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Research Article

First Problem of Stokes for Generalized Burgers' Fluids

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The velocity field and the adequate shear stress corresponding to the first problem of Stokes for generalized Burgers' fluids are determined in simple forms by means of integral transforms. The solutions that have been obtained, presented as a sum of steady and transient solutions, satisfy all imposed initial and boundary conditions. They can be easily reduced to the similar solutions for Burgers, Oldroyd-B, Maxwell, and second-grade and Newtonian fluids. Furthermore, as a check of our calculi, for small values of the corresponding material parameters, their diagrams are almost identical to those corresponding to the known solutions for Newtonian and Oldroyd-B fluids. Finally, the influence of the rheological parameters on the fluid motions, as well as a comparison between models, is graphically illustrated. The non-Newtonian effects disappear in time, and the required time to reach steady-state is the lowest for Newtonian fluids.

1. Introduction

There is evidence that the interest of the workers in non-Newtonian fluids is on the leading edge during the last few years. Many researchers have the opinion that flows of such fluids are important in industry and technology. Several investigations in the field cite a wide variety of applications in rheological problems in biological sciences, geophysics, and chemical and petroleum industries [1]. It is an established fact that unlike the Newtonian fluids, the flows of non-Newtonian fluids cannot be analyzed by a single constitutive equation. This is due to the rheological properties of non-Newtonian fluids. The understanding of flows of such fluids has progressed via a number of theoretical, computational, and experimental efforts. The resulting equations of such fluids are in general of higher order than the Navier-Stokes equation and one needs additional conditions for a unique solution [2, 3]. Specifically to obtain an analytic solution for such flows is not an easy task. In spite of several challenges,

many investigations regarding the analytic solutions for flows of non-Newtonian fluids have been performed [4–19].

Many models are accorded to describe the rheological behavior of non-Newtonian fluids [20, 21]. They are usually classified as fluids of differential, rate and integral type. Amongst the non-Newtonian fluids, the rate-type fluids are those which take into account the elastic and memory effects. The simplest subclasses of rate type fluids are those of Maxwell and Oldroyd-B fluids. But these fluid models do not exhibit rheological properties of many real fluids such as asphalt in geomechanics and cheese in food products. Recently, a thermodynamic framework has been put into place to develop the one-dimensional rate type model known as Burgers' model [22] to the frame-indifferent three-dimensional form by Murali Krishnan and Rajagopal [23]. This model has been successfully used to describe the motion of the earth's mantle. The Burgers' model is the preferred model to describe the response of asphalt and asphalt concrete [24]. This model is mostly used to model other geological structures, such as Olivine rocks [25] and the propagation of seismic waves in the interior of the earth [26]. In the literature, the vast majority of the flows of the rate-type models has been discussed using Maxwell and Oldroyd-B models. However, the Burgers' model has not received much attention in spite of its diverse applications. We here mention some of the studies [27–33] made by using Burgers' model.

The purpose of this work is to established exact solutions corresponding to the first problem of Stokes for generalized Burgers' fluids. Actually, we determine the velocity and the adequate shear stress corresponding to the motion of such a fluid over a plane wall, which initially is at rest and is suddenly moved in its own plane withe a constant velocity. The general solutions, obtained by means of Fourier sine and Laplace transforms, are presented under integral form in terms of the elementary functions and can be reduced to the similar solutions for Burgers fluids. As a check of their correctness, we also showed that for small values of the rheological parameters λ_1, λ_2, λ_3, and λ_4 or λ_2 and λ_4 only, the diagrams of the general solutions are almost identical to those corresponding to the known solutions for Newtonian, respectively, Oldroyd-B fluids. The influence of the material parameters on the fluid motion, as well as a comparison between some models, is also underlined by graphical illustrations. The non-Newtonian effects disappear in time and the Newtonian fluid flows faster.

2. Basic Governing Equations

The Cauchy stress tensor $\mathbf{T}$ for an incompressible generalized Burgers' fluid is characterized by the following constitutive equations [30–33]:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} + \lambda_1 \frac{\delta \mathbf{S}}{\delta t} + \lambda_2 \frac{\delta^2 \mathbf{S}}{\delta t^2} = \mu \left[\mathbf{A} + \lambda_3 \frac{\delta \mathbf{A}}{\delta t} + \lambda_4 \frac{\delta^2 \mathbf{A}}{\delta t^2} \right], \tag{2.1}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor ($\mathbf{L}$ being the velocity gradient), μ is the dynamic viscosity, λ_1 and λ_3 ($< \lambda_1$) are relaxation and retardation times, λ_2 and λ_4 are new material parameters of the generalized Burgers' fluid (having the dimension of t^2), and $\delta/\delta t$ denotes the upper convected derivative defined in [30–33].

