{I_\omega} g(s) \Delta s = \frac{1}{\omega} \int_\kappa^{\kappa+\omega} g(s) \Delta s, \end{aligned} \quad (2.7)$$

where $g \in C_{\text{rd}}(\mathbb{T})$ is an ω -periodic real function, that is, $g(t + \omega) = g(t)$ for all $t \in \mathbb{T}$.

Lemma 2.7 (see [26]). *Assume that $\{f_n\}_{n \in \mathbb{N}}$ is a function on J such that*

- (i) $\{f_n\}_{n \in \mathbb{N}}$ is uniformly bounded on J ,
- (ii) $\{f_n^\Delta\}_{n \in \mathbb{N}}$ is uniformly bounded on J .

Then there is a subsequence of $\{f_n\}_{n \in \mathbb{N}}$ which converges uniformly on J .

3. Existence of at Least 2^n Periodic Solutions

In this section, by using Mawhin's continuation theorem and the skills of inequalities, we will show the existence of periodic solutions of (1.3). To do so, we need to make some preparations.

Let X and Z be real normed vector spaces. Let $L : \text{Dom } L \subset X \rightarrow Z$ be a linear mapping and $N : X \times [0, 1] \rightarrow Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \text{Ker } L = \text{codim } \text{Im } L < \infty$ and $\text{Im } L$ is closed in Z . If L is a Fredholm mapping of index zero, then there exists continuous projectors $P : X \rightarrow X$ and $Q : Z \rightarrow Z$ such that $\text{Im } P = \text{Ker } L$ and $\text{Ker } Q = \text{Im } L = \text{Im}(I - Q)$, and $X = \text{Ker } L \oplus \text{Ker } P$, $Z = \text{Im } L \oplus \text{Im } Q$. It follows that $L|_{\text{Dom } L \cap \text{Ker } P} : (I - P)X \rightarrow \text{Im } L$ is invertible and its inverse is denoted by K_P . If Ω is a bounded open subset of X , the mapping N is called L -compact on $\overline{\Omega} \times [0, 1]$, if $QN(\overline{\Omega} \times [0, 1])$ is bounded and $K_P(I - Q)N : \overline{\Omega} \times [0, 1] \rightarrow X$ is compact. Because $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$.

The Mawhin's continuous theorem [17, page 40] is given as follows.

Lemma 3.1 (see [17]). *Let L be a Fredholm mapping of index zero and let N be L -compact on $\overline{\Omega} \times [0, 1]$. Assume*

- (a) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda N(x, \lambda)$ is such that $x \notin \partial\Omega \cap \text{Dom } L$;*
- (b) *$QN(x, 0)x \neq 0$ for each $x \in \partial\Omega \cap \text{Ker } L$;*
- (c) *$\deg(JQN(x, 0), \Omega \cap \text{Ker } L, 0) \neq 0$.*

Then $Lx = N(x, 1)$ has at least one solution in $\overline{\Omega} \cap \text{Dom } L$.

For simplicity, we need to introduce some notations as follows:

$$\begin{aligned}
 g_i &= \frac{1}{2} \sum_{k=1}^q \ln(1 + g_{ik}) \quad (1 \leq i \leq n), \quad G_i = \frac{1}{2} \sum_{k=1}^q [\ln(1 + g_{ik}) + |\ln(1 + g_{ik})|] \quad (1 \leq i \leq n), \\
 l_1^\pm &= \frac{(\bar{a}_1 + (2g_1/\omega))e^{\bar{a}_1\omega + G_1} \pm \sqrt{((\bar{a}_1 + (2g_1/\omega))e^{\bar{a}_1\omega + G_1})^2 - 4\bar{b}_1\bar{h}_1e^{\bar{a}_1\omega + G_1}}}{2\bar{b}_1}, \\
 A_1^\pm &= \frac{(\bar{a}_1 + (2g_1/\omega) - \bar{c}_{12}l_2^\pm) \pm \sqrt{(\bar{a}_1 + (2g_1/\omega) - \bar{c}_{12}l_2^\pm)^2 - 4\bar{b}_1\bar{h}_1e^{\bar{a}_1\omega + G_1}}}{2\bar{b}_1}, \\
 B_1^\pm &= \frac{(\bar{a}_1 + (2g_1/\omega) - \bar{c}_{12}l_2^\pm)e^{-(\bar{a}_1\omega + G_1)} \pm \sqrt{((\bar{a}_1 + (2g_1/\omega) - \bar{c}_{12}l_2^\pm)e^{-(\bar{a}_1\omega + G_1)})^2 - 4\bar{b}_1\bar{h}_1e^{-(\bar{a}_1\omega + G_1)}}}{2\bar{b}_1}, \\
 l_i^\pm &= \frac{(\bar{c}_{i,i-1} + (2g_i/\omega))e^{\bar{c}_{i,i-1}\omega + G_i} \pm \sqrt{((\bar{c}_{i,i-1} + (2g_i/\omega))e^{\bar{c}_{i,i-1}\omega + G_i})^2 - 4\bar{b}_i\bar{h}_ie^{\bar{c}_{i,i-1}\omega + G_i}}}{2\bar{b}_i} \\
 &\hspace{15em} (2 \leq i \leq n),
 \end{aligned}$$

$$\begin{aligned}
A_i^\pm &= \frac{\left((2g_i/\omega) - \bar{d}_i - \bar{c}_{i,i+1}l_{i+1}^+\right) \pm \sqrt{\left((2g_i/\omega) - \bar{d}_i - \bar{c}_{i,i+1}l_{i+1}^+\right)^2 - 4\bar{b}_i\bar{h}_ie^{\bar{c}_{i,i-1}\omega+G_i}}}{2\bar{b}_i} \\
&\quad (2 \leq i \leq n-1), \\
B_i^\pm &= \frac{1}{2\bar{b}_i} \left[\pm \sqrt{\left(\left(\frac{2g_i}{\omega} - \bar{d}_i - \bar{c}_{i,i+1}l_{i+1}^+\right)e^{-(\bar{c}_{i,i-1}\omega+G_i)}\right)^2 - 4\bar{b}_i\bar{h}_ie^{-(\bar{c}_{i,i-1}\omega+G_i)}} \right. \\
&\quad \left. + \left(\frac{2g_i}{\omega} - \bar{d}_i - \bar{c}_{i,i+1}l_{i+1}^+\right)e^{-(\bar{c}_{i,i-1}\omega+G_i)} \right] \quad (2 \leq i \leq n-1), \\
B_n^\pm &= \frac{\left((2g_n/\omega) - \bar{d}_n\right)e^{-(\bar{c}_{n,n-1}\omega+G_n)} \pm \sqrt{\left(\left((2g_n/\omega) - \bar{d}_n\right)e^{-(\bar{c}_{n,n-1}\omega+G_n)}\right)^2 - 4\bar{b}_n\bar{h}_ne^{-(\bar{c}_{n,n-1}\omega+G_n)}}}{2\bar{b}_n}, \\
A_n^\pm &= \frac{\left((2g_n/\omega) - \bar{d}_n\right) \pm \sqrt{\left((2g_n/\omega) - \bar{d}_n\right)^2 - 4\bar{b}_n\bar{h}_ne^{\bar{c}_{n,n-1}\omega+G_n}}}{2\bar{b}_n}, \\
H_1 &= l_1^- e^{-(\bar{a}_1\omega+G_1)}, \quad H_i = l_i^- e^{-(\bar{c}_{i,i-1}\omega+G_i)} \quad (2 \leq i \leq n).
\end{aligned} \tag{3.1}$$

Throughout this paper, we need the following assumptions:

$$\begin{aligned}
(H_1) \quad &(\bar{a}_1 + (2g_1/\omega) - \bar{c}_{12}l_2^+)e^{-(1/2)(\bar{a}_1\omega+G_1)} > 2\sqrt{\bar{b}_1\bar{h}_1}, \\
(H_2) \quad &((2g_i/\omega) - \bar{d}_i - \bar{c}_{i,i+1}l_{i+1}^+)e^{-(1/2)(\bar{c}_{i,i-1}\omega+G_i)} > 2\sqrt{\bar{b}_i\bar{h}_i} \quad (2 \leq i \leq n-1), \\
(H_3) \quad &((2g_n/\omega) - \bar{d}_n)e^{-(1/2)(\bar{c}_{n,n-1}\omega+G_n)} > 2\sqrt{\bar{b}_n\bar{h}_n}.
\end{aligned}$$

The following results will play an important role in the proof of our main result.

Lemma 3.2. *Let $\mathbb{T}$ be a ω -periodic time scale. $\{t_k\}_1^\infty \subset \mathbb{T}$ is a strictly increasing sequence with $t_1 > 0$ and $\lim_{k \rightarrow \infty} t_k = \infty$. $\kappa = \min\{[0, \infty) \cap \mathbb{T}\}$, $I_\omega = [\kappa, \kappa + \omega] \cap \mathbb{T}$, $I_\omega \cap \{t_k\}_{k=1}^\infty = \{t_1, t_2, \dots, t_q\}$. Suppose $\varphi : \mathbb{T} \rightarrow \mathbb{R}$ be a ω -periodic function which is rd-continuous for $t \neq t_k$, exists the right limit $\varphi(t_k^+)$ and left limit $\varphi(t_k^-) = \varphi(t_k)$ in the sense of time scales. Then*

$$0 \leq \sup_{s \in I_\omega} \varphi(s) - \inf_{s \in I_\omega} \varphi(s) \leq \frac{1}{2} \left[\int_{\kappa}^{\kappa+\omega} |\varphi^\Delta(s)| \Delta s + \sum_{k=1}^q |\Delta \varphi(t_k)| \right]. \tag{3.2}$$

Proof. First, take $t_* \in I_\omega$ such that $\psi(t_*) = \inf_{s \in I_\omega} \psi(s)$. Next, take $t^* \in [t_*, t_* + \omega] \cap \mathbb{T}$ such that $\psi(t^*) = \sup_{s \in I_\omega} \psi(s)$. Then we have

$$\begin{aligned}
 \psi(t^*) &= \psi(t_*) + (\psi(t^*) - \psi(t_*)) \\
 &= \psi(t_*) + \left(\psi(t^*) - \sum_{t_* \leq t_k < t^*} \Delta\psi(t_k) - \psi(t_*) \right) + \sum_{t_* \leq t_k < t^*} \Delta\psi(t_k) \\
 &= \psi(t_*) + \int_{t_*}^{t^*} \psi(s) \Delta s + \sum_{t_* \leq t_k < t^*} \Delta\psi(t_k) \\
 &\leq \psi(t_*) + \int_{t_*}^{t^*} |\psi(s)| \Delta s + \sum_{t_* \leq t_k < t^*} |\Delta\psi(t_k)|.
 \end{aligned} \tag{3.3}$$

Noticing that $\psi(t^*) = \psi(t^* - \omega) = \psi(t_*) + (\psi(t^*) - \psi(t_* - \omega))$, similar to the above, we can get

$$\psi(t^*) \leq \psi(t_*) + \int_{t^* - \omega}^{t_*} |\psi(s)| \Delta s + \sum_{t^* - \omega \leq t_k < t_*} |\Delta\psi(t_k)|. \tag{3.4}$$

From (3.3) and (3.4), we get

$$\begin{aligned}
 2\psi(t^*) &\leq 2\psi(t_*) + \int_{t_*}^{t^*} |\psi(s)| \Delta s + \int_{t^* - \omega}^{t_*} |\psi(s)| \Delta s + \sum_{t_* \leq t_k < t^*} |\Delta\psi(t_k)| + \sum_{t^* - \omega \leq t_k < t_*} |\Delta\psi(t_k)| \\
 &= 2\psi(t_*) + \int_0^\omega |\psi(s)| \Delta s + \sum_{0 \leq t_k < \omega} |\Delta\psi(t_k)|,
 \end{aligned} \tag{3.5}$$

that is

$$\psi(t^*) \leq \psi(t_*) + \frac{1}{2} \left[\int_\kappa^{\kappa + \omega} |\psi(s)| \Delta s + \sum_{k=1}^q |\Delta\psi(t_k)| \right], \tag{3.6}$$

which implies that (3.2) holds. This completes the proof. $\square$

Lemma 3.3. Let $x > 0$, $y > 0$, $z > 0$ and $x > 2\sqrt{yz}$, for the functions $f(x, y, z) = (x + \sqrt{x^2 - 4yz})/2z$ and $g(x, y, z) = (x - \sqrt{x^2 - 4yz})/2z$, the following assertions hold.

