Research Article

Two-Step Newton-Tikhonov Method for Hammerstein-Type Equations: Finite-Dimensional Realization

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Finite-dimensional realization of a Two-Step Newton-Tikhonov method is considered for obtaining a stable approximate solution to nonlinear ill-posed Hammerstein-type operator equations $KF(x) = f$. Here $F : D(F) \subseteq X \rightarrow X$ is nonlinear monotone operator, $K : X \rightarrow Y$ is a bounded linear operator, X is a real Hilbert space, and Y is a Hilbert space. The error analysis for this method is done under two general source conditions, the first one involves the operator K and the second one involves the Fréchet derivative of F at an initial approximation x_0 of the the solution $\hat{x}$: balancing principle of Pereverzev and Schock (2005) is employed in choosing the regularization parameter and order optimal error bounds are established. Numerical illustration is given to confirm the reliability of our approach.

1. Introduction

Tikhonov's regularization (e.g., [1]) method has been used extensively to stabilize the approximate solution of nonlinear ill-posed problems. In recent years, increased emphasis has been placed on iterative regularization procedures [2, 3] for obtaining the approximate solution of such problems. In this paper, we examine the use of iterative regularization procedures for Hammerstein-type [4, 5] equations of the form

$$KF(x) = f, \quad (1.1)$$

where $F : D(F) \subseteq X \rightarrow X$, is nonlinear monotone operator, $F'(\cdot)^{-1}$ does not exists, and $K : X \rightarrow Y$ is a bounded linear operator. Throughout this paper, $D(F)$ is the domain of F , $F'(\cdot)$ is the Fréchet derivative of F , X is a real Hilbert space, and Y is a Hilbert space. The inner product and the corresponding norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

Recall that [6] the operator F is said to be a monotone operator if $\langle F(x) - F(y), x - y \rangle \geq 0$, for all $x, y \in D(F)$.

It is assumed throughout that $f^\delta \in Y$ are the available noisy data with

$$\|f - f^\delta\| \leq \delta, \quad (1.2)$$

and F possesses a uniformly bounded Fréchet derivative for each $x \in D(F)$ (cf. [7]), that is,

$$\|F'(x)\| \leq M, \quad x \in D(F) \quad (1.3)$$

for some M .

Observe that the solution x of (1.1) with f^δ in place of f can be obtained by first solving

$$Kz = f^\delta \quad (1.4)$$

for z and then solving the nonlinear problem

$$F(x) = z. \quad (1.5)$$

In [4, 5, 8] this was exploited. In [4], z is approximated with z_α^δ where

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1} K^* f^\delta, \quad \alpha > 0, \delta > 0, \quad (1.6)$$

and then solve (1.5) iteratively using the following Newton-type procedure:

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_0)^{-1} \left(F(x_{n,\alpha}^\delta) - z_\alpha^\delta \right) \quad (1.7)$$

with $x_{0,\alpha}^\delta := x_0$ and obtained local linear convergence. Here and below $x_0 \in D(F)$ is a known initial approximation of the solution $\hat{x}$ of (1.1) such that $\|x_0 - \hat{x}\| \leq \rho$.

In [8], to solve (1.5), George and Kunhanandan used the iteration

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_{n,\alpha}^\delta)^{-1} \left(F(x_{n,\alpha}^\delta) - z_\alpha^\delta \right), \quad x_{0,\alpha}^\delta := x_0, \quad (1.8)$$

where

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1} K^* (f^\delta - KF(x_0)) + F(x_0), \quad (1.9)$$

and obtained local quadratic convergence.

A sequence (x_n) in X with $\lim x_n = x^*$ is said to be convergent of order $p > 1$, if there exist positive reals β and γ such that, for all $n \in N$,

$$\|x_n - x^*\| \leq \beta e^{-\gamma p^n}. \quad (1.10)$$

If a sequence (x_n) satisfies $\|x_n - x^*\| \leq \beta q^n$, $0 < q < 1$, then (x_n) is said to be linearly convergent.

As in [8], it is assumed that the solution $\hat{x}$ of (1.1) satisfies

$$\|F(\hat{x}) - F(x_0)\| = \min\{\|F(x) - F(x_0)\| : KF(x) = f, x \in D(F)\}. \quad (1.11)$$

The regularization parameter α is chosen from a finite set

$$D_N = \{\alpha_i : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N\} \quad (1.12)$$

using the adaptive method considered by Pereverzev and Schock in [9].

In [10], Argyros and Hilout considered a method called Two-Step Directional Newton Method (TSDNM) for approximating a zero x^* of a differentiable function F defined on a convex subset $\mathfrak{D}$ of a Hilbert space H with values in $\mathbb{R}$. Motivated by TSDNM, in [11], we propose a Two-Step Newton-Tikhonov Methods (TSNTM) for solving (1.1).

In fact, in [11] we consider two cases of F , in the first case we assume that $F'(x_0)^{-1}$ exist and in the second case we assume F is monotone. In this paper we consider the finite-dimensional realization of the second case, that is, F is monotone. The finite-dimensional realization of the method and associated algorithm are proposed for which local-cubic convergence is established theoretically and validated numerically.

The organization of this paper is as follows. Section 2 deals with Discretized Tikhonov regularization and Section 3 investigates the convergence of the Discretized TSNTM. Section 4 discusses the algorithm and finally the paper ends with a numerical example in Section 5.

2. Discretized Tikhonov Regularization

This section deals with discretized Tikhonov regularized solution $z_\alpha^{h,\delta}$ of (1.4) and (an a priori and an a posteriori) error estimate for $\|F(\hat{x}) - z_\alpha^{h,\delta}\|$ using an error estimate for $\|F(\hat{x}) - z_\alpha\|$ from [8].

The following assumption is used in [8] to obtain the error estimate.

Assumption 2.1. There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \rightarrow (0, \infty)$ with $a \geq \|K\|^2$ satisfying

$$(i) \lim_{\lambda \rightarrow 0} \varphi(\lambda) = 0,$$

(ii)

$$\sup_{\lambda > 0} \frac{\alpha \varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha) \quad \forall \lambda \in (0, a], \quad (2.1)$$

(iii)

$$F(\hat{x}) - F(x_0) = \varphi(K^*K)w \quad (2.2)$$

for some $w \in X$ such that $\|w\| \leq 1$.

Remark 2.2. The functions

$$\varphi(\lambda) := \lambda^\nu, \quad \lambda > 0, \quad (2.3)$$

for $0 < \nu \leq 1$ and

$$\varphi(\lambda) = \begin{cases} \left[\ln \frac{1}{\lambda} \right]^{-p}, & 0 < \lambda \leq e^{-(p+1)}, \\ 0 & \text{otherwise} \end{cases} \quad (2.4)$$

for $p \geq 0$ satisfy the above assumption (see [12]).