- (1) $f(x, y, z)$ and $g(x, y, z)$ are monotonically increasing and monotonically decreasing on the variable $x \in (0, \infty)$, respectively.
- (2) $f(x, y, z)$ and $g(x, y, z)$ are monotonically decreasing and monotonically increasing on the variable $y \in (0, \infty)$, respectively.
- (3) $f(x, y, z)$ and $g(x, y, z)$ are monotonically decreasing and monotonically increasing on the variable $z \in (0, \infty)$, respectively.

Proof. In fact, for all $x > 0, y > 0, z > 0$, we have

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{x + \sqrt{x^2 - 4yz}}{2z\sqrt{x^2 - 4yz}} > 0, & \frac{\partial g}{\partial x} &= \frac{\sqrt{x^2 - 4yz} - x}{2z\sqrt{x^2 - 4yz}} < 0, & \frac{\partial f}{\partial y} &= \frac{-1}{\sqrt{x^2 - 4yz}} < 0, \\ \frac{\partial g}{\partial y} &= \frac{1}{\sqrt{x^2 - 4yz}} > 0, & \frac{\partial f}{\partial z} &= \frac{-(x + \sqrt{x^2 - 4yz})^2}{4z^2\sqrt{x^2 - 4yz}} < 0, & \frac{\partial g}{\partial z} &= \frac{(x - \sqrt{x^2 - 4yz})^2}{4z^2\sqrt{x^2 - 4yz}} > 0. \end{aligned} \quad (3.7)$$

By the relationship of the derivative and the monotonicity, the above assertions obviously hold. The proof of Lemma 3.3 is complete. $\square$

Lemma 3.4. Assume that (H_1) , (H_2) , and (H_3) hold, then we have the following inequalities:

$$(1) \ln l_i^+ > \ln A_i^+ > \ln A_i^- > \ln l_i^- > \ln H_i \quad (1 \leq i \leq n);$$

$$(2) \ln l_i^+ > \ln B_i^+ > \ln B_i^- > \ln l_i^- > \ln H_i \quad (1 \leq i \leq n);$$

$$(3) \min\{\ln A_i^+, \ln B_i^+\} > \max\{\ln A_i^-, \ln B_i^-\} \quad (1 \leq i \leq n).$$

Proof. (1) Since

$$\begin{aligned} \left(\bar{a}_1 + \frac{2g_1}{\omega}\right)e^{\bar{a}_1\omega + G_1} &> \bar{a}_1 + \frac{2g_1}{\omega} - \bar{c}_{12}l_2^+ > \left(\bar{a}_1 + \frac{2g_1}{\omega} - \bar{c}_{12}l_2^+\right)e^{-(1/2)(\bar{a}_1\omega + G_1)} > 2\sqrt{\bar{b}_1\bar{h}_1} > 0, \\ \left(\bar{c}_{i,i-1} + \frac{2g_i}{\omega}\right)e^{\bar{c}_{i,i-1}\omega + G_i} &> \frac{2g_i}{\omega} - \bar{d}_i - \bar{c}_{i,i+1}l_{i+1}^+ > \left(\frac{2g_i}{\omega} - \bar{d}_i - \bar{c}_{i,i+1}l_{i+1}^+\right)e^{-(1/2)(\bar{c}_{i,i-1}\omega + G_i)} \\ &> 2\sqrt{\bar{b}_i\bar{h}_i} > 0, \\ \left(\bar{c}_{n,n-1} + \frac{2g_n}{\omega}\right)e^{\bar{c}_{n,n-1}\omega + G_n} &> \frac{2g_n}{\omega} - \bar{d}_n > \left(\frac{2g_n}{\omega} - \bar{d}_n\right)e^{-(1/2)(\bar{c}_{n,n-1}\omega + G_n)} > 2\sqrt{\bar{b}_n\bar{h}_n} > 0, \\ 0 &< \bar{b}_i, \quad 0 < \bar{h}_i, \quad i = 1, 2, \dots, n. \end{aligned} \quad (3.8)$$

Applying Lemma 3.3, we have

$$\begin{aligned}
0 < H_1 &= l_1^- e^{-(\bar{a}_1 \omega + G_1)} < l_1^- = g\left(\left(\bar{a}_1 + \frac{2g_1}{\omega}\right) e^{\bar{a}_1 \omega + G_1}, \bar{b}_1, \bar{h}_1 e^{\bar{a}_1 \omega + G_1}\right) \\
&< g\left(\bar{a}_1 + \frac{2g_1}{\omega} - \bar{c}_{12} l_2^+, \bar{b}_1, \bar{h}_1 e^{\bar{a}_1 \omega + G_1}\right) = A_1^- < A_1^+ = f\left(\bar{a}_1 + \frac{2g_1}{\omega} - \bar{c}_{12} l_2^+, \bar{b}_1, \bar{h}_1 e^{\bar{a}_1 \omega + G_1}\right) \\
&< f\left(\left(\bar{a}_1 + \frac{2g_1}{\omega}\right) e^{\bar{a}_1 \omega + G_1}, \bar{b}_1, \bar{h}_1 e^{\bar{a}_1 \omega + G_1}\right) = l_1^+, \\
0 < H_i &= l_i^- e^{-(\bar{c}_{i,i-1} \omega + G_i)} < l_i^- = g\left(\left(\bar{c}_{i,i-1} + \frac{2g_i}{\omega}\right) e^{\bar{c}_{i,i-1} \omega + G_i}, \bar{b}_i, \bar{h}_i e^{\bar{c}_{i,i-1} \omega + G_i}\right) \\
&< g\left(\frac{2g_i}{\omega} - \bar{d}_i - \bar{c}_{i,i+1} l_{i+1}^+, \bar{b}_i, \bar{h}_i e^{\bar{c}_{i,i-1} \omega + G_i}\right) = A_i^- \\
&< A_i^+ = f\left(\frac{2g_i}{\omega} - \bar{d}_i - \bar{c}_{i,i+1} l_{i+1}^+, \bar{b}_i, \bar{h}_i e^{\bar{c}_{i,i-1} \omega + G_i}\right) \\
&< f\left(\left(\bar{c}_{i,i-1} + \frac{2g_i}{\omega}\right) e^{\bar{c}_{i,i-1} \omega + G_i}, \bar{b}_i, \bar{h}_i e^{\bar{c}_{i,i-1} \omega + G_i}\right) = l_i^+, \quad i = 2, 3, \dots, n-1, \\
0 < H_n &= l_n^- e^{-(\bar{c}_{n,n-1} \omega + G_n)} < l_n^- = g\left(\left(\bar{c}_{n,n-1} + \frac{2g_n}{\omega}\right) e^{\bar{c}_{n,n-1} \omega + G_n}, \bar{b}_n, \bar{h}_n e^{\bar{c}_{n,n-1} \omega + G_n}\right) \\
&< g\left(\frac{2g_n}{\omega} - \bar{d}_n, \bar{b}_n, \bar{h}_n e^{\bar{c}_{n,n-1} \omega + G_n}\right) = A_n^- < A_n^+ = f\left(\frac{2g_n}{\omega} - \bar{d}_n, \bar{b}_n, \bar{h}_n e^{\bar{c}_{n,n-1} \omega + G_n}\right) \\
&< f\left(\left(\bar{c}_{n,n-1} + \frac{2g_n}{\omega}\right) e^{\bar{c}_{n,n-1} \omega + G_n}, \bar{b}_n, \bar{h}_n e^{\bar{c}_{n,n-1} \omega + G_n}\right) = l_n^+.
\end{aligned} \tag{3.9}$$

Thus, we have $\ln H_i < \ln l_i^- < \ln A_i^- < \ln A_i^+ < \ln l_i^+$, $i = 1, 2, \dots, n$. The proof of (1) of Lemma 3.4 is complete. For (2) of Lemma 3.4, we similarly have

$$\begin{aligned}
0 < H_1 &= l_1^- e^{-(\bar{a}_1 \omega + G_1)} < l_1^- = g\left(\left(\bar{a}_1 + \frac{2g_1}{\omega}\right) e^{\bar{a}_1 \omega + G_1}, \bar{b}_1 e^{\bar{a}_1 \omega + G_1}, \bar{h}_1\right) \\
&< g\left(\bar{a}_1 + \frac{2g_1}{\omega} - \bar{c}_{12} l_2^+, \bar{b}_1 e^{\bar{a}_1 \omega + G_1}, \bar{h}_1\right) = B_1^- < B_1^+ = f\left(\bar{a}_1 + \frac{2g_1}{\omega} - \bar{c}_{12} l_2^+, \bar{b}_1 e^{\bar{a}_1 \omega + G_1}, \bar{h}_1\right) \\
&< f\left(\left(\bar{a}_1 + \frac{2g_1}{\omega}\right) e^{\bar{a}_1 \omega + G_1}, \bar{b}_1 e^{\bar{a}_1 \omega + G_1}, \bar{h}_1\right) = l_1^+, \\
0 < H_i &= l_i^- e^{-(\bar{c}_{i,i-1} \omega + G_i)} < l_i^- = g\left(\left(\bar{c}_{i,i-1} + \frac{2g_i}{\omega}\right) e^{\bar{c}_{i,i-1} \omega + G_i}, \bar{b}_i e^{\bar{c}_{i,i-1} \omega + G_i}, \bar{h}_i\right) \\
&< g\left(\frac{2g_i}{\omega} - \bar{d}_i - \bar{c}_{i,i+1} l_{i+1}^+, \bar{b}_i e^{\bar{c}_{i,i-1} \omega + G_i}, \bar{h}_i\right) = B_i^- \\
&< B_i^+ = f\left(\frac{2g_i}{\omega} - \bar{d}_i - \bar{c}_{i,i+1} l_{i+1}^+, \bar{b}_i e^{\bar{c}_{i,i-1} \omega + G_i}, \bar{h}_i\right) \\
&< f\left(\left(\bar{c}_{i,i-1} + \frac{2g_i}{\omega}\right) e^{\bar{c}_{i,i-1} \omega + G_i}, \bar{b}_i e^{\bar{c}_{i,i-1} \omega + G_i}, \bar{h}_i\right) = l_i^+, \quad i = 2, 3, \dots, n-1,
\end{aligned}$$

$$\begin{aligned}
0 < H_n &= l_n^- e^{-(\bar{c}_{n,n-1}\omega + G_n)} < l_n^- = g\left(\left(\bar{c}_{n,n-1} + \frac{2g_n}{\omega}\right)e^{\bar{c}_{n,n-1}\omega + G_n}, \bar{b}_n e^{\bar{c}_{n,n-1}\omega + G_n}, \bar{h}_n\right) \\
&< g\left(\frac{2g_n}{\omega} - \bar{d}_n, \bar{b}_n e^{\bar{c}_{n,n-1}\omega + G_n}, \bar{h}_n\right) = B_n^- < B_n^+ = f\left(\frac{2g_n}{\omega} - \bar{d}_n, \bar{b}_n e^{\bar{c}_{n,n-1}\omega + G_n}, \bar{h}_n\right) \\
&< f\left(\left(\bar{c}_{n,n-1} + \frac{2g_n}{\omega}\right)e^{\bar{c}_{n,n-1}\omega + G_n}, \bar{b}_n e^{\bar{c}_{n,n-1}\omega + G_n}, \bar{h}_n\right) = l_n^+,
\end{aligned} \tag{3.10}$$

which imply that $\ln H_i < \ln l_i^- < \ln B_i^- < \ln B_i^+ < \ln l_i^+$, $i = 1, 2, \dots, n$; that is, the inequality (2) of Lemma 3.4 holds. Let us now to prove the inequality (3) of Lemma 3.4. In fact, since

$$\begin{aligned}
\max\{\ln A_i^-, \ln B_i^-\} &= \frac{\ln A_i^- + \ln B_i^- + |\ln A_i^- - \ln B_i^-|}{2} \quad (1 \leq i \leq n), \\
\min\{\ln A_i^+, \ln B_i^+\} &= \frac{\ln A_i^+ + \ln B_i^+ - |\ln A_i^+ - \ln B_i^+|}{2} \quad (1 \leq i \leq n),
\end{aligned} \tag{3.11}$$

we have

$$\begin{aligned}
&\min\{\ln A_i^+, \ln B_i^+\} - \max\{\ln A_i^-, \ln B_i^-\} \\
&= \frac{1}{2} [\ln A_i^+ + \ln B_i^+ - \ln A_i^- - \ln B_i^- - |\ln A_i^- - \ln B_i^-| - |\ln A_i^+ - \ln B_i^+|].
\end{aligned} \tag{3.12}$$

If $A_i^+ \geq B_i^+$, then $A_i^- \leq B_i^-$, and

$$\min\{\ln A_i^+, \ln B_i^+\} - \max\{\ln A_i^-, \ln B_i^-\} = \ln B_i^+ - \ln B_i^- > 0; \tag{3.13}$$

If $A_i^+ \leq B_i^+$, then $A_i^- \geq B_i^-$, and

$$\min\{\ln A_i^+, \ln B_i^+\} - \max\{\ln A_i^-, \ln B_i^-\} = \ln A_i^+ - \ln A_i^- > 0. \tag{3.14}$$

So, we drive $\min\{\ln A_i^+, \ln B_i^+\} > \max\{\ln A_i^-, \ln B_i^-\}$. From the above, we have completed the proof of Lemma 3.4. $\square$

Theorem 3.5. Assume that (H_1) , (H_2) , and (H_3) hold. Then system (1.3) has at least 2^n ω -periodic solutions.

Proof. Let $C[I_\omega; t_1, \dots, t_q] = \{u : I_\omega \rightarrow \mathbb{R}^n \text{ is a piecewise continuous map with first-class discontinuity points in } I_\omega \cap \{t_k\}_{k=1}^\infty, \text{ and at each discontinuity point, it is continuous on the left}\}$. Take

$$X = \left\{ u = (u_1, u_2, \dots, u_n)^T \in C[I_\omega; t_1, \dots, t_q] : u_i(t + \omega) = u_i(t) \right\}, \quad Z = X \times \mathbb{R}^{n \times (q+1)} \tag{3.15}$$

and define

$$\begin{aligned}\|u\|_0 &= \sum_{i=1}^n \sup_{t \in I_{\omega}} |u_i(t)|, \quad u \in X, \\ \|z\|_1 &= \|u\|_0 + \sum_{j=1}^q |\xi_j|, \quad z = (u, \xi_1, \xi_2, \dots, \xi_q) \in Z.\end{aligned}\tag{3.16}$$

Then both $(X, \|\cdot\|_0)$ and $(Z, \|\cdot\|_1)$ are Banach spaces. Let

$$\begin{aligned}\text{Dom } L &= \left\{ u \in X : u^\Delta \in X \right\}, \quad L : \text{Dom } L \longrightarrow Z, \quad N : X \longrightarrow Z, \\ Lu &= \left(\begin{pmatrix} u_1^\Delta \\ u_2^\Delta \\ \vdots \\ u_n^\Delta \end{pmatrix}, \begin{pmatrix} \Delta u_1(t_1) \\ \Delta u_2(t_1) \\ \vdots \\ \Delta u_n(t_1) \end{pmatrix}, \dots, \begin{pmatrix} \Delta u_1(t_q) \\ \Delta u_2(t_q) \\ \vdots \\ \Delta u_n(t_q) \end{pmatrix} \right), \\ N(u, \lambda) &= \left(\begin{pmatrix} F_1(t, \lambda) \\ \vdots \\ F_i(t, \lambda) \\ \vdots \\ F_n(t, \lambda) \end{pmatrix}, \begin{pmatrix} \ln(1 + g_{11}) \\ \vdots \\ \ln(1 + g_{i1}) \\ \vdots \\ \ln(1 + g_{n1}) \end{pmatrix}, \dots, \begin{pmatrix} \ln(1 + g_{1q}) \\ \vdots \\ \ln(1 + g_{iq}) \\ \vdots \\ \ln(1 + g_{nq}) \end{pmatrix} \right),\end{aligned}\tag{3.17}$$

where

$$\begin{aligned}F_1(t, \lambda) &= a_1(t) - b_1(t)e^{u_1(t-\tau_{11}(t))} - \lambda c_{12}(t)e^{u_2(t-\tau_{12}(t))} - h_1(t)e^{-u_1(t)}, \\ F_i(t, \lambda) &= -\lambda d_i(t) - b_i(t)e^{u_i(t-\tau_{ii}(t))} + c_{i,i-1}(t)e^{u_{i-1}(t-\tau_{i,i-1}(t))} \\ &\quad - \lambda c_{i,i+1}(t)e^{u_{i+1}(t-\tau_{i,i+1}(t))} - h_i(t)e^{-u_i(t)}, \quad i = 2, 3, \dots, n-1, \\ F_n(t, \lambda) &= -\lambda d_n(t) - b_n(t)e^{u_n(t-\tau_{nn}(t))} + c_{n,n-1}(t)e^{u_{n-1}(t-\tau_{n,n-1}(t))} - h_n(t)e^{-u_n(t)}.\end{aligned}$$

$$P : X \longrightarrow X, \quad Q : Z \longrightarrow Z,$$

$$P(u = (u_1, u_2, \dots, u_n)^T) = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n)^T, \quad u \in X,\tag{3.18}$$

$$Q\left(\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \left\{ \begin{pmatrix} m_{1k} \\ m_{2k} \\ \vdots \\ m_{nk} \end{pmatrix} \right\}_{k=1}^q\right) = \left(\begin{pmatrix} \bar{u}_1 + \frac{1}{\omega} \sum_{k=1}^q m_{1k} \\ \bar{u}_2 + \frac{1}{\omega} \sum_{k=1}^q m_{2k} \\ \vdots \\ \bar{u}_n + \frac{1}{\omega} \sum_{k=1}^q m_{nk} \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}_{k=1}^q\right).$$

Obviously,

$$\begin{aligned} \text{Ker } L &= \left\{ u = (u_1, u_2, \dots, u_n)^T \in X : (u_1(t), u_2(t), \dots, u_n(t)) = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n, t \in \mathbb{T} \right\}, \\ \text{Im } L &= \left\{ \left(\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \left\{ \begin{pmatrix} m_{1k} \\ m_{2k} \\ \vdots \\ m_{nk} \end{pmatrix} \right\}_{k=1}^q \right) \in Z : \begin{aligned} &\omega \bar{u}_1 + \sum_{k=1}^q m_{1k} = 0 \\ &\omega \bar{u}_2 + \sum_{k=1}^q m_{2k} = 0 \\ &\vdots \\ &\omega \bar{u}_n + \sum_{k=1}^q m_{nk} = 0 \end{aligned} \right\}. \end{aligned} \quad (3.19)$$

Since $\text{Im } L$ is closed in Z , $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L$, and $\dim \text{Ker } L = n = \text{codim Im } L$, we know that L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : \text{Im } L \rightarrow \text{Ker } P \cap \text{Dom } L$ is given by

$$\begin{aligned} K_P \left(z = \left(\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \left\{ \begin{pmatrix} m_{1k} \\ m_{2k} \\ \vdots \\ m_{nk} \end{pmatrix} \right\}_{k=1}^q \right) \right) \\ = \begin{pmatrix} \int_{\kappa}^t u_1(s) \Delta s + \sum_{\kappa < t_k < t} m_{1k} - \frac{1}{\omega} \sum_{k=1}^q m_{1k} - \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \int_{\kappa}^t u_1(s) \Delta s \Delta t \\ \int_{\kappa}^t u_2(s) \Delta s + \sum_{\kappa < t_k < t} m_{2k} - \frac{1}{\omega} \sum_{k=1}^q m_{2k} - \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \int_{\kappa}^t u_2(s) \Delta s \Delta t \\ \vdots \\ \int_{\kappa}^t u_n(s) \Delta s + \sum_{\kappa < t_k < t} m_{nk} - \frac{1}{\omega} \sum_{k=1}^q m_{nk} - \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \int_{\kappa}^t u_n(s) \Delta s \Delta t \end{pmatrix}. \end{aligned} \quad (3.20)$$