Theorem 2.3 (cf. [8], Theorem 4.3). *Let $z_\alpha := z_\alpha^0$ be as in (1.9) and Assumption 2.1 holds. Then*

$$\|F(\hat{x}) - z_\alpha\| \leq \varphi(\alpha). \quad (2.5)$$

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections on X . Let

$$\begin{aligned} \varepsilon_h &:= \|K(I - P_h)\|, \\ \tau_h &:= \|F'(x)(I - P_h)\|, \quad \forall x \in D(F) \end{aligned} \quad (2.6)$$

and $\{b_h : h > 0\}$ is such that $\lim_{h \rightarrow 0} (\|(I - P_h)x_0\|/b_h) = 0$, $\lim_{h \rightarrow 0} (\|(I - P_h)F(x_0)\|/b_h) = 0$, and $\lim_{h \rightarrow 0} b_h = 0$. We assume that $\varepsilon_h \rightarrow 0$ and $\tau_h \rightarrow 0$ as $h \rightarrow 0$. The above assumption is satisfied if, $P_h \rightarrow I$ pointwise and if K and $F'(x)$ are compact operators. Further we assume that $\varepsilon_h < \varepsilon_0$, $\tau_h \leq \tau_0$, $b_h \leq b_0$, and $\delta \in (0, \delta_0]$ where $\delta_0 + \varepsilon_0 < (2/(2M + 3))\sqrt{\alpha_0}$. The discretized Tikhonov regularization method for the regularized equation (1.4) consists of solving the equation

$$(P_h K^* K P_h + \alpha P_h) \left(z_\alpha^{h,\delta} - P_h F(x_0) \right) = P_h K^* \left[f^\delta - K F(x_0) \right]. \quad (2.7)$$

Theorem 2.4. *Suppose assumptions in Theorem 2.3 hold. Let $z_\alpha^{h,\delta}$ be as in (2.7) and $b_h \leq (\delta + \varepsilon_h)/\sqrt{\alpha}$. Then*

$$\left\| F(\hat{x}) - z_\alpha^{h,\delta} \right\| \leq C \left(\varphi(\alpha) + \left(\frac{\delta + \varepsilon_h}{\sqrt{\alpha}} \right) \right), \quad (2.8)$$

where $C = (1/2) \max\{M\rho, 1\} + 1$.

Proof. Let $z_\alpha = (K^*K + \alpha I)^{-1}K^*(f - KF(x_0)) + F(x_0)$. Then

$$\begin{aligned}
\|z_\alpha - z_\alpha^h\| &= \|(K^*K + \alpha I)^{-1}K^*(f - KF(x_0)) \\
&\quad - (P_h K^* K P_h + \alpha I)^{-1}P_h K^*(f - KF(x_0)) + F(x_0) - P_h F(x_0)\| \\
&\leq \|(P_h K^* K P_h + \alpha P_h)^{-1}P_h K^*(K P_h - K)(K^*K + \alpha I)^{-1}K^*K[F(\hat{x}) - F(x_0)]\| \\
&\quad + \|(I - P_h)F(x_0)\| \\
&\leq \|F(\hat{x}) - F(x_0)\| \frac{\varepsilon_h}{2\sqrt{\alpha}} + b_h \\
&\leq \left\| \int_0^1 F'(x_0 + t(\hat{x} - x_0))(\hat{x} - x_0) dt \right\| \frac{\varepsilon_h}{2\sqrt{\alpha}} + b_h \\
&\leq M \rho \frac{\varepsilon_h}{2\sqrt{\alpha}} + b_h, \\
\|z_\alpha^h - z_\alpha^{h,\delta}\| &= \|(P_h K^* K P_h + \alpha I)^{-1}P_h K^*(f - f^\delta)\| \\
&\leq \frac{\delta}{2\sqrt{\alpha}}.
\end{aligned} \tag{2.9}$$

Now the result follows from (2.9), Theorem 2.3 and the following triangle inequality:

$$\|F(\hat{x}) - z_{\alpha,h}^\delta\| \leq \|F(\hat{x}) - z_\alpha\| + \|z_\alpha - z_\alpha^h\| + \|z_\alpha^h - z_\alpha^{h,\delta}\|. \tag{2.10}$$

□

2.1. A Priori Choice of the Parameter

Note that the estimate $\varphi(\alpha) + (\delta + \varepsilon_h)/\sqrt{\alpha}$ in (2.8) is of optimal order for the choice $\alpha := \alpha(\delta, h)$ which satisfies $\varphi(\alpha(\delta, h)) = (\delta + \varepsilon_h)/\sqrt{\alpha(\delta, h)}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}$, $0 < \lambda \leq a$. Then we have $\delta + \varepsilon_h = \sqrt{\alpha(\delta, h)}\varphi(\alpha(\delta, h)) = \psi(\varphi(\alpha(\delta, h)))$ and

$$\alpha(\delta, h) = \varphi^{-1}(\psi^{-1}(\delta + \varepsilon_h)). \tag{2.11}$$

So the relation (2.8) leads to $\|F(\hat{x}) - z_\alpha^{h,\delta}\| \leq 2C\psi^{-1}(\delta + \varepsilon_h)$.

2.2. An Adaptive Choice of the Parameter

In this subsection, we consider the balancing principle established by Pereverzev and Shock [9] for choosing the parameter α . Let

$$D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N\} \tag{2.12}$$

be the set of possible values of the parameter α .

Let

$$l := \max \left\{ i : \varphi(\alpha_i) \leq \frac{\delta + \varepsilon_h}{\sqrt{\alpha_i}} \right\} < N, \quad (2.13)$$

$$k = \max \{ i : \alpha_i \in D_N^+ \}, \quad (2.14)$$

where $D_N^+ = \{ \alpha_i \in D_N : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \leq 4C(\delta + \varepsilon_h)/\sqrt{\alpha_j}, j = 0, 1, 2, \dots, i-1 \}$.

We use the following theorem, proof of which is analogous to the proof of Theorem 4.3 in [8], for our error analysis.

Theorem 2.5 (cf. [8], Theorem 4.3). *Let l be as in (2.13), let k be as in (2.14), and let $z_{\alpha_k}^{h,\delta}$ be as in (2.7) with $\alpha = \alpha_k := \mu^k \alpha_0$, $\mu > 1$. Then $l \leq k$ and*

$$\|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\| \leq C \left(2 + \frac{4\mu}{\mu-1} \right) \mu \psi^{-1}(\delta + \varepsilon_h). \quad (2.15)$$

3. Discretized Two-Step Newton Method (DTSNM)

We need the following assumptions for the convergence of DTSNM and to obtain the error estimate.

Assumption 3.1 (cf. [7], Assumption 3 (A3)). There exists a constant $k_0 \geq 0$ such that for every $x, u \in D(F)$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v)$, $\|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|$.

Assumption 3.2. There exists a continuous, strictly monotonically increasing function $\varphi_1 : (0, b] \rightarrow (0, \infty)$ with $b \geq \|F'(x_0)\|$ satisfying

- (i) $\lim_{\lambda \rightarrow 0} \varphi_1(\lambda) = 0$,
- (ii)

$$\sup_{\lambda > 0} \frac{\alpha \varphi_1(\lambda)}{\lambda + \alpha} \leq \varphi_1(\alpha) \quad \forall \lambda \in (0, b], \quad (3.1)$$

- (iii) there exists $v \in X$ with $\|v\| \leq 1$ (cf. [6]) such that

$$x_0 - \hat{x} = \varphi_1(F'(x_0))v, \quad (3.2)$$

- (iv) for each $x \in B_r(x_0) := \{x : \|x - x_0\| < r\}$ there exists a bounded linear operator $G(x, x_0)$ (cf. [13]) such that

$$F'(x) = F'(x_0)G(x, x_0) \quad (3.3)$$

with $\|G(x, x_0)\| \leq K_1$.