Then

$$QN(u, \lambda) = \left(\begin{pmatrix} \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} F_1(s, \lambda) \Delta s + \frac{2g_1}{\omega} \\ \vdots \\ \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} F_n(s, \lambda) \Delta s + \frac{2g_n}{\omega} \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}_{k=1}^q \right),$$

$$\begin{aligned}
K_P(I - Q)N(u, \lambda) = & \begin{pmatrix} \int_{\kappa}^t F_1(s, \lambda) \Delta s + \sum_{\kappa < t_k < t} \ln(1 + g_{1k}) \\ \vdots \\ \int_{\kappa}^t F_n(s, \lambda) \Delta s + \sum_{\kappa < t_k < t} \ln(1 + g_{nk}) \end{pmatrix} \\
& + \left(\frac{1}{2} - \frac{t}{\omega} \right) \begin{pmatrix} \int_{\kappa}^{\kappa+\omega} F_1(s, \lambda) \Delta s + 2g_1 \\ \vdots \\ \int_{\kappa}^{\kappa+\omega} F_n(s, \lambda) \Delta s + 2g_n \end{pmatrix} \\
& - \frac{1}{\omega} \begin{pmatrix} \int_{\kappa}^{\kappa+\omega} \int_{\kappa}^t F_1(s, \lambda) \Delta s \Delta t + 2g_1 \\ \vdots \\ \int_{\kappa}^{\kappa+\omega} \int_{\kappa}^t F_n(s, \lambda) \Delta s \Delta t + 2g_n \end{pmatrix},
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
F_1(s, \lambda) &= a_1(s) - b_1(s)e^{u_1(s-\tau_{11}(s))} - \lambda c_{12}(s)e^{u_2(s-\tau_{12}(s))} - h_1(s)e^{-u_1(s)}, \\
F_i(s, \lambda) &= -\lambda d_i(s) - b_i(s)e^{u_i(s-\tau_{ii}(s))} + c_{i,i-1}(s)e^{u_{i-1}(s-\tau_{i,i-1}(s))} \\
&\quad - \lambda c_{i,i+1}(s)e^{u_{i+1}(s-\tau_{i,i+1}(s))} - h_i(s)e^{-u_i(s)}, \quad i = 2, 3, \dots, n-1, \\
F_n(s, \lambda) &= -\lambda d_n(s) - b_n(s)e^{u_n(s-\tau_{nn}(s))} + c_{n,n-1}(s)e^{u_{n-1}(s-\tau_{n,n-1}(s))} - h_n(s)e^{-u_n(s)}.
\end{aligned} \tag{3.22}$$

By the Lebesgue convergence theorem, QN and $K_P(I - Q)N$ are continuous. Next, we show that $K_P(I - Q)N$ maps bounded sets into relatively compact sets. Let $\Omega \subset X$ be an arbitrary bounded set in X , then there exists a number $R > 0$ such that $\|u\| < R$ for any $u \in \Omega$. We prove that $\overline{K_P(I - Q)N(\overline{\Omega})}$ is relatively compact. In fact, for any $\{u_\nu = (u_1^\nu, u_2^\nu, \dots, u_n^\nu)^T\}_{\nu \in \mathbb{N}} \in \Omega$ and $t \in I_\omega$, we have

$$\begin{aligned}
\|K_P(I - Q)N(u_\nu, \lambda)\| &= \sum_{i=1}^n \sup_{t \in I_\omega} \left| \int_{\kappa}^t F_i(s, \lambda) \Delta s + \sum_{\kappa < t_k < t} \ln(1 + g_{ik}) + \left(\frac{1}{2} - \frac{t}{\omega} \right) \right. \\
&\quad \times \left. \int_{\kappa}^{\kappa+\omega} F_i(s, \lambda) \Delta s + 2g_i - \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \int_{\kappa}^t F_i(s, \lambda) \Delta s \Delta t + 2g_i \right|.
\end{aligned} \tag{3.23}$$

Since $\|u_\nu\| < R$, it follows from the periodicity that there exists a constant W_1 such that $\|K_P(I - Q)N(u_\nu, \lambda)\| \leq W_1$. For any $\{u_\nu = (u_1^\nu, u_2^\nu, \dots, u_n^\nu)^T\}_{\nu \in \mathbb{N}} \in \Omega$ and $t \in I_\omega \cap (t_k, t_{k+1}]$, $k = 1, 2, \dots, q-1$, we have

$$(K_P(I - Q)N(u_\nu, \lambda))^\Delta = \begin{pmatrix} F_1(s, \lambda) \\ \vdots \\ F_n(s, \lambda) \end{pmatrix} - \frac{1}{\omega} \begin{pmatrix} \int_\kappa^{\kappa+\omega} F_1(s, \lambda) \Delta s + 2g_1 \\ \vdots \\ \int_\kappa^{\kappa+\omega} F_n(s, \lambda) \Delta s + 2g_n \end{pmatrix}. \quad (3.24)$$

Similarly, there has a constant W_2 such that

$$\sum_{i=1}^n \sup_{t \in I_\omega \cap (t_k, t_{k+1}]} |(K_P(I - Q)N(u_\nu, \lambda))^\Delta| \leq W_2, \quad k = 1, 2, \dots, q-1. \quad (3.25)$$

It follows from the definition of X that the right limit of $u_\nu(t)$ exists at each t_k for $k = 1, 2, \dots, q$. $\nu \in \mathbb{N}$. Without loss of generality, here, we can assume that $t_1 = \kappa$ and $t_q = \kappa + \omega$, for if $t_1 > \kappa$ or $t_q < \kappa + \omega$, we only need to add intervals $[\kappa, t_1] \cap \mathbb{T}$ or $[t_q, \kappa + \omega] \cap \mathbb{T}$ into the consideration. Consider the following functions:

$$\begin{aligned} (H(u_\nu, \lambda))(t) &= \begin{cases} (K_P(I - Q)N(u_\nu, \lambda))(t), & t \in I_\omega \cap (t_k, t_{k+1}], \quad k = 1, \dots, q-1, \\ (K_P(I - Q)N(u_\nu, \lambda))(t_k + 0), & t = t_k, \quad k = 1, \dots, q, \end{cases} \\ (H(u_\nu, \lambda))^\Delta(t) &= \begin{cases} (K_P(I - Q)N(u_\nu, \lambda))^\Delta(t), & t \in I_\omega \cap (t_k, t_{k+1}], \quad k = 1, \dots, q-1, \\ (K_P(I - Q)N(u_\nu, \lambda))^\Delta(t_k + 0), & t = t_k, \quad k = 1, \dots, q, \end{cases} \end{aligned} \quad (3.26)$$

where $\nu \in \mathbb{N}$. Then on each interval $[t_k, t_{k+1}]$, $\{(H_k u_\nu) = (H u_\nu)|_{[t_k, t_{k+1}]}\}_{\nu \in \mathbb{N}}$, and $\{(H_k u_\nu)^\Delta = (H u_\nu)^\Delta|_{[t_k, t_{k+1}]}\}_{\nu \in \mathbb{N}}$ are uniformly bounded on $[t_k, t_{k+1}] \cap \mathbb{T}$, $k = 1, 2, \dots, q-1$. By Lemma 2.7, there exists a subsequence $\{H_1 u_{\nu_1^{(1)}}\}_{\nu_1^{(1)} \in \mathbb{N}}$ of $\{H_1 u_\nu\}_{\nu \in \mathbb{N}}$ converges uniformly on $[t_1, t_2] \cap \mathbb{T}$. Similarly, there is a subsequence $\{H_2 u_{\nu_2^{(2)}}\}_{\nu_2^{(2)} \in \mathbb{N}}$ of $\{H_1 u_{\nu_1^{(1)}}\}_{\nu_1^{(1)} \in \mathbb{N}}$ converges uniformly on $[t_2, t_3] \cap \mathbb{T}$. Repeating such a process, we can obtain that there is a subsequence $\{H_{q-1} u_{\nu_{q-1}^{(q-1)}}\}_{\nu_{q-1}^{(q-1)} \in \mathbb{N}}$ of $\{H_{q-2} u_{\nu_{q-2}^{(q-2)}}\}_{\nu_{q-2}^{(q-2)} \in \mathbb{N}}$ converges uniformly on $[t_{q-1}, t_q] \cap \mathbb{T}$. Obviously, $\{H_{q-1} u_{\nu_{q-1}^{(q-1)}}\}_{\nu_{q-1}^{(q-1)} \in \mathbb{N}}$ is a subsequence of $\{H u_\nu\}_{\nu \in \mathbb{N}}$ that converges uniformly on $I_\omega = [\kappa, \kappa + \omega] \cap \mathbb{T}$. Therefore, $\{K_P(I - Q)N_{q-1} u_{\nu_{q-1}^{(q-1)}}\}_{\nu_{q-1}^{(q-1)} \in \mathbb{N}}$ is a subsequence of $\{K_P(I - Q)N u_\nu\}_{\nu \in \mathbb{N}}$ that converges uniformly on I_ω , namely, $\overline{K_P(I - Q)N(\overline{\Omega})}$ is relatively compact. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L -compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$.

In order to use Lemma 3.1, we have to find at least 2^n appropriate open bounded subsets of X . Considering the operator equation $Lu = \lambda N(u, \lambda)$, $\lambda \in (0, 1)$, we have

$$\begin{aligned}
 u_1^\Delta(t) &= \lambda \left(a_1(t) - b_1(t)e^{u_1(t-\tau_{11}(t))} - \lambda c_{12}(t)e^{u_2(t-\tau_{12}(t))} - h_1(t)e^{-u_1(t)} \right), \\
 &\vdots \\
 u_i^\Delta(t) &= \lambda \left(-\lambda d_i(t) - b_i(t)e^{u_i(t-\tau_{ii}(t))} + c_{i,i-1}(t)e^{u_{i-1}(t-\tau_{i,i-1}(t))} - \lambda c_{i,i+1}(t)e^{u_{i+1}(t-\tau_{i,i+1}(t))} - h_i(t)e^{-u_i(t)} \right), \\
 &\qquad\qquad\qquad 2 \leq i \leq n-1, \\
 &\vdots \\
 u_n^\Delta(t) &= \lambda \left(-\lambda d_n(t) - b_n(t)e^{u_n(t-\tau_{nn}(t))} + c_{n,n-1}(t)e^{u_{n-1}(t-\tau_{n,n-1}(t))} - h_n(t)e^{-u_n(t)} \right), \\
 \Delta u_j(t_k) &= \lambda \ln(1 + g_{jk}), \quad 1 \leq j \leq n.
 \end{aligned} \tag{3.27}$$

Assume that $u \in X$ is an ω -periodic solution of system (3.27) for some $\lambda \in (0, 1)$. Integrating (3.27) from κ to $\kappa + \omega$, we have

$$\begin{aligned}
 -2g_1 &= \int_{\kappa}^{\kappa+\omega} \left\{ a_1(t) - b_1(t)e^{u_1(t-\tau_{11}(t))} - \lambda c_{12}(t)e^{u_2(t-\tau_{12}(t))} - h_1(t)e^{-u_1(t)} \right\} \Delta t, \\
 &\vdots \\
 -2g_i &= \int_{\kappa}^{\kappa+\omega} \left\{ -\lambda d_i(t) - b_i(t)e^{u_i(t-\tau_{ii}(t))} + c_{i,i-1}(t)e^{u_{i-1}(t-\tau_{i,i-1}(t))} \right. \\
 &\qquad\qquad\qquad \left. - \lambda c_{i,i+1}(t)e^{u_{i+1}(t-\tau_{i,i+1}(t))} - h_i(t)e^{-u_i(t)} \right\} \Delta t, \quad 2 \leq i \leq n-1, \\
 &\vdots \\
 -2g_n &= \int_{\kappa}^{\kappa+\omega} \left\{ -\lambda d_n(t) - b_n(t)e^{u_n(t-\tau_{nn}(t))} + c_{n,n-1}(t)e^{u_{n-1}(t-\tau_{n,n-1}(t))} - h_n(t)e^{-u_n(t)} \right\} \Delta t.
 \end{aligned} \tag{3.28}$$