First we consider a DTSNM for approximating the zero $x_{c,\alpha_k}^{h,\delta}$ of

$$P_h \left(F(x) + \frac{\alpha_k}{c} (x - x_0) \right) = P_h z_{\alpha_k}^{h,\delta} \quad (3.4)$$

and then we show that $x_{c,\alpha_k}^{h,\delta}$ is an approximation to the solution $\hat{x}$ of (1.1) where $c \leq \alpha_k$. For an initial guess $x_0 \in X$ and for $R(x) := P_h F'(x) P_h + (\alpha_k/c) P_h$, the DTSNM is defined as

$$y_{n,\alpha_k}^{h,\delta} = x_{n,\alpha_k}^{h,\delta} - R(x_{n,\alpha_k}^{h,\delta})^{-1} P_h \left[F(x_{n,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta} + \frac{\alpha_k}{c} (x_{n,\alpha_k}^{h,\delta} - x_{0,\alpha_k}^{h,\delta}) \right], \quad (3.5)$$

$$x_{n+1,\alpha_k}^{h,\delta} = y_{n,\alpha_k}^{h,\delta} - R(x_{n,\alpha_k}^{h,\delta})^{-1} P_h \left[F(y_{n,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta} + \frac{\alpha_k}{c} (y_{n,\alpha_k}^{h,\delta} - x_{0,\alpha_k}^{h,\delta}) \right], \quad (3.6)$$

where $x_{0,\alpha_k}^{h,\delta} := P_h x_0$. Note that with the above notation

$$\begin{aligned} \left\| R(x_{n,\alpha_k}^{h,\delta})^{-1} P_h F'(x_{n,\alpha_k}^{h,\delta}) \right\| &= \left\| \left(P_h F'(x_{n,\alpha_k}^{h,\delta}) P_h + \frac{\alpha_k}{c} P_h \right)^{-1} P_h F'(x_{n,\alpha_k}^{h,\delta}) \right\| \\ &= \left\| \left(P_h F'(x_{n,\alpha_k}^{h,\delta}) P_h + \frac{\alpha_k}{c} P_h \right)^{-1} P_h F'(x_{n,\alpha_k}^{h,\delta}) [P_h + I - P_h] \right\| \\ &\leq \left\| \left(P_h F'(x_{n,\alpha_k}^{h,\delta}) P_h + \frac{\alpha_k}{c} P_h \right)^{-1} P_h F'(x_{n,\alpha_k}^{h,\delta}) P_h \right\| \\ &\quad + \left\| \left(P_h F'(x_{n,\alpha_k}^{h,\delta}) P_h + \frac{\alpha_k}{c} P_h \right)^{-1} P_h F'(x_{n,\alpha_k}^{h,\delta}) (I - P_h) \right\| \\ &\leq 1 + \frac{\left\| P_h F'(x_{n,\alpha_k}^{h,\delta}) (I - P_h) \right\|}{\alpha_k/c} \\ &\leq 1 + \tau_h \leq 1 + \tau_0. \end{aligned} \quad (3.7)$$

Let

$$e_{n,\alpha_k}^{h,\delta} := \left\| y_{n,\alpha_k}^{h,\delta} - x_{n,\alpha_k}^{h,\delta} \right\|, \quad \forall n \geq 0. \quad (3.8)$$

and let k_0 be such that

$$\frac{k_0^2}{8} (4 + 3k_0(1 + \tau_0))(1 + \tau_0)^2 < 1. \quad (3.9)$$

Remark 3.3. Note that the above assumption is satisfied if we choose $k_0 < \min\{1, (1/(1 + \tau_0))\sqrt{8/4 + 3(1 + \tau_0)}\}$.

Let $g : (0, 1) \rightarrow (0, 1)$ be the function defined by

$$g(t) = \frac{k_0^2}{8} (4 + 3k_0(1 + \tau_0)t)(1 + \tau_0)^2 t^2 \quad \forall t \in (0, 1). \quad (3.10)$$

Let $\|\hat{x} - x_0\| \leq \rho$, with

$$\begin{aligned} \rho &< \frac{1}{M} \left(1 - \left(\frac{3}{2} + M \right) \frac{\delta_0 + \varepsilon_0}{\sqrt{\alpha_0}} \right), \\ \gamma_\rho &:= M\rho + \left(\frac{3}{2} + M \right) \left(\frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}} \right). \end{aligned} \quad (3.11)$$

Theorem 3.4. Let $e_{n,\alpha_k}^{h,\delta}$ and g be as in (3.8) and (3.10), respectively, and let $x_{n,\alpha_k}^{h,\delta}$ and $y_{n,\alpha_k}^{h,\delta}$ be as in (3.6) and (3.5), respectively, with $\delta \in (0, \delta_0]$, $\alpha = \alpha_k$ and $\varepsilon_h \in (0, \varepsilon_0]$. Then the following holds:

(a)

$$\left\| x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta} \right\| \leq (1 + \tau_0) \frac{k_0 e_{n-1,\alpha_k}^{h,\delta}}{2} \left\| y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta} \right\|; \quad (3.12)$$

(b)

$$\left\| x_{n,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta} \right\| \leq \left(1 + (1 + \tau_0) \frac{k_0 e_{n-1,\alpha_k}^{h,\delta}}{2} \right) \left\| y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta} \right\|; \quad (3.13)$$

(c)

$$\left\| y_{n,\alpha_k}^{h,\delta} - x_{n,\alpha_k}^{h,\delta} \right\| \leq g(e_{n-1,\alpha_k}^{h,\delta}) \left\| y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta} \right\|; \quad (3.14)$$

(d)

$$g(e_{n,\alpha_k}^{h,\delta}) \leq g(\gamma_\rho)^{3^n}, \quad \forall n \geq 0; \quad (3.15)$$

(e)

$$e_{n,\alpha_k}^{h,\delta} \leq g(\gamma_\rho)^{(3^n-1)/2} \gamma_\rho, \quad \forall n \geq 0. \quad (3.16)$$

Proof. Observe that

$$\begin{aligned} x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta} &= y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta} - R(x_{n-1,\alpha_k}^{h,\delta})^{-1} \\ &\quad \times P_h \left(F(y_{n-1,\alpha_k}^{h,\delta}) - F(x_{n-1,\alpha_k}^{h,\delta}) + \frac{\alpha_k}{c} (y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}) \right) \\ &= R(x_{n-1,\alpha_k}^{h,\delta})^{-1} \left[R(x_{n-1,\alpha_k}^{h,\delta}) (y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}) \right. \\ &\quad \left. - P_h \left(F(y_{n-1,\alpha_k}^{h,\delta}) - F(x_{n-1,\alpha_k}^{h,\delta}) \right) - \frac{\alpha_k}{c} (y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}) \right] \end{aligned}$$

$$\begin{aligned}
&= R(x_{n-1,\alpha_k}^{h,\delta})^{-1} \left[\left(P_h F'(x_{n-1,\alpha_k}^{h,\delta}) P_h + \frac{\alpha_k}{c} P_h \right) (y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}) \right. \\
&\quad \left. - P_h \left(F(y_{n-1,\alpha_k}^{h,\delta}) - F(x_{n-1,\alpha_k}^{h,\delta}) \right) - \frac{\alpha_k}{c} (y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}) \right] \\
&= R(x_{n-1,\alpha_k}^{h,\delta})^{-1} P_h \\
&\quad \times \int_0^1 \left[F'(x_{n-1,\alpha_k}^{h,\delta}) - F'(x_{n-1,\alpha_k}^{h,\delta} + t(y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta})) \right] P_h (y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}) dt.
\end{aligned} \tag{3.17}$$

Now by Assumption 3.1 and (3.7) we have

$$\begin{aligned}
\|x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta}\| &\leq (1 + \tau_0) \left\| \int_0^1 \Phi(x_{n-1,\alpha_k}^{h,\delta}, x_{n-1,\alpha_k}^{h,\delta} + t(y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}), y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}) dt \right\| \\
&\leq (1 + \tau_0) \frac{k_0}{2} \|y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}\|^2.
\end{aligned} \tag{3.18}$$

This proves (a). Now (b) follows from (a) and the triangle inequality

$$\|x_{n,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}\| \leq \|x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta}\| + \|y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}\|. \tag{3.19}$$