Then

$$\begin{aligned}
 &\int_{\kappa}^{\kappa+\omega} \left\{ b_1(t)e^{u_1(t-\tau_{11}(t))} + \lambda c_{12}(t)e^{u_2(t-\tau_{12}(t))} + h_1(t)e^{-u_1(t)} \right\} \Delta t = \bar{a}_1\omega + 2g_1 \\
 &\vdots
 \end{aligned}$$

$$\begin{aligned}
& \int_{\kappa}^{\kappa+\omega} \left\{ \lambda d_i(t) + b_i(t) e^{u_i(t-\tau_{ii}(t))} + \lambda c_{i,i+1}(t) e^{u_{i+1}(t-\tau_{i,i+1}(t))} + h_i(t) e^{-u_i(t)} \right\} \Delta t \\
&= \int_{\kappa}^{\kappa+\omega} c_{i,i-1}(t) e^{u_{i-1}(t-\tau_{i,i-1}(t))} \Delta t + 2g_i, \quad 2 \leq i \leq n-1, \\
&\vdots \\
& \int_{\kappa}^{\kappa+\omega} \left\{ \lambda d_n(t) + b_n(t) e^{u_n(t-\tau_{nn}(t))} + h_n(t) e^{-u_n(t)} \right\} \Delta t = \int_{\kappa}^{\kappa+\omega} c_{n,n-1}(t) e^{u_{n-1}(t-\tau_{n,n-1}(t))} \Delta t + 2g_n.
\end{aligned} \tag{3.29}$$

Furthermore, Note that $u = (u_1, u_2, \dots, u_n)^T \in X$, and there exists $\xi_i, \eta_i \in [\kappa, \kappa + \omega] \cap \mathbb{T} = I_\omega$, $i = 1, 2, \dots, n$, such that $u_i(\xi_i) = \sup_{t \in I_\omega} u_i(t)$, $u_i(\eta_i) = \inf_{t \in I_\omega} u_i(t)$.

On the one hand, according to the first equation of (3.27) and (3.29), we have

$$\int_{\kappa}^{\kappa+\omega} |u_1^\Delta(t)| \Delta t < 2\bar{a}_1\omega + 2g_1. \tag{3.30}$$

In the light of the first of equation of (3.29), (3.30), and Lemma 3.2, we get

$$\begin{aligned}
\omega \bar{b}_1 e^{u_1(\xi_1) - (\bar{a}_1\omega + G_1)} + \omega \bar{h}_1 e^{-u_1(\xi_1)} &\leq \omega \bar{b}_1 e^{u_1(\xi_1) - (1/2)[\int_{\kappa}^{\kappa+\omega} |u_1^\Delta(s)| \Delta s + \sum_{k=1}^q |\Delta u_1(t_k)|]} + \omega \bar{h}_1 e^{-u_1(\xi_1)} \\
&< \omega \bar{b}_1 e^{u_1(\eta_1)} + \omega \bar{h}_1 e^{-u_1(\xi_1)} \\
&< \int_{\kappa}^{\kappa+\omega} \left\{ b_1(t) e^{u_1(t-\tau_{11}(t))} + \lambda c_{12}(t) e^{u_2(t-\tau_{12}(t))} + h_1(t) e^{-u_1(t)} \right\} \Delta t \\
&= \bar{a}_1\omega + 2g_1,
\end{aligned} \tag{3.31}$$

namely,

$$\bar{b}_1 e^{2u_1(\xi_1)} - \left(\bar{a}_1 + \frac{2g_1}{\omega} \right) e^{\bar{a}_1\omega + G_1} e^{u_1(\xi_1)} + \bar{h}_1 e^{\bar{a}_1\omega + G_1} < 0, \tag{3.32}$$

which implies that

$$\ln l_1^- < u_1(\xi_1) < \ln l_1^+. \tag{3.33}$$

Applying Lemma 3.2, we drive

$$\ln H_1 = \ln l_1^- - (\bar{a}_1\omega + G_1) < u_1(\eta) \leq u_1(\xi_1) < \ln l_1^+. \tag{3.34}$$

According to (3.29) and (3.30), we get

$$\int_{\kappa}^{\kappa+\omega} \left| u_2^{\Delta}(t) \right| \Delta t < 2\bar{c}_{21}\omega + 2g_2. \quad (3.35)$$

By (3.29) and (3.35), we have

$$\begin{aligned} & \omega \bar{b}_2 e^{u_2(\xi_2) - (\bar{c}_{21}\omega + G_2)} + \omega \bar{h}_2 e^{-u_2(\xi_2)} \\ & \leq \omega \bar{b}_2 e^{u_2(\xi_2) - (1/2)[\int_{\kappa}^{\kappa+\omega} |u_2^{\Delta}(s)| \Delta s + \sum_{k=1}^q |\Delta u_2(t_k)|]} + \omega \bar{h}_2 e^{-u_2(\xi_2)} < \omega \bar{b}_2 e^{u_2(\eta_2)} + \omega \bar{h}_2 e^{-u_2(\xi_2)} \\ & < \int_{\kappa}^{\kappa+\omega} \left\{ \lambda d_2(t) + b_2(t) e^{u_2(t-\tau_{22}(t))} + \lambda c_{23}(t) e^{u_3(t-\tau_{23}(t))} + h_2(t) e^{-u_2(t)} \right\} \Delta t \\ & = \int_{\kappa}^{\kappa+\omega} c_{21}(t) e^{u_1(t-\tau_{21}(t))} \Delta t + 2g_1 < \omega \bar{c}_{21} l_1^+ + 2g_2, \end{aligned} \quad (3.36)$$

that is

$$\bar{b}_2 e^{2u_2(\xi_2)} - \left(\bar{c}_{21} + \frac{2g_2}{\omega} \right) e^{\bar{c}_{21}\omega + G_2} e^{u_2(\xi_2)} + \bar{h}_2 e^{\bar{c}_{21}\omega + G_2} < 0, \quad (3.37)$$

which implies that

$$\ln l_2^- < u_2(\xi_2) < \ln l_2^+. \quad (3.38)$$

Applying Lemma 3.2, we drive

$$\ln H_2 = \ln l_2^- - (\bar{c}_{21}\omega + G_2) < u_2(\eta) \leq u_2(\xi_2) < \ln l_2^+. \quad (3.39)$$

By deducing for $i = 3, 4, \dots, n$, we obtain

$$\begin{aligned} & \int_{\kappa}^{\kappa+\omega} \left| u_i^{\Delta}(t) \right| \Delta t < 2\bar{c}_{i,i-1}\omega + 2g_i, \\ & \omega \bar{b}_i e^{u_i(\xi_i) - (\bar{c}_{i,i-1}\omega + G_i)} + \omega \bar{h}_i e^{-u_i(\xi_i)} \\ & \leq \omega \bar{b}_i e^{u_i(\xi_i) - (1/2)[\int_{\kappa}^{\kappa+\omega} |u_i^{\Delta}(s)| \Delta s + \sum_{k=1}^q |\Delta u_i(t_k)|]} + \omega \bar{h}_i e^{-u_i(\xi_i)} < \omega \bar{b}_i e^{u_i(\eta_i)} + \omega \bar{h}_i e^{-u_i(\xi_i)} \\ & < \int_{\kappa}^{\kappa+\omega} \left\{ \lambda d_i(t) + b_i(t) e^{u_i(t-\tau_{ii}(t))} + \lambda c_{i,i+1}(t) e^{u_{i+1}(t-\tau_{i,i+1}(t))} + h_i(t) e^{-u_i(t)} \right\} \Delta t \\ & = \int_{\kappa}^{\kappa+\omega} c_{i,i-1}(t) e^{u_{i-1}(t-\tau_{i,i-1}(t))} \Delta t + 2g_{i-1} < \omega \bar{c}_{i,i-1} l_{i-1}^+ + 2g_i, \end{aligned} \quad (3.40)$$

namely,

$$\bar{b}_i e^{2u_i(\xi_i)} - \left(\bar{c}_{i,i-1} + \frac{2g_i}{\omega} \right) e^{\bar{c}_{i,i-1}\omega + G_i} e^{u_i(\xi_i)} + \bar{h}_i e^{\bar{c}_{i,i-1}\omega + G_i} < 0, \quad (3.41)$$

which implies that

$$\ln l_i^- < u_i(\xi_i) < \ln l_i^+, \quad i = 3, 4, \dots, n. \quad (3.42)$$

Applying Lemma 3.2, we drive

$$\ln H_i = \ln l_i^- - (\bar{c}_{i,i-1}\omega + G_i) < u_i(\eta) \leq u_i(\xi_i) < \ln l_i^+, \quad i = 3, 4, \dots, n. \quad (3.43)$$

On the other hand, in the light of the first equation of (3.29) and (3.30), we have

$$\begin{aligned} \bar{a}_1\omega + 2g_1 &= \int_{\kappa}^{\kappa+\omega} \left\{ b_1(t) e^{u_1(t-\tau_{11}(t))} + \lambda c_{12}(t) e^{u_2(t-\tau_{12}(t))} + h_1(t) e^{-u_1(t)} \right\} \Delta t \\ &< \bar{b}_1\omega e^{u_1(\xi_1)} + \bar{c}_{12}\omega l_2^+ + \bar{h}_1\omega e^{-u_1(\eta_1)} \\ &< \bar{b}_1\omega e^{u_1(\xi_1)} + \bar{c}_{12}\omega l_2^+ + \bar{h}_1\omega e^{-u_1(\xi_1) + (1/2)[\int_{\kappa}^{\kappa+\omega} |u_1^\Delta(s)| \Delta s + \sum_{k=1}^q |\Delta u_1(t_k)|]} \\ &< \bar{b}_1\omega e^{u_1(\xi_1)} + \bar{c}_{12}\omega l_2^+ + \bar{h}_1\omega e^{-u_1(\xi_1) + a_1\omega + G_1}, \end{aligned} \quad (3.44)$$

namely,

$$\bar{b}_1 e^{2u_1(\xi_1)} - \left(\bar{a}_1 + \frac{2g_1}{\omega} - \bar{c}_{12}l_2^+ \right) e^{u_1(\xi_1)} + \bar{h}_1 e^{\bar{a}_1\omega + G_1} > 0, \quad (3.45)$$

which implies that

$$\begin{aligned} \ln A_1^+ &< u_1(\xi_1) \quad \text{or} \quad u_1(\xi_1) < \ln A_1^-, \\ \bar{a}_1\omega + 2g_1 &= \int_{\kappa}^{\kappa+\omega} \left\{ b_1(t) e^{u_1(t-\tau_{11}(t))} + \lambda c_{12}(t) e^{u_2(t-\tau_{12}(t))} + h_1(t) e^{-u_1(t)} \right\} \Delta t \\ &< \omega \left(\bar{b}_1 e^{u_1(\xi_1)} + \bar{c}_{12}l_2^+ + \bar{h}_1 e^{-u_1(\eta_1)} \right) \\ &< \omega \left(\bar{b}_1 e^{u_1(\eta_1) + (1/2)[\int_{\kappa}^{\kappa+\omega} |u_1^\Delta(s)| \Delta s + \sum_{k=1}^q |\Delta u_1(t_k)|]} + \bar{c}_{12}l_2^+ + \bar{h}_1 e^{-u_1(\eta_1)} \right) \\ &< \bar{b}_1\omega e^{u_1(\eta_1) + \bar{a}_1\omega + G_1} + \bar{c}_{12}\omega l_2^+ + \bar{h}_1\omega e^{-u_1(\eta_1)}, \end{aligned} \quad (3.46)$$

that is,

$$\bar{b}_1 e^{2u_1(\eta_1)} - \left(\bar{a}_1 + \frac{2g_1}{\omega} - \bar{c}_{12}l_2^+ \right) e^{-(\bar{a}_1\omega + G_1)} e^{u_1(\eta_1)} + \bar{h}_1 e^{-(\bar{a}_1\omega + G_1)} > 0, \quad (3.47)$$

which implies that

$$\ln B_1^+ < u_1(\eta_1) \quad \text{or} \quad u_1(\eta_1) < \ln B_1^-. \quad (3.48)$$