To prove (c) we observe that

$$\begin{aligned}
e_{n,\alpha_k}^{h,\delta} &= \left\| x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta} - R(x_{n,\alpha_k}^{h,\delta})^{-1} P_h \left(F(x_{n,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta} + \frac{\alpha_k}{c} (x_{n,\alpha_k}^{h,\delta} - x_{0,\alpha_k}^{h,\delta}) \right) \right. \\
&\quad \left. + R(x_{n-1,\alpha_k}^{h,\delta})^{-1} P_h \left(F(y_{n-1,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta} + \frac{\alpha_k}{c} (y_{n-1,\alpha_k}^{h,\delta} - x_{0,\alpha_k}^{h,\delta}) \right) \right\| \\
&= \left\| x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta} - R(x_{n,\alpha_k}^{h,\delta})^{-1} P_h \left(F(x_{n,\alpha_k}^{h,\delta}) - F(y_{n-1,\alpha_k}^{h,\delta}) + \frac{\alpha_k}{c} (x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta}) \right) \right. \\
&\quad \left. + \left[R(x_{n-1,\alpha_k}^{h,\delta})^{-1} - R(x_{n,\alpha_k}^{h,\delta})^{-1} \right] \times P_h \left(F(y_{n-1,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta} + \frac{\alpha_k}{c} (y_{n-1,\alpha_k}^{h,\delta} - x_{0,\alpha_k}^{h,\delta}) \right) \right\| \\
&:= \Gamma_1 + \Gamma_2,
\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
\Gamma_1 &:= \left\| x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta} - R(x_{n,\alpha_k}^{h,\delta})^{-1} P_h \left(F(x_{n,\alpha_k}^{h,\delta}) - F(y_{n-1,\alpha_k}^{h,\delta}) + \frac{\alpha_k}{c} (x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta}) \right) \right\|, \\
\Gamma_2 &:= \left\| \left[R(x_{n-1,\alpha_k}^{h,\delta})^{-1} - R(x_{n,\alpha_k}^{h,\delta})^{-1} \right] P_h \left(F(y_{n-1,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta} + \frac{\alpha_k}{c} (y_{n-1,\alpha_k}^{h,\delta} - x_{0,\alpha_k}^{h,\delta}) \right) \right\|.
\end{aligned} \tag{3.21}$$

Note that

$$\begin{aligned}
\Gamma_1 &= \left\| R(x_{n,\alpha_k}^{h,\delta})^{-1} \left[P_h F'(x_{n,\alpha_k}^{h,\delta}) P_h (x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta}) - P_h (F(x_{n,\alpha_k}^{h,\delta}) - F(y_{n-1,\alpha_k}^{h,\delta})) \right] \right\| \\
&= \left\| R(x_{n,\alpha_k}^{h,\delta})^{-1} P_h \int_0^1 \left[F'(x_{n,\alpha_k}^{h,\delta}) - F'(y_{n-1,\alpha_k}^{h,\delta} + t(x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta})) \right] \times P_h (x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta}) dt \right\| \\
&\leq (1 + \tau_0) \left\| \int_0^1 \Phi(x_{n,\alpha_k}^{h,\delta}, y_{n-1,\alpha_k}^{h,\delta} + t(x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta}), x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta}) dt \right\| \\
&\leq (1 + \tau_0) \frac{k_0}{2} \|x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta}\|^2.
\end{aligned} \tag{3.22}$$

The last but one step follows from Assumption 3.1 and (3.7). Similarly one can prove that

$$\Gamma_2 \leq (1 + \tau_0) k_0 \|x_{n,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}\| \|x_{n,\alpha_k}^{h,\delta} - y_{n-1,\alpha_k}^{h,\delta}\|. \tag{3.23}$$

Thus from (3.20), (3.22), (3.23), (a) and (b) we have

$$\begin{aligned}
e_{n,\alpha_k}^{h,\delta} &\leq (1 + \tau_0)^2 \left(\frac{k_0^2}{2} + \frac{3k_0^3(1 + \tau_0)}{8} \|y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}\| \right) \\
&\quad \times \|y_{n-1,\alpha_k}^{h,\delta} - x_{n-1,\alpha_k}^{h,\delta}\|^3 \\
&\leq g(e_{n-1,\alpha_k}^{h,\delta}) e_{n-1,\alpha_k}^{h,\delta}.
\end{aligned} \tag{3.24}$$

Again since for $\mu \in (0, 1)$, $g(\mu t) \leq \mu^2 g(t)$, for all $t \in (0, 1)$, by (3.24) we have

$$g(e_{n,\alpha_k}^{h,\delta}) \leq g(e_{0,\alpha_k}^{h,\delta})^{3^n}, \tag{3.25}$$

$$\begin{aligned}
e_{n,\alpha_k}^{h,\delta} &\leq g^3(e_{n-2,\alpha_k}^{h,\delta}) e_{n-1,\alpha_k}^{h,\delta} \leq g^3(e_{n-2,\alpha_k}^{h,\delta}) g^3(e_{n-3,\alpha_k}^{h,\delta}) e_{n-2,\alpha_k}^{h,\delta} \cdots g(e_{0,\alpha_k}^{h,\delta}) e_{0,\alpha_k}^{h,\delta} \\
&\leq g(e_{0,\alpha_k}^{h,\delta})^{3^{n-1} + 3^{n-2} + \cdots + 1} e_{0,\alpha_k}^{h,\delta} \\
&\leq g(e_{0,\alpha_k}^{h,\delta})^{(3^n - 1)/2} e_{0,\alpha_k}^{h,\delta}
\end{aligned} \tag{3.26}$$

provided $e_{n,\alpha_k}^{h,\delta} < 1$, for all $n \geq 0$. From (3.26) it is clear that, $e_{n,\alpha_k}^{h,\delta} \leq 1$ if $e_{0,\alpha_k}^{h,\delta} \leq 1$. This can be seen as follows:

$$\begin{aligned}
e_{0,\alpha_k}^{h,\delta} &= \|y_{0,\alpha_k}^{h,\delta} - x_{0,\alpha_k}^{h,\delta}\| = \|y_{0,\alpha_k}^{h,\delta} - P_h x_0\| \\
&= \left\| \left(P_h F'(P_h x_0) P_h + \frac{\alpha_k}{c} \right)^{-1} P_h (F(P_h x_0) - z_{\alpha_k}^{h,\delta}) \right\| \\
&\leq \|F(P_h x_0) - z_{\alpha_k}^h\| + \|z_{\alpha_k}^h - z_{\alpha_k}^{h,\delta}\|,
\end{aligned}$$

$$\begin{aligned}
\|F(P_h x_0) - z_{\alpha_k}^h\| &\leq \|F(P_h x_0) - F(x_0)\| + \|F(x_0) - z_{\alpha_k}\| + \|z_{\alpha_k} - z_{\alpha_k}^h\| \\
&\leq \left\| \int_0^1 F'(x_0 + t(P_h x_0 - x_0))(P_h x_0 - x_0) dt \right\| \\
&\quad + \|(K^* K + \alpha_k I)^{-1} K^* K(F(\hat{x}) - F(x_0))\| + \|z_{\alpha_k} - z_{\alpha_k}^h\| \\
&\leq M b_h + \|F(\hat{x}) - F(x_0)\| + \|z_{\alpha_k} - z_{\alpha_k}^h\| \\
&\leq M b_h + M \rho + \|z_{\alpha_k} - z_{\alpha_k}^h\|.
\end{aligned} \tag{3.27}$$

Therefore by (3.27) and (2.9) we have

$$\begin{aligned}
e_{0,\alpha_k}^{h,\delta} &\leq (M+1)b_h + \left(1 + \frac{\varepsilon_h}{2\sqrt{\alpha_k}}\right) M\rho + \frac{\delta}{2\sqrt{\alpha_k}} \\
&\leq (M+1) \frac{\varepsilon_h + \delta}{\sqrt{\alpha_k}} + M\rho + \frac{1}{2} \max\{M\rho, 1\} \frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}} \\
&\leq (M+1) \frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}} + M\rho + \frac{\varepsilon_0 + \delta_0}{2\sqrt{\alpha_0}} \\
&\leq \gamma_\rho < 1.
\end{aligned} \tag{3.28}$$

Now since g is monotonic increasing and $e_{0,\alpha_k}^{h,\delta} \leq \gamma_\rho$, we have $g(e_{0,\alpha_k}^{h,\delta}) \leq g(\gamma_\rho)$. This completes the proof of the theorem. $\square$

Theorem 3.5. Let $r = (1/(1 - g(\gamma_\rho)) + (1 + \tau_0)(k_0/2)(\gamma_\rho/(1 - g(\gamma_\rho)^2)))\gamma_\rho$ and the assumptions of Theorem 3.4 hold. Then $x_{n,\alpha_k}^{h,\delta}, y_{n,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$, for all $n \geq 0$.

Proof. Note that by (b) of Theorem 3.4 we have

$$\begin{aligned}
\|x_{1,\alpha_k}^{h,\delta} - P_h x_0\| &\leq \left[1 + (1 + \tau_0) \frac{k_0}{2} e_{0,\alpha_k}^{h,\delta}\right] e_{0,\alpha_k}^{h,\delta} \\
&\leq \left[1 + (1 + \tau_0) \frac{k_0}{2} \gamma_\rho\right] \gamma_\rho \\
&\leq r,
\end{aligned} \tag{3.29}$$

that is, $x_1 \in B_r(P_h x_0)$. Again note that by (3.29) and (c) of Theorem 3.4 we have

$$\begin{aligned}
\|y_{1,\alpha_k}^{h,\delta} - P_h x_0\| &\leq \|y_{1,\alpha_k}^{h,\delta} - x_{1,\alpha_k}^{h,\delta}\| + \|x_{1,\alpha_k}^{h,\delta} - P_h x_0\| \\
&\leq \left(1 + g(e_{0,\alpha_k}^{h,\delta}) + (1 + \tau_0) \frac{k_0}{2} e_{0,\alpha_k}^{h,\delta}\right) e_{0,\alpha_k}^{h,\delta} \\
&\leq \left(1 + g(\gamma_\rho) + (1 + \tau_0) \frac{k_0}{2} \gamma_\rho\right) \gamma_\rho \\
&\leq r,
\end{aligned} \tag{3.30}$$

that is, $y_{1,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$. Further by (3.29) and (b) of Theorem 3.4 we have

$$\begin{aligned}
 \|x_{2,\alpha_k}^{h,\delta} - P_h x_0\| &\leq \|x_{2,\alpha_k}^{h,\delta} - x_{1,\alpha_k}^{h,\delta}\| + \|x_{1,\alpha_k}^{h,\delta} - P_h x_0\| \\
 &\leq \left(1 + (1 + \tau_0) \frac{k_0}{2} e_{1,\alpha_k}^{h,\delta}\right) e_{1,\alpha_k}^{h,\delta} \\
 &\quad + \left(1 + (1 + \tau_0) \frac{k_0}{2} e_{0,\alpha_k}^{h,\delta}\right) e_{0,\alpha_k}^{h,\delta} \\
 &\leq \left(1 + g(e_{0,\alpha_k}^{h,\delta}) + (1 + \tau_0) \frac{k_0}{2} e_{0,\alpha_k}^{h,\delta} \left(1 + g(e_{0,\alpha_k}^{h,\delta})^2\right)\right) e_{0,\alpha_k}^{h,\delta} \\
 &\leq r.
 \end{aligned} \tag{3.31}$$

The last but one step follows from the monotonicity of g and (3.28).

And by (3.31) and (c) of Theorem 3.4 we have

$$\begin{aligned}
 \|y_{2,\alpha_k}^{h,\delta} - P_h x_0\| &\leq \|y_{2,\alpha_k}^{h,\delta} - x_{2,\alpha_k}^{h,\delta}\| + \|x_{2,\alpha_k}^{h,\delta} - P_h x_0\| \\
 &\leq g(e_{1,\alpha_k}^{h,\delta}) e_{1,\alpha_k}^{h,\delta} + \left(1 + g(e_{0,\alpha_k}^{h,\delta}) + (1 + \tau_0) \frac{k_0}{2} e_{0,\alpha_k}^{h,\delta} \left(1 + g(e_{0,\alpha_k}^{h,\delta})^2\right)\right) e_{0,\alpha_k}^{h,\delta} \\
 &\leq g(e_{0,\alpha_k}^{h,\delta})^4 e_{0,\alpha_k}^{h,\delta} + \left(1 + g(e_{0,\alpha_k}^{h,\delta}) + (1 + \tau_0) \frac{k_0}{2} e_{0,\alpha_k}^{h,\delta} \left(1 + g(e_{0,\alpha_k}^{h,\delta})^2\right)\right) e_{0,\alpha_k}^{h,\delta} \\
 &\leq \left(1 + g(\gamma_\rho) + g(\gamma_\rho)^2 + (1 + \tau_0) \frac{k_0}{2} \gamma_\rho \left(1 + g(\gamma_\rho)^2\right)\right) \gamma_\rho \\
 &\leq r,
 \end{aligned} \tag{3.32}$$

that is, $x_{2,\alpha_k}^{h,\delta}, y_{2,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$. Continuing this way one can prove that $x_{n,\alpha_k}^{h,\delta}, y_{n,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$, for all $n \geq 0$. This completes the proof. $\square$

The main result of this section is the following theorem.

Theorem 3.6. Let $y_{n,\alpha_k}^{h,\delta}$ and $x_{n,\alpha_k}^{h,\delta}$ be as in (3.5) and (3.6), respectively, and let assumptions of Theorem 3.5 hold. Then $(x_{n,\alpha_k}^{h,\delta})$ is a Cauchy sequence in $B_r(P_h x_0)$ and converges to $x_{c,\alpha_k}^{h,\delta} \in B_r(P_h x_0)$. Further $P_h[F(x_{c,\alpha_k}^{h,\delta}) + (\alpha_k/c)(x_{c,\alpha_k}^{h,\delta} - x_0)] = P_h z_{\alpha_k}^{h,\delta}$ and

$$\|x_{n,\alpha_k}^{h,\delta} - x_{c,\alpha_k}^{h,\delta}\| \leq \overline{C_0} e^{-\gamma_1 3^n}, \tag{3.33}$$

where $\overline{C_0} = (1/(1 - g(\gamma_\rho)^3) + (1 + \tau_0)(k_0 \gamma_\rho / 2)(1/(1 - (g(\gamma_\rho)^2)^3))g(\gamma_\rho)^{3^n}) \gamma_\rho$ and $\gamma_1 = -\log g(\gamma_\rho)$.

Proof. Using the relation (b) and (e) of Theorem 3.4 and (3.28), we obtain

$$\begin{aligned}
\|x_{n+m,\alpha_k}^{h,\delta} - x_{n,\alpha_k}^{h,\delta}\| &\leq \sum_{i=0}^{m-1} \|x_{n+i+1,\alpha_k}^{h,\delta} - x_{n+i,\alpha_k}^{h,\delta}\| \\
&\leq \sum_{i=0}^{m-1} \left(1 + (1 + \tau_0) \frac{k_0 e_{n+i,\alpha_k}^{h,\delta}}{2}\right) e_{n+i,\alpha_k}^{h,\delta} \\
&\leq \sum_{i=0}^{m-1} \left(1 + (1 + \tau_0) \frac{k_0 e_{0,\alpha_k}^{h,\delta}}{2} g(e_{0,\alpha_k}^{h,\delta})^{3^{n+i}}\right) g(e_{0,\alpha_k}^{h,\delta})^{3^{n+i}} e_{0,\alpha_k}^{h,\delta} \\
&\leq \left[\left(1 + g(\gamma_\rho)^3 + g(\gamma_\rho)^{3^2} + \cdots + g(\gamma_\rho)^{3^m}\right) + (1 + \tau_0) \frac{k_0 \gamma_\rho}{2} \right. \\
&\quad \left. \times \left(1 + (g(\gamma_\rho)^2)^3 + (g(\gamma_\rho)^2)^{3^2} + \cdots + (g(\gamma_\rho)^2)^{3^m}\right) \times g(\gamma_\rho)^{3^n} \right] g(\gamma_\rho)^{3^n} \gamma_\rho \\
&\leq \overline{C}_0 g(\gamma_\rho)^{3^n} \\
&\leq \overline{C}_0 e^{-\gamma_1 3^n}.
\end{aligned} \tag{3.34}$$