By (3.29) and (3.35), we have

$$\begin{aligned} 2g_2 &< \int_{\kappa}^{\kappa+\omega} c_{21}(t) e^{u_1(t-\tau_{21}(t))} \Delta t + 2g_2 \\ &= \int_{\kappa}^{\kappa+\omega} \left\{ \lambda d_2(t) + b_2(t) e^{u_2(t-\tau_{22}(t))} + \lambda c_{23}(t) e^{u_3(t-\tau_{23}(t))} + h_2(t) e^{-u_2(t)} \right\} \Delta t \\ &< \bar{b}_2 \omega e^{u_2(\xi_2)} + (\bar{d}_2 + \bar{c}_{23} l_3^+) \omega + \bar{h}_2 \omega e^{-u_2(\eta_2)} \\ &< \bar{b}_2 \omega e^{u_2(\xi_2)} + (\bar{d}_2 + \bar{c}_{23} l_3^+) \omega + \bar{h}_2 \omega e^{-u_2(\xi_2) + (1/2) [\int_{\kappa}^{\kappa+\omega} |u_2^\Delta(s)| \Delta s + \sum_{k=1}^q |\Delta u_2(t_k)|]} \\ &< \bar{b}_2 \omega e^{u_2(\xi_2)} + (\bar{d}_2 + \bar{c}_{23} l_3^+) \omega + \bar{h}_2 \omega e^{-u_2(\xi_2) + \bar{c}_{21} \omega + G_2}, \end{aligned} \quad (3.49)$$

namely,

$$\bar{b}_2 e^{2u_2(\xi_2)} - \left(\frac{2g_2}{\omega} - \bar{d}_2 - \bar{c}_{23} l_3^+ \right) e^{u_2(\xi_2)} + \bar{h}_2 e^{\bar{c}_{21} \omega + G_2} > 0, \quad (3.50)$$

which implies that

$$\begin{aligned} \ln A_2^+ &< u_2(\xi_2) \quad \text{or} \quad u_2(\xi_2) < \ln A_2^-, \\ 2g_2 &< \int_{\kappa}^{\kappa+\omega} c_{21}(t) e^{u_1(t-\tau_{21}(t))} \Delta t + 2g_2 \\ &= \int_{\kappa}^{\kappa+\omega} \left\{ \lambda d_2(t) + b_2(t) e^{u_2(t-\tau_{22}(t))} + \lambda c_{23}(t) e^{u_3(t-\tau_{23}(t))} + h_2(t) e^{-u_2(t)} \right\} \Delta t \\ &< \bar{b}_2 \omega e^{u_2(\xi_2)} + (\bar{d}_2 + \bar{c}_{23} l_3^+) \omega + \bar{h}_2 \omega e^{-u_2(\eta_2)} \\ &< \bar{b}_2 \omega e^{u_2(\eta_2) + (1/2) [\int_{\kappa}^{\kappa+\omega} |u_2^\Delta(s)| \Delta s + \sum_{k=1}^q |\Delta u_2(t_k)|]} + (\bar{d}_2 + \bar{c}_{23} l_3^+) \omega + \bar{h}_2 \omega e^{-u_2(\eta_2)} \\ &< \bar{b}_2 \omega e^{u_2(\eta_2) + \bar{c}_{21} \omega + G_2} + (\bar{d}_2 + \bar{c}_{23} l_3^+) \omega + \bar{h}_2 \omega e^{-u_2(\eta_2)}, \end{aligned} \quad (3.51)$$

that is,

$$\bar{b}_2 e^{2u_2(\eta_2)} - \left(\frac{2g_2}{\omega} - \bar{d}_2 - \bar{c}_{23} l_3^+ \right) e^{-\bar{c}_{21} \omega + G_2} e^{u_2(\eta_2)} + \bar{h}_2 e^{-(\bar{c}_{21} \omega + G_2)} > 0, \quad (3.52)$$

which implies that

$$\ln B_2^+ < u_2(\eta_2) \quad \text{or} \quad u_2(\eta_2) < \ln B_2^-. \quad (3.53)$$

By deducing for $i = 3, 4, \dots, n-1$, we obtain

$$\begin{aligned}
2g_i &< \int_{\kappa}^{\kappa+\omega} c_{i,i-1}(t) e^{u_{i-1}(t-\tau_{i,i-1}(t))} \Delta t + 2g_i \\
&= \int_{\kappa}^{\kappa+\omega} \left\{ \lambda d_i(t) + b_i(t) e^{u_i(t-\tau_{ii}(t))} + \lambda c_{i,i+1}(t) e^{u_{i+1}(t-\tau_{i,i+1}(t))} + h_i(t) e^{-u_i(t)} \right\} \Delta t \\
&< \bar{b}_i \omega e^{u_i(\xi_i)} + \left(\bar{d}_i + \bar{c}_{i,i+1} l_{i+1}^+ \right) \omega + \bar{h}_i \omega e^{-u_i(\eta_i)} \\
&< \bar{b}_i \omega e^{u_i(\xi_i)} + \left(\bar{d}_i + \bar{c}_{i,i+1} l_{i+1}^+ \right) \omega + \bar{h}_i \omega e^{-u_i(\xi_i) + (1/2) [\int_{\kappa}^{\kappa+\omega} |u_i^\Delta(s)| \Delta s + \sum_{k=1}^q |\Delta u_i(t_k)|]} \\
&< \bar{b}_i \omega e^{u_i(\xi_i)} + \left(\bar{d}_i + \bar{c}_{i,i+1} l_{i+1}^+ \right) \omega + \bar{h}_i \omega e^{-u_i(\xi_i) + \bar{c}_{i,i-1} \omega + G_i},
\end{aligned} \tag{3.54}$$

namely,

$$\bar{b}_i e^{2u_i(\xi_i)} - \left(\frac{2g_i}{\omega} - \bar{d}_i - \bar{c}_{i,i+1} l_{i+1}^+ \right) e^{u_i(\xi_i)} + \bar{h}_i e^{\bar{c}_{i,i-1} \omega + G_i} > 0, \tag{3.55}$$

which implies that

$$\ln A_i^+ < u_i(\xi_i) \quad \text{or} \quad u_i(\xi_i) < \ln A_i^-,$$

$$\begin{aligned}
2g_i &< \int_{\kappa}^{\kappa+\omega} c_{i,i-1}(t) e^{u_{i-1}(t-\tau_{i,i-1}(t))} \Delta t + 2g_i \\
&= \int_{\kappa}^{\kappa+\omega} \left\{ \lambda d_i(t) + b_i(t) e^{u_i(t-\tau_{ii}(t))} + \lambda c_{i,i+1}(t) e^{u_{i+1}(t-\tau_{i,i+1}(t))} + h_i(t) e^{-u_i(t)} \right\} \Delta t \\
&< \bar{b}_i \omega e^{u_i(\xi_i)} + \left(\bar{d}_i + \bar{c}_{i,i+1} l_{i+1}^+ \right) \omega + \bar{h}_2 \omega e^{-u_i(\eta_i)} \\
&< \bar{b}_i \omega e^{u_i(\eta_i) + (1/2) [\int_{\kappa}^{\kappa+\omega} |u_i^\Delta(s)| \Delta s + \sum_{k=1}^q |\Delta u_i(t_k)|]} + \left(\bar{d}_i + \bar{c}_{i,i+1} l_{i+1}^+ \right) \omega + \bar{h}_i \omega e^{-u_i(\eta_i)} \\
&< \bar{b}_i \omega e^{u_i(\eta_i) + \bar{c}_{i,i-1} \omega + G_i} + \left(\bar{d}_i + \bar{c}_{i,i+1} l_{i+1}^+ \right) \omega + \bar{h}_i \omega e^{-u_i(\eta_i)},
\end{aligned} \tag{3.56}$$

that is,

$$\bar{b}_i e^{2u_i(\eta_i)} - \left(\frac{2g_i}{\omega} - \bar{d}_i - \bar{c}_{i,i+1} l_{i+1}^+ \right) e^{u_i(\eta_i)} + \bar{h}_i e^{-(\bar{c}_{i,i-1} \omega + G_i)} > 0, \tag{3.57}$$

which implies that

$$\ln B_i^+ < u_i(\eta_i) \quad \text{or} \quad u_i(\eta_i) < \ln B_i^-. \tag{3.58}$$

By the final equation of (3.29), we get

$$\begin{aligned}
 2g_n &< \int_{\kappa}^{\kappa+\omega} c_{n,n-1}(t) e^{u_{n-1}(t-\tau_{n,n-1}(t))} \Delta t + 2g_n \\
 &= \int_{\kappa}^{\kappa+\omega} \left\{ \lambda d_n(t) + b_n(t) e^{u_n(t-\tau_{nn}(t))} + h_n(t) e^{-u_n(t)} \right\} \Delta t < \bar{d}_n \omega + \bar{b}_n \omega e^{u_n(\xi_n)} + \bar{h}_n \omega e^{-u_n(\eta_n)} \\
 &< \bar{d}_n \omega + \bar{b}_n \omega e^{u_n(\xi_n)} + \bar{h}_n \omega e^{-u_n(\xi_n) + (1/2) [\int_{\kappa}^{\kappa+\omega} |u_n^{\Delta}(s)| \Delta s + \sum_{k=1}^q |\Delta u_n(t_k)|]} \\
 &< \bar{d}_n \omega + \bar{b}_n \omega e^{u_n(\xi_n)} + \bar{h}_n \omega e^{-u_n(\xi_n) + \bar{c}_{n,n-1} \omega + G_n},
 \end{aligned} \tag{3.59}$$

namely,

$$\bar{b}_n e^{2u_n(\xi_n)} - \left(\frac{2g_n}{\omega} - \bar{d}_n \right) e^{u_n(\xi_n)} + \bar{h}_n e^{\bar{c}_{n,n-1} \omega + G_n} > 0, \tag{3.60}$$

which implies that

$$\ln A_n^+ < u_n(\xi_n) \quad \text{or} \quad u_n(\xi_n) < \ln A_n^-,$$

$$\begin{aligned}
 2g_n &< \int_{\kappa}^{\kappa+\omega} c_{n,n-1}(t) e^{u_{n-1}(t-\tau_{n,n-1}(t))} \Delta t + 2g_n \\
 &= \int_{\kappa}^{\kappa+\omega} \left\{ \lambda d_n(t) + b_n(t) e^{u_n(t-\tau_{nn}(t))} + h_n(t) e^{-u_n(t)} \right\} \Delta t < \bar{d}_n \omega + \bar{b}_n \omega e^{u_n(\xi_n)} + \bar{h}_n \omega e^{-u_n(\eta_n)} \\
 &< \bar{d}_n \omega + \bar{b}_n \omega e^{u_n(\eta_n) + (1/2) [\int_{\kappa}^{\kappa+\omega} |u_n^{\Delta}(s)| \Delta s + \sum_{k=1}^q |\Delta u_n(t_k)|]} + \bar{h}_n \omega e^{-u_n(\eta_n)} \\
 &< \bar{d}_n \omega + \bar{b}_n \omega e^{u_n(\eta_n) + \bar{c}_{n,n-1} \omega + G_n} + \bar{h}_n \omega e^{-u_n(\eta_n)},
 \end{aligned} \tag{3.61}$$

namely,

$$\bar{b}_n e^{2u_n(\eta_n)} - \left(\frac{2g_n}{\omega} - \bar{d}_n \right) e^{u_n(\eta_n)} + \bar{h}_n e^{-(\bar{c}_{n,n-1} \omega + G_n)} > 0, \tag{3.62}$$

which implies that

$$\ln B_n^+ < u_n(\eta_n) \quad \text{or} \quad u_n(\eta_n) < \ln B_n^-, \tag{3.63}$$

In view of (3.34)–(3.63) and Lemma 3.3, we have

$$\min\{\ln A_i^+, \ln B_i^+\} < u_i(\eta_i) < u_i(\xi_i) < \ln l_i^+ \quad (1 \leq i \leq n) \tag{3.64}$$

or

$$\ln H_i < u_i(\eta_i) < u_i(\xi_i) < \max\{\ln A_i^-, \ln B_i^-\} \quad (1 \leq i \leq n), \quad (3.65)$$

which imply that, for all $t \in \mathbb{T}$,

$$\min\{\ln A_i^+, \ln B_i^+\} < u_i(t) < \ln l_i^+ \quad \text{or} \quad \ln H_i^- < u_i(t) < \max\{\ln A_i^-, \ln B_i^-\}. \quad (3.66)$$

For convenience, we denote

$$G_i = (\ln H_i^-, \max\{\ln A_i^-, \ln B_i^-\}), \quad E_i = (\min\{\ln A_i^+, \ln B_i^+\}, \ln l_i^+) \quad (1 \leq i \leq n). \quad (3.67)$$

Clearly, $l_i^\pm$ ($i = 1, 2, \dots, n$), $A_i^\pm$ ($i = 1, 2, \dots, n$), and $B_i^\pm$ ($i = 1, 2, \dots, n$) are independent of λ . For each $i = 1, 2, \dots, n$, we choose an interval between two intervals G_i and E_i and denote it as Δ_i , then define the set

$$\left\{ u = (u_1, u_2, \dots, u_n)^T \in X : u_i(t) \in \Delta_i, \quad t \in \mathbb{T}, \quad i = 1, 2, \dots, n \right\}. \quad (3.68)$$

Obviously, the number of the above sets is 2^n . We denote these sets as Ω_k , $k = 1, 2, \dots, 2^n$. Ω_k , $k = 1, 2, \dots, 2^n$ are bounded open subsets of X , $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$. Thus Ω_k ($k = 1, 2, \dots, 2^n$) satisfies the requirement (a) in Lemma 3.1.

Now we show that (b) of Lemma 3.1 holds; that is, we prove when $u \in \partial\Omega_k \cap \text{Ker } L = \partial\Omega_k \cap \mathbb{R}^n$, $QN(u, 0) \neq (0, 0, \dots, 0)^T$, $k = 1, 2, \dots, 2^n$. If it is not true, then when $u \in \partial\Omega_k \cap \text{Ker } L = \partial\Omega_k \cap \mathbb{R}^n$, $i = 1, 2, \dots, 2^n$, constant vector $u = (u_1, u_2, \dots, u_n)^T$ with $u \in \partial\Omega_k$, $k = 1, 2, \dots, 2^n$, satisfies

$$\begin{aligned} & -2g_1 = \int_{\kappa}^{\kappa+\omega} \{a_1(t) - b_1(t)e^{u_1} - h_1(t)e^{-u_1}\} \Delta t, \\ & \vdots \\ & -2g_i = \int_{\kappa}^{\kappa+\omega} \{-b_i(t)e^{u_i} + c_{i,i-1}(t)e^{u_{i-1}} - h_i(t)e^{-u_i}\} \Delta t, \quad 2 \leq i \leq n-1, \\ & \vdots \\ & -2g_n = \int_{\kappa}^{\kappa+\omega} \{-b_n(t)e^{u_n} + c_{n,n-1}(t)e^{u_{n-1}} - h_n(t)e^{-u_n}\} \Delta t, \end{aligned} \quad (3.69)$$

that is,

$$\begin{aligned}
 -\frac{2g_1}{\omega} &= \bar{a}_1 - \bar{b}_1 e^{u_1} - \bar{h}_1 e^{-u_1}, \\
 &\vdots \\
 -\frac{2g_i}{\omega} &= -\bar{b}_i e^{u_i} + \bar{c}_{i,i-1} e^{u_{i-1}} - \bar{h}_i e^{-u_i}, \quad 2 \leq i \leq n-1, \\
 &\vdots \\
 -\frac{2g_n}{\omega} &= -\bar{b}_n e^{u_n} + \bar{c}_{n,n-1} e^{u_{n-1}} - \bar{h}_n e^{-u_n}.
 \end{aligned} \tag{3.70}$$

Similar to the process of (3.29)–(3.66), we obtain

$$\ln H_i < u_i^- < \max\{\ln A_i^-, \ln B_i^-\} < \min\{\ln A_i^+, \ln B_i^+\} < u_i^+ < \ln I_i^+ \quad (1 \leq i \leq n). \tag{3.71}$$

Then u belongs to one of $\Omega_k \cap \mathbb{R}^n$, $k = 1, 2, \dots, 2^n$. This contradicts the fact that $u \in \partial\Omega_k \cap \mathbb{R}^n$, $k = 1, 2, \dots, 2^n$. This proves that (b) in Lemma 3.1 holds.

Finally, in order to show that (c) in Lemma 3.1 holds, we only prove that for $u \in \partial\Omega_k \cap \text{Ker } L = \partial\Omega_k \cap \mathbb{R}^n$, $k = 1, 2, \dots, 2^n$, then it holds that $\deg\{JQN(u, 0), \Omega_k \cap \text{Ker } L, (0, 0, \dots, 0)^T\} \neq 0$. To this end, we define the mapping $\phi : \text{Dom } L \times [0, 1] \rightarrow X$ by

$$\phi(u, \mu) = \mu QN(u, 0) + (1 - \mu)G(u), \tag{3.72}$$

here $\mu \in [0, 1]$ is a parameter and $G(u)$ is defined by

$$G(u) = \left(\begin{pmatrix} \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} (a_1(s) - b_1(s)e^{u_1(s)} - h_1(s)e^{-u_1(s)}) \Delta s + \frac{2g_1}{\omega} \\ \vdots \\ \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} (c_{i,i-1}(s)l_{i-1}^+ - b_i(s)e^{u_i(s)} - h_i(s)e^{-u_i(s)}) \Delta s + \frac{2g_i}{\omega} \\ \vdots \\ \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} (c_{n,n-1}(s)l_{n-1}^+ - b_n(s)e^{u_n(s)} - h_n(s)e^{-u_n(s)}) \Delta s + \frac{2g_n}{\omega} \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{k=1}^q \right\} \right), \tag{3.73}$$

where $i = 2, 3, \dots, n-1$. We show that for $u \in \partial\Omega_k \cap \text{Ker } L = \partial\Omega_k \cap \mathbb{R}^n$, $k = 1, 2, \dots, 2^n$, $\mu \in [0, 1]$, then it holds that $\phi(u, \mu) \neq (0, 0, \dots, 0)^T$. Otherwise, parameter μ and constant vector $u = (u_1, u_2, \dots, u_n)^T \in \mathbb{R}^n$ satisfy $\phi(u, \mu) = (0, 0, \dots, 0)^T$, that is,

$$\begin{aligned}
 0 &= \mu \int_{\kappa}^{\kappa+\omega} \left[a_1(s) - b_1(s)e^{u_1} - h_1(s)e^{-u_1} + \frac{2g_1}{\omega} \right] \Delta s \\
 &\quad + (1-\mu) \int_{\kappa}^{\kappa+\omega} \left[a_1(s) - b_1(s)e^{u_1} - h_1(s)e^{-u_1} + \frac{2g_1}{\omega} \right] \Delta s, \\
 &\quad \vdots \\
 0 &= \mu \int_{\kappa}^{\kappa+\omega} \left[c_{i,i-1}(s)e^{u_{i-1}} - b_i(s)e^{u_i} - h_i(s)e^{-u_i} + \frac{2g_i}{\omega} \right] \Delta s \\
 &\quad + (1-\mu) \int_{\kappa}^{\kappa+\omega} \left[c_{i,i-1}(s)l_{i-1}^+ - b_i(s)e^{u_i} - h_i(s)e^{-u_i} + \frac{2g_i}{\omega} \right] \Delta s, \\
 &\quad \vdots \\
 0 &= \mu \int_{\kappa}^{\kappa+\omega} \left[c_{n,n-1}(s)e^{u_{n-1}} - b_n(s)e^{u_n} - h_n(s)e^{-u_n} + \frac{2g_n}{\omega} \right] \Delta s \\
 &\quad + (1-\mu) \int_{\kappa}^{\kappa+\omega} \left[c_{n,n-1}(s)l_{n-1}^+ - b_n(s)e^{u_n} - h_n(s)e^{-u_n} + \frac{2g_n}{\omega} \right] \Delta s,
 \end{aligned} \tag{3.74}$$

where $i = 2, 3, \dots, n-1$, that is,

$$\begin{aligned}
 \bar{a}_1 - \bar{b}_1 e^{u_1} - \bar{h}_1 e^{-u_1} &= -\frac{2g_1}{\omega}, \\
 &\quad \vdots \\
 \bar{c}_{i,i-1} l_{i-1}^+ - \bar{b}_i e^{u_i} - \bar{h}_i e^{-u_i} &= \mu(\bar{c}_{i,i-1} l_{i-1}^+ - \bar{c}_{i,i-1} e^{u_{i-1}}) - \frac{2g_i}{\omega}, \\
 &\quad \vdots \\
 \bar{c}_{n,n-1} l_{n-1}^+ - \bar{b}_n e^{u_n} - \bar{h}_n e^{-u_n} &= \mu(\bar{c}_{n,n-1} l_{n-1}^+ - \bar{c}_{n,n-1} e^{u_{n-1}}) - \frac{2g_n}{\omega},
 \end{aligned} \tag{3.75}$$

where $i = 2, 3, \dots, n-1$. Following the argument of (3.29)–(3.66), we obtain

$$\ln H_i < u_i^- < \max\{\ln A_i^-, \ln B_i^-\} < \min\{\ln A_i^+, \ln B_i^+\} < u_i^+ < \ln l_i^+ \quad (1 \leq i \leq n). \tag{3.76}$$