Thus $(x_{n,\alpha_k}^{h,\delta})$ is a Cauchy sequence in $B_r(P_h x_0)$ and hence it converges, say, to $x_{c,\alpha_k}^{h,\delta} \in \overline{B_r(P_h x_0)}$. Observe that from (3.5)

$$\begin{aligned}
\left\| P_h \left(F(x_{n,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta} \right) + \frac{\alpha_k}{c} (x_{n,\alpha_k}^{h,\delta} - P_h x_0) \right\| &= \left\| R(x_{n,\alpha_k}^{h,\delta}) (x_{n,\alpha_k}^{h,\delta} - y_{n,\alpha_k}^{h,\delta}) \right\| \\
&\leq \left\| R(x_{n,\alpha_k}^{h,\delta}) \right\| \|y_{n,\alpha_k}^{h,\delta} - x_{n,\alpha_k}^{h,\delta}\| \\
&\leq \left(\left\| P_h F'(x_{n,\alpha_k}^{h,\delta}) P_h \right\| + \frac{\alpha_k}{c} \right) e_{n,\alpha_k}^{h,\delta} \\
&\leq \left(\left\| P_h F'(x_{n,\alpha_k}^{h,\delta}) P_h \right\| + \frac{\alpha_k}{c} \right) \\
&\quad \times g(e_{0,\alpha_k}^{h,\delta})^{3^n} e_{0,\alpha_k}^{h,\delta} \\
&\leq \left(M + \frac{\alpha_k}{c} \right) g(\gamma_\rho)^{3^n} \gamma_\rho.
\end{aligned} \tag{3.35}$$

Now by letting $n \rightarrow \infty$ in (3.35) we obtain $P_h F(x_{c,\alpha_k}^{h,\delta}) + (\alpha_k/c)(x_{c,\alpha_k}^{h,\delta} - P_h x_0) = P_h z_{\alpha_k}^{h,\delta}$. This completes the proof. $\square$

Hereafter we assume that $r < 1/k_0$ and $K_1 < (1 - k_0 r)/(1 - c)$. The proof of the following theorem is analogous to the proof of Theorem 3.14 in [11] but for the sake of completeness we give the proof.

Theorem 3.7 (cf. [11], Theorem 3.14). *Suppose x_{c,α_k}^δ is the solution of*

$$F(x) + \frac{\alpha_k}{c}(x - x_0) = z_{\alpha_k}^\delta \tag{3.36}$$

and Assumptions 3.1 and 3.2 holds, then

$$\|\hat{x} - x_{c,\alpha_k}^\delta\| \leq \frac{\varphi_1(\alpha_k) + (2 + 4\mu/(\mu - 1))\mu\psi^{-1}(\delta + \varepsilon_h)}{1 - (1 - c)K_1 - k_0r}. \quad (3.37)$$

Proof. Note that $c(F(x_{c,\alpha_k}^\delta) - z_{\alpha_k}^\delta) + \alpha_k(x_{c,\alpha_k}^\delta - x_0) = 0$, so

$$\begin{aligned} (F'(x_0) + \alpha_k I)(x_{c,\alpha_k}^\delta - \hat{x}) &= (F'(x_0) + \alpha_k I)(x_{c,\alpha_k}^\delta - \hat{x}) \\ &\quad - c(F(x_{c,\alpha_k}^\delta) - z_{\alpha_k}^\delta) - \alpha_k(x_{c,\alpha_k}^\delta - x_0) \\ &= \alpha_k(x_0 - \hat{x}) - c(F(\hat{x}) - z_{\alpha_k}^\delta) + F'(x_0)(x_{c,\alpha_k}^\delta - \hat{x}) \\ &\quad - c[F(x_{c,\alpha_k}^\delta) - F(\hat{x})]. \end{aligned} \quad (3.38)$$

Thus

$$\begin{aligned} \|x_{c,\alpha_k}^\delta - \hat{x}\| &\leq \|\alpha_k(F'(x_0) + \alpha_k I)^{-1}(x_0 - \hat{x})\| + \|(F'(x_0) + \alpha_k I)^{-1}c(F(\hat{x}) - z_{\alpha_k}^\delta)\| \\ &\quad + \|(F'(x_0) + \alpha_k I)^{-1}[F'(x_0)(x_{c,\alpha_k}^\delta - \hat{x}) - c(F(x_{c,\alpha_k}^\delta) - F(\hat{x}))]\| \\ &\leq \|\alpha_k(F'(x_0) + \alpha_k I)^{-1}(x_0 - \hat{x})\| + \|F(\hat{x}) - z_{\alpha_k}^\delta\| + \Gamma, \end{aligned} \quad (3.39)$$

where $\Gamma := \|(F'(x_0) + \alpha_k I)^{-1} \int_0^1 [F'(x_0) - cF'(\hat{x} + t(x_{c,\alpha_k}^\delta - \hat{x}))](x_{c,\alpha_k}^\delta - \hat{x}) dt\|$. So by Assumption 3.2, we obtain

$$\begin{aligned} \Gamma &\leq \left\| (F'(x_0) + \alpha_k I)^{-1} \int_0^1 [F'(x_0) - F'(\hat{x} + t(x_{c,\alpha_k}^\delta - \hat{x}))] \times (x_{c,\alpha_k}^\delta - \hat{x}) dt \right\| \\ &\quad + (1 - c) \left\| (F'(x_0) + \alpha_k I)^{-1} F'(x_0) \times \int_0^1 G(\hat{x} + t(x_{c,\alpha_k}^\delta - \hat{x}), x_0)(x_{c,\alpha_k}^\delta - \hat{x}) dt \right\| \\ &\leq k_0r \|x_{c,\alpha_k}^\delta - \hat{x}\| + (1 - c)K_1 \|x_{c,\alpha_k}^\delta - \hat{x}\| \end{aligned} \quad (3.40)$$

and hence by (3.39) and (3.40) we have

$$\begin{aligned} \|x_{c,\alpha_k}^\delta - \hat{x}\| &\leq \frac{\|\alpha_k(F'(x_0) + \alpha_k I)^{-1}(x_0 - \hat{x})\| + \|F(\hat{x}) - z_{\alpha_k}^\delta\|}{1 - (1 - c)K_1 - k_0r} \\ &\leq \frac{\varphi_1(\alpha_k) + (2 + 4\mu/(\mu - 1))\mu\psi^{-1}(\delta + \varepsilon_h)}{1 - (1 - c)K_1 - k_0r}. \end{aligned} \quad (3.41)$$

This completes the proof of the theorem. $\square$

Theorem 3.8. Suppose $x_{c,\alpha_k}^{h,\delta}$ is the solution of (3.4) and Assumption 2.1 and Theorem 3.7 hold. In addition if $\tau_0 < 1$, then

$$\|x_{c,\alpha_k}^{h,\delta} - x_{c,\alpha_k}^\delta\| \leq \frac{2}{1-\tau_0} \left(\frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}} \right). \quad (3.42)$$

Proof. Suppose x_{c,α_k}^δ and $x_{c,\alpha_k}^{h,\delta}$ are the solutions of (3.36) and (3.4), respectively, then by (3.36) we have,

$$P_h F(x_{c,\alpha_k}^\delta) + \frac{\alpha_k}{c} (P_h x_{c,\alpha_k}^{h,\delta} - P_h x_0) = P_h z_{\alpha_k}^\delta. \quad (3.43)$$

So from (3.4) and (3.43),

$$P_h [F(x_{c,\alpha_k}^{h,\delta}) - F(x_{c,\alpha_k}^\delta)] + \frac{\alpha_k}{c} P_h (x_{c,\alpha_k}^{h,\delta} - x_{c,\alpha_k}^\delta) = P_h (z_{\alpha_k}^{h,\delta} - z_{\alpha_k}^\delta). \quad (3.44)$$

Let $M_f = \int_0^1 F'(x_{c,\alpha_k}^\delta + t(x_{c,\alpha_k}^{h,\delta} - x_{c,\alpha_k}^\delta)) dt$. Then by (3.44) we have

$$P_h [M_f (x_{c,\alpha_k}^{h,\delta} - x_{c,\alpha_k}^\delta)] + \frac{\alpha_k}{c} P_h (x_{c,\alpha_k}^{h,\delta} - x_{c,\alpha_k}^\delta) = P_h (z_{\alpha_k}^{h,\delta} - z_{\alpha_k}^\delta) \quad (3.45)$$

and hence

$$\begin{aligned} \|x_{c,\alpha_k}^{h,\delta} - x_{c,\alpha_k}^\delta\| &\leq \|z_{\alpha_k}^{h,\delta} - z_{\alpha_k}^\delta\| + \|M_f(P_h - I)\| \|x_{c,\alpha_k}^{h,\delta} - x_{c,\alpha_k}^\delta\| \\ &\leq \|z_{\alpha_k}^{h,\delta} - z_{\alpha_k}^\delta\| + \tau_0 \|x_{c,\alpha_k}^{h,\delta} - x_{c,\alpha_k}^\delta\|. \end{aligned} \quad (3.46)$$

Thus

$$\begin{aligned} \|x_{c,\alpha_k}^{h,\delta} - x_{c,\alpha_k}^\delta\| &\leq \frac{1}{1-\tau_0} \|z_{\alpha_k}^{h,\delta} - z_{\alpha_k}^\delta\| \\ &\leq \frac{1}{1-\tau_0} [\|z_{\alpha_k}^{h,\delta} - z_{\alpha_k}^h\| + \|z_{\alpha_k}^h - z_{\alpha_k}\| + \|z_{\alpha_k} - z_{\alpha_k}^\delta\|]. \end{aligned} \quad (3.47)$$

Now the result follows from (2.9), (3.47) and the relation

$$\|z_{\alpha_k} - z_{\alpha_k}^\delta\| \leq \frac{\delta}{2\sqrt{\alpha_k}}. \quad (3.48)$$

The following theorem is a consequence of Theorems 3.6, 3.7, and 3.8. □

Theorem 3.9. Let $x_{n,\alpha_k}^{h,\delta}$ be as in (3.6) and let assumptions in Theorems 3.6, 3.7 and 3.8 hold. Then

$$\|\hat{x} - x_{n,\alpha_k}^{h,\delta}\| \leq \overline{C}_0 e^{-\gamma_1 3^n} + \frac{\varphi_1(\alpha_k) + (2 + 4\mu/(\mu-1))\mu\psi^{-1}(\delta + \varepsilon_h)}{1 - (1-c)K_1 - k_0 r} + \frac{2}{1-\tau_0} \left(\frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}} \right), \quad (3.49)$$

where $\overline{C}_0$ and γ_1 are as in Theorem 3.6.

Theorem 3.10. Let $x_{n,\alpha_k}^{h,\delta}$ be as in (3.6) and let assumptions in Theorem 3.9 hold. Further let $\varphi_1(\alpha_k) \leq \varphi(\alpha_k)$ and

$$n_k := \min \left\{ n : e^{-\gamma_1 3^n} \leq \frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}} \right\}. \quad (3.50)$$

Then

$$\left\| \hat{x} - x_{n,\alpha_k}^{h,\delta} \right\| = O\left(\psi^{-1}(\delta + \varepsilon_h)\right). \quad (3.51)$$

4. Algorithm

Note that for $i, j \in \{0, 1, 2, \dots, N\}$,

$$z_{\alpha_i}^{h,\delta} - z_{\alpha_j}^{h,\delta} = (\alpha_j - \alpha_i)(P_h K^* K P_h + \alpha_j I)^{-1} (P_h K^* K P_h + \alpha_i I)^{-1} P_h K^* (f^\delta - KF(x_0)). \quad (4.1)$$

Therefore the balancing principle algorithm associated with the choice of the parameter specified in Section 2 involves the following steps:

Step 1. Choose α_0 such that $\delta_0 + \varepsilon_0 < 2\sqrt{\alpha_0}/(2M + 3)$ and $\mu > 1$.

Step 2. $\alpha_i = \mu^{2i} \alpha_0$.

Step 3. Solve for w_i :

$$(P_h K^* K P_h + \alpha_i I) w_i = P_h K^* (f^\delta - KF(x_0)). \quad (4.2)$$

Step 4. Solve for $j < i$, $z_{ij}^{h,\delta} : (P_h K^* K P_h + \alpha_j I) z_{ij}^{h,\delta} = (\alpha_j - \alpha_i) w_i$.

Step 5. If $\|z_{ij}^{h,\delta}\| > 4C(\delta + \varepsilon_h)/\sqrt{\alpha_j}$, then take $k = i - 1$.

Step 6. Otherwise, repeat with $i + 1$ in place of i .

Step 7. Choose $n_k = \min\{n : e^{-\gamma_1 3^n} \leq (\delta + \varepsilon_h)/\sqrt{\alpha_k}\}$.

Step 8. Solve $x_{n_k,\alpha_k}^{h,\delta}$ using the iteration (3.6).

In the next section we consider an example to illustrate the above algorithm. The computational results provided endorse the reliability and effectiveness of our method.

5. Example

In this section we consider an example satisfying the assumptions made in this paper and give the numerical illustration. Consider the operator $KF : L^2(0,1) \rightarrow L^2(0,1)$ where $K : L^2(0,1) \rightarrow L^2(0,1)$ is defined by

$$K(x)(t) = \int_0^1 k(t,s)x(s)ds \quad (5.1)$$

and $F : D(F) \subseteq L^2(0,1) \rightarrow L^2(0,1)$ defined by

$$F(u) := \int_0^1 k(t,s)u^3(s)ds, \quad (5.2)$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1, \\ (1-s)t, & 0 \leq t \leq s \leq 1. \end{cases} \quad (5.3)$$

Then for all $x(t), y(t) : x(t) > y(t)$ (see [7], Section 4.3):

$$\langle F(x) - F(y), x - y \rangle = \int_0^1 \left[\int_0^1 k(t,s)(x^3 - y^3)(s)ds \right] (x - y)(t)dt \geq 0. \quad (5.4)$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3 \int_0^1 k(t,s)(u(s))^2 w(s)ds. \quad (5.5)$$

Let V_n be a sequence of finite-dimensional subspaces of X and let $P_h = P_{1/n}$ denote the orthogonal projection on X with range $R(P_h) = V_n$. We assume that $\dim V_n = n + 1$ and $\|P_h x - x\| \rightarrow 0$ as $h \rightarrow 0$ for all $x \in X$. We choose the linear splines $\{v_1, v_2, \dots, v_{n+1}\}$ in a uniform grid of $n + 1$ points in $[0, 1]$ as a basis of V_n .