Equation (3.76) gives that u belongs to one of $\Omega_k \cap \mathbb{R}^n$, $k = 1, 2, \dots, 2^n$. This contradicts the fact that $u \in \partial\Omega_k \cap \mathbb{R}^n$, $k = 1, 2, \dots, 2^n$. This proves $\phi(u, \mu) \neq (0, 0, \dots, 0)^T$ holds. Note that the system of algebraic equations

$$\begin{aligned} \bar{a}_1 - \bar{b}_1 e^{x_1} - \bar{h}_1 e^{-x_1} &= -\frac{2g_1}{\omega}, \\ &\vdots \\ \bar{c}_{i,i-1} l_{i-1}^+ - \bar{b}_i e^{x_i} - \bar{h}_i e^{-x_i} &= -\frac{2g_i}{\omega}, \\ &\vdots \\ \bar{c}_{n,n-1} l_{n-1}^+ - \bar{b}_n e^{x_n} - \bar{h}_n e^{-x_n} &= -\frac{2g_n}{\omega}, \end{aligned} \quad (3.77)$$

has 2^n distinct solutions since (H_1) , (H_2) , and (H_3) hold, $(x_1^*, x_2^*, \dots, x_n^*) = (\ln \hat{x}_1, \ln \hat{x}_2, \dots, \ln \hat{x}_n)$, where $x_1^\pm = ((\bar{a}_1 + (2g_1/\omega)) \pm \sqrt{(\bar{a}_1 + (2g_1/\omega))^2 - 4\bar{b}_1\bar{h}_1})/2\bar{b}_1$, $x_k^\pm = ((\bar{c}_{k,k-1} l_{k-1}^+ + (2g_k/\omega)) \pm \sqrt{(\bar{c}_{k,k-1} l_{k-1}^+ + (2g_k/\omega))^2 - 4\bar{b}_k\bar{h}_k})/2\bar{b}_k$ ($k = 2, 3, \dots, n$), $\hat{x}_i = x_i^-$ or $\hat{x}_i = x_i^+$, $i = 1, 2, \dots, n$. Similar to the proof of Lemma 3.3, it is easy to verify that

$$\ln H_i^- < \ln x_i^- < \max\{\ln B_i^-, \ln A_i^-\} < \min\{\ln A_i^+, \ln B_i^+\} < \ln x_i^+ < \ln l_i^+, \quad i = 1, 2, \dots, n. \quad (3.78)$$

Therefore, $(x_1^*, x_2^*, \dots, x_n^*)$ uniquely belongs to the corresponding Ω_k . Since $\text{Ker } L = \text{Im } Q$, we can take $J = I$. A direct computation gives, for $k = 1, 2, \dots, 2^n$,

$$\begin{aligned} \deg\{JQN(u, 0), \Omega_k \cap \text{Ker } L, (0, 0, \dots, 0)^T\} &= \deg\{\phi(u, 1), \Omega_k \cap \text{Ker } L, (0, 0, \dots, 0)^T\} \\ &= \deg\{\phi(u, 0), \Omega_k \cap \text{Ker } L, (0, 0, \dots, 0)^T\} \\ &= \text{sign} \left[\prod_{i=1}^n \left(-\bar{b}_i x_i^* + \frac{\bar{h}_i}{x_i^*} \right) \right]. \end{aligned} \quad (3.79)$$

Since $(\bar{a}_1 + (2g_1/\omega)) - \bar{b}_1 x_1^* - (\bar{h}_1/x_1^*) = 0$, $(\bar{c}_{i,i-1} l_{i-1}^+ + (2g_i/\omega)) - \bar{b}_i x_i^* - (\bar{h}_i/x_i^*) = 0$ ($i = 2, 3, \dots, n$), then

$$\begin{aligned} \deg\{JQN(u, 0), \Omega_k \cap \text{Ker } L, (0, 0, \dots, 0)^T\} \\ = \text{sign} \left[\prod_{i=2}^n \left(\bar{a}_1 + \frac{2g_1}{\omega} - 2\bar{b}_1 x_1^* \right) \left(\bar{c}_{i,i-1} l_{i-1}^+ + \frac{2g_i}{\omega} - 2\bar{b}_i x_i^* \right) \right] = \pm 1, \quad k = 1, 2, \dots, 2^n. \end{aligned} \quad (3.80)$$

So far, we have proved that Ω_k ($k = 1, 2, \dots, 2^n$) satisfies all the assumptions in Lemma 3.1. Hence, system (1.3) has at least 2^n different ω -periodic solutions. This completes the proof of Theorem 3.5. $\square$

If $n = 2$, then (1.3) is changed into the predator-prey system (3.81). The following Corollary 3.6 gives the existence of multiple periodic solutions for system (3.81).

Corollary 3.6. *For the following predator-prey system on ω -periodic time scales $\mathbb{T}$,*

$$\begin{aligned} u_1^\Delta(t) &= a_1(t) - b_1(t)e^{u_1(t-\tau_{11}(t))} - c_{12}(t)e^{u_2(t-\tau_{12}(t))} - h_1(t)e^{-u_1(t)}, \quad t \neq t_k, \\ u_2^\Delta(t) &= -d_2(t) - b_2(t)e^{u_2(t-\tau_{22}(t))} + c_{21}(t)e^{u_1(t-\tau_{21}(t))} - h_2(t)e^{-u_2(t)}, \quad t \neq t_k, \\ \Delta u_i(t_k) &= u_i(t_k^+) - u_i(t_k^-) = \ln(1 + g_{ik}), \quad i = 1, 2, \quad t = t_k \end{aligned} \quad (3.81)$$

if the inequalities $(\bar{a}_1 + (2g_1/\omega) - \bar{c}_{12}l_2^+)e^{-(1/2)(\bar{a}_1\omega + G_1)} > 2\sqrt{\bar{b}_1\bar{h}_1}$ and $((2g_2/\omega) - \bar{d}_2)e^{-(1/2)(\bar{c}_{21}\omega + G_2)} > 2\sqrt{\bar{b}_2\bar{h}_2}$ hold, then system (3.81) at least exist four ω -periodic solutions.

4. Illustrative Examples

Consider the following time-delay predator-prey system with impulses on time scales:

$$\begin{aligned} u_1^\Delta(t) &= a_1(t) - b_1(t)e^{u_1(t-\tau_{11}(t))} - c_{12}(t)e^{u_2(t-\tau_{12}(t))} - h_1(t)e^{-u_1(t)}, \\ u_2^\Delta(t) &= -d_2(t) - b_2(t)e^{u_2(t-\tau_{22}(t))} + c_{21}(t)e^{u_1(t-\tau_{21}(t))} - h_2(t)e^{-u_2(t)}, \\ \Delta x_1(1) &= -2, \quad \Delta x_1(3) = 2, \quad \Delta x_2(1) = -1, \quad \Delta x_2(3) = 3, \end{aligned} \quad (4.1)$$

where $a_1(t) = (3 + \sin(0.5\pi t))/2$, $b_1(t) = (2 + \sin(0.5\pi t))/10e^4$, $h_1(t) = (4 + \cos(0.5\pi t))/20e^4$, $\tau_{11}(t) = |\sin(0.5\pi t)|$, $\tau_{12}(t) = |\cos(0.5\pi t)|$, $d_2(t) = (2 + \cos(0.5\pi t))/25$, $b_2(t) = (5 + \cos(0.5\pi t))/10$, $h_2(t) = (2 + \cos(0.5\pi t))/25e^2$, $c_{21}(t) = (2 + \sin(\pi t))/8$, $\tau_{22}(t) = |\cos(\pi t)|$, $\tau_{21}(t) = |\sin(\pi t)|$.

If $\mathbb{T} = \mathbb{R}$, then (4.1) reduces to the following system:

$$\begin{aligned} \dot{u}_1(t) &= a_1(t) - b_1(t)e^{u_1(t-\tau_{11}(t))} - c_{12}(t)e^{u_2(t-\tau_{12}(t))} - h_1(t)e^{-u_1(t)}, \\ \dot{u}_2(t) &= -d_2(t) - b_2(t)e^{u_2(t-\tau_{22}(t))} + c_{21}(t)e^{u_1(t-\tau_{21}(t))} - h_2(t)e^{-u_2(t)}, \\ \Delta x_1(1) &= -2, \quad \Delta x_1(3) = 2, \quad \Delta x_2(1) = -1, \quad \Delta x_2(3) = 3. \end{aligned} \quad (4.2)$$

Let $x_i(t) = e^{u_i(t)}$ ($i = 1, 2$), then (4.2) can be changed into the following system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)(a_1(t) - b_1(t)x_1(t - \tau_{11}(t)) - c_{12}(t)x_2(t - \tau_{12}(t))) - h_1(t), \\ \dot{x}_2(t) &= x_2(t)(-d_2(t) - b_2(t)x_2(t - \tau_{22}(t)) + c_{21}(t)x_1(t - \tau_{21}(t))) - h_2(t), \\ \Delta x_1(1) &= e^{-2} - 1, \quad \Delta x_1(3) = e^2 - 1, \quad \Delta x_2(1) = e^{-1} - 1, \quad \Delta x_2(3) = e^3 - 1. \end{aligned} \quad (4.3)$$

Since

$$\begin{aligned}
 l_1^\pm &= \frac{(\bar{a}_1 + (2g_1/\omega))e^{\bar{a}_1\omega+G_1} \pm \sqrt{((\bar{a}_1 + (2g_1/\omega))e^{\bar{a}_1\omega+G_1})^2 - 4\bar{b}_1\bar{h}_1e^{\bar{a}_1\omega+G_1}}}{2\bar{b}_1} \\
 &= \frac{(15e^8 \pm 3\sqrt{225e^{16} - 16})}{4e^{-4}}, \\
 l_2^\pm &= \frac{(\bar{c}_{21} + (2g_2/\omega))e^{\bar{c}_{21}\omega+G_2} \pm \sqrt{((\bar{c}_{21} + (2g_2/\omega))e^{\bar{c}_{21}\omega+G_2})^2 - 4\bar{b}_2\bar{h}_2e^{\bar{c}_{21}\omega+G_2}}}{2\bar{b}_2} \\
 &= \frac{15e^4 \pm \sqrt{225e^8 - 64}}{20e^{-2}}.
 \end{aligned} \tag{4.4}$$

Take $c_{12}(t) = c_{12}(t + 2\pi) > 0$, such that $\bar{c}_{12} = 1/l_2^+$, then

$$\begin{aligned}
 \left(\bar{a}_1 + \frac{2g_1}{\omega} - \bar{c}_{12}l_2^+\right)e^{-(1/2)(\bar{a}_1\omega+G_1)} &= \frac{1}{2}e^{-4} > \frac{2}{5}e^{-4} = 2\sqrt{\bar{b}_1\bar{h}_1}, \\
 \left(\frac{2g_2}{\omega} - \bar{d}_2\right)e^{-(1/2)(\bar{c}_{21}\omega+G_2)} &= \frac{21}{25}e^{-2} > \frac{2}{5}e^{-2} = 2\sqrt{\bar{b}_2\bar{h}_2}.
 \end{aligned} \tag{4.5}$$

Therefore, all conditions of Corollary 3.6 are satisfied. By Corollary 3.6, system (4.1) has at least four 4-periodic solutions.

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