Since $w_i \in V_n$, w_i is of the form $\sum_{i=1}^{n+1} \lambda_i v_i$ for some scalars $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$. It can be seen that w_i is a solution of (4.2) if and only if $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{n+1})^T$ is the unique solution of

$$(M_n + \alpha_i B_n) \bar{\lambda} = \bar{a}, \quad (5.6)$$

where

$$\begin{aligned} M_n &= (\langle K v_i, K v_j \rangle), \quad i, j = 1, 2, \dots, n+1, \\ B_n &= (\langle v_i, v_j \rangle), \quad i, j = 1, 2, \dots, n+1, \\ \bar{a} &= (\langle P_h K^*(f^\delta - KF(x_0)), v_i \rangle)^T, \quad i = 1, 2, \dots, n+1. \end{aligned} \quad (5.7)$$

Observe that $z_{ij}^{h,\delta}$ in Step 4 of algorithm is again in V_n and hence $z_{ij}^{h,\delta} = \sum_{k=1}^{n+1} \mu_k^{ij} v_k$ for some $\mu_k^{ij}, k = 1, 2, \dots, n+1$. One can see that for $j < i$, $z_{ij}^{h,\delta}$ is a solution of

$$(P_h K^* K P_h + \alpha_j I) z_{ij}^{h,\delta} = (\alpha_j - \alpha_i) w_i \quad (5.8)$$

if and only if $\overline{\mu^{ij}} = (\mu_1^{ij}, \mu_2^{ij}, \dots, \mu_{n+1}^{ij})^T$ is the unique solution of

$$(M_n + \alpha_j B_n) \overline{\mu^{ij}} = \bar{b}, \quad (5.9)$$

where

$$\bar{b} = (\langle (\alpha_j - \alpha_i) w_i, v_i \rangle)^T. \quad (5.10)$$

Compute $z_{ij}^{h,\delta}$ till $\|z_{ij}^{h,\delta}\| > 4C(\delta + \varepsilon_h)/\sqrt{\alpha_j}$ and fix $k = i - 1$. Now choose $n_k = \min\{n : e^{-\gamma_1 3^n} \leq (\delta + \varepsilon_h)/\sqrt{\alpha_k}\}$.

Let $\xi^n = (\xi_1^n, \xi_2^n, \dots, \xi_{n+1}^n)$, $\eta^n = (\eta_1^n, \eta_2^n, \dots, \eta_{n+1}^n)$, $y_{n,\alpha_k}^{h,\delta} = \sum_{i=1}^{n+1} \xi_i^n v_i$ and $x_{n,\alpha_k}^{h,\delta} = \sum_{i=1}^{n+1} \eta_i^n v_i$. Then from (3.5) we have

$$\begin{aligned} \left(P_h F'(x_{n,\alpha_k}^{h,\delta}) + \frac{\alpha_k}{c} \right) \sum_{i=1}^{n+1} (\xi_i^n - \eta_i^n) v_i &= \sum_{i=1}^{n+1} \lambda_i v_i - \sum_{i=1}^{n+1} P_h F(x_{n,\alpha_k}^{h,\delta}) v_i \\ &+ \frac{\alpha_k}{c} \sum_{i=1}^{n+1} (x_0(t_i) - \eta_i^n) v_i, \end{aligned} \quad (5.11)$$

where $t_1, t_2, \dots, t_{n+1}$ are the grid points.

Observe that $(y_{n,\alpha_k}^{h,\delta} - x_{n,\alpha_k}^{h,\delta})$ is a solution of (3.5) if and only if $(\overline{\xi^n - \eta^n}) = (\xi_1^n - \eta_1^n, \xi_2^n - \eta_2^n, \dots, \xi_{n+1}^n - \eta_{n+1}^n)^T$ is the unique solution of

$$\left(Q_n + \frac{\alpha_k}{c} B_n \right) (\overline{\xi^n - \eta^n}) = B_n \left[\bar{\lambda} - F_{h1} + \frac{\alpha_k}{c} (X_0 - \overline{\eta^n}) \right], \quad (5.12)$$

where $Q_n = \langle F'(x_{n,\alpha_k}^{h,\delta}) v_i, v_j \rangle$, $i, j = 1, 2, \dots, n+1$

$$F_{h1} = \left[F(x_{n,\alpha_k}^{h,\delta})(t_1), F(x_{n,\alpha_k}^{h,\delta})(t_2), \dots, F(x_{n,\alpha_k}^{h,\delta})(t_{n+1}) \right]^T \quad (5.13)$$

and $X_0 = [x_0(t_1), x_0(t_2), \dots, x_0(t_{n+1})]^T$.

Further from (3.6) it follows that

$$\left(P_h F'(x_{n,\alpha_k}^{h,\delta}) + \frac{\alpha_k}{c} \right) (x_{n+1,\alpha_k}^{h,\delta} - y_{n,\alpha_k}^{h,\delta}) = P_h \left[z_{\alpha_k}^{h,\delta} - F(y_{n,\alpha_k}^{h,\delta}) + \frac{\alpha_k}{c} (x_{0,\alpha_k}^{h,\delta} - y_{n,\alpha_k}^{h,\delta}) \right]. \quad (5.14)$$

Figure 1: Curve of the exact and approximate solutions.

Thus $(x_{n+1, \alpha_k}^{h, \delta} - y_{n, \alpha_k}^{h, \delta})$ is a solution of (5.14) if and only if $(\overline{\eta^{n+1} - \xi^n}) = (\eta_1^{n+1} - \xi_1^n, \eta_2^{n+1} - \xi_2^n, \dots, \eta_{n+1}^{n+1} - \xi_{n+1}^n)^T$ is the unique solution of

$$\left(Q_n + \frac{\alpha_k}{c} B_n\right) (\overline{\eta^{n+1} - \xi^n}) = B_n \left[\bar{\lambda} - F_{h2} + \frac{\alpha_k}{c} (X_0 - \bar{\xi}^n)\right], \quad (5.15)$$

where $F_{h2} = [F(y_{n, \alpha_k}^{h, \delta})(t_1), F(y_{n, \alpha_k}^{h, \delta})(t_2), \dots, F(y_{n, \alpha_k}^{h, \delta})(t_{n+1})]^T$.

5.1. Numerical Example

Example 5.1. To illustrate the method discussed in the above section, we consider the space $X = Y = L^2[0, 1]$ and the Fredholm integral operator $K : L^2[0, 1] \rightarrow L^2[0, 1]$. The algorithm

Figure 2: Curve of exact and approximate solutions.

in Section 5 is applied by choosing V_n as the space of linear splines in a uniform grid of $n + 1$ points in $[0, 1]$.

In our computation, we take $f(t) = (1/36\pi^2)(27 \sin \pi t - \sin 3\pi t) + (1/36\pi)(27t^2 \cos \pi t - 3t^2 \cos 3\pi t + 6t \cos 3\pi t - 3 \cos 3\pi t - 27t \cos \pi t)$ and $f^\delta = f + \delta$. Then the exact solution

$$\hat{x}(t) = \sin \pi t. \quad (5.16)$$

We use

$$x_0(t) = \sin \pi t + \frac{3}{4\pi^2} \left(1 + t\pi^2 - t^2\pi^2 - \cos^2(\pi t) \right) \quad (5.17)$$

Table 1

n	k	α_k	$\ x_k - \hat{x}\ $	$\ x_k - \hat{x}\ /(\delta + \varepsilon_h)^{1/2}$
8	4	0.1790	0.0363	0.1388
16	4	0.1729	0.0432	0.1669
32	4	0.1714	0.0450	0.1742
64	4	0.1710	0.0455	0.1761
128	4	0.1709	0.0456	0.1765
256	4	0.1709	0.0456	0.1767
512	4	0.1709	0.0456	0.1767
1024	4	0.1709	0.0456	0.1767

as our initial guess, so that the function $x_0 - \hat{x}$ satisfies the source condition

$$x_0 - \hat{x} = F'(\hat{x})1 = \varphi_1(F'(x_0))G(x_0, \hat{x}), \quad (5.18)$$

where $\varphi_1(\lambda) = \lambda$. Thus we expect to have an accuracy of order at least $O((\delta + \varepsilon_h)^{1/2})$.

We choose $\alpha_0 = (1.5)\delta^2$, $\mu = 1.5$, $\delta = 0.0667 = c$, $\varepsilon_h = 1/10n^2$, $\rho = 0.19$, $\gamma_\rho = 0.8173$, and $g(\gamma_\rho) = 0.54$ approximately. For all n the number of iteration $n_k = 3$ in this example. The results of the computation are presented in Table 1. The plots of the exact and the approximate solution obtained are given in Figures 1 and 2.

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