This model includes as special cases the Burgers' model (for $\lambda_4 = 0$), Oldroyd-B model (for $\lambda_2 = \lambda_4 = 0$), Maxwell model (for $\lambda_2 = \lambda_3 = \lambda_4 = 0$), and the Newtonian fluid model when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. In some special flows, like those to be here considered, the governing equations corresponding to generalized Burgers' fluids resemble those for second-grade fluids.

For the problem under consideration, we assume a velocity field $\mathbf{V}$ and an extra-stress tensor $\mathbf{S}$ of the form

$$\mathbf{V} = \mathbf{V}(y,t) = u(y,t)\mathbf{i}, \qquad \mathbf{S} = \mathbf{S}(y,t), \tag{2.2}$$

where $\mathbf{i}$ is the unit vector along the x-coordinate direction. For these flows, the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{V}(y,0) = \mathbf{0}, \qquad \mathbf{S}(y,0) = \frac{\partial \mathbf{S}(y,0)}{\partial t} = \mathbf{0}. \tag{2.3}$$

Equations (2.1) and (2.3) imply $S_{yy} = S_{yz} = S_{zz} = S_{xz} = 0$, and the meaningful equation

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right)\tau(y,t) = \mu\left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right)\frac{\partial u(y,t)}{\partial y}, \tag{2.4}$$

where $\tau(y,t) = S_{xy}(y,t)$ is the nonzero shear stress. In the absence of body forces, the balance of linear momentum reduces to

$$\frac{\partial \tau(y,t)}{\partial y} - \frac{\partial p}{\partial x} = \rho \frac{\partial u(y,t)}{\partial t}, \quad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0. \tag{2.5}$$

Eliminating τ between (2.4) and (2.5) and assuming that there is no pressure gradient in the flow direction, we find the governing equation under the form

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right)\frac{\partial u(y,t)}{\partial t} = \nu\left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 u(y,t)}{\partial y^2}; \quad y,t > 0. \tag{2.6}$$

Consider an incompressible generalized Burgers' fluid occupying the space above a flat plate perpendicular to the y-axis. Initially, the fluid is at rest and at the moment $t = 0^+$ the plate is impulsively brought to the constant velocity U in its plane. Due to the shear, the fluid above the plate is gradually moved. Its velocity is of the form $(2.3)_1$, while the governing equations are given by (2.6) and (2.4). The relevant problem under initial and boundary conditions [34–36] is

$$u(y,0) = \frac{\partial u(y,0)}{\partial t} = \frac{\partial^2 u(y,0)}{\partial t^2} = 0, \quad y > 0, \tag{2.7}$$

$$u(0,t) = UH(t), \quad t \geq 0, \tag{2.8}$$

where $H(t)$ is the Heaviside function. Moreover, the natural conditions

$$u(y,t), \frac{\partial u(y,t)}{\partial y} \longrightarrow 0 \quad \text{as } y \longrightarrow \infty,\ t > 0 \tag{2.9}$$

have to be also satisfied. They are consequences of the fact that the fluid is at rest at infinity and there is no shear in the free stream.

3. Solution of the Problem

In order to determine the exact solution, we shall use the Fourier sine transforms [37]. Multiplying both sides of (2.6) by $\sqrt{2/\pi}\sin(y\xi)$, integrating the result with respect to y from 0 to infinity, and taking into account the boundary conditions (2.8) and (2.9), we find that

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right)\frac{\partial u_s}{\partial t} + \nu\xi^2\left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right)u_s = \nu\xi U\sqrt{\frac{2}{\pi}}\left[H(t) + \lambda_3\delta(t) + \lambda_4\delta'(t)\right], \tag{3.1}$$

where $\delta(t)$ and $\delta'(t)$ are delta function and its derivative and the Fourier sine transform $u_s = u_s(\xi,t)$ of $u(y,t)$ defined by

$$u_s(\xi,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(y,t)\sin(y\xi)\,dy, \tag{3.2}$$

has to satisfy the initial conditions

$$u_s(\xi,0) = \frac{\partial u_s(\xi,0)}{\partial t} = \frac{\partial^2 u_s(\xi,0)}{\partial t^2} = 0, \quad \xi > 0. \tag{3.3}$$

By applying the Laplace transform to (3.1) and having in mind the initial conditions (2.7), we find that

$$\overline{u}_s(\xi,q) = \nu\xi U\sqrt{\frac{2}{\pi}}\frac{\lambda_4 q^2 + \lambda_3 q + 1}{q\left[\lambda_2 q^3 + \{\lambda_1 + \lambda_4\nu\xi^2\}q^2 + \{1 + \lambda_3\nu\xi^2\}q + \nu\xi^2\right]}. \tag{3.4}$$

Now, for a more suitable presentation of the final results, we rewrite (3.4) in the following equivalent form:

$$\overline{u}_s(\xi,q) = U\sqrt{\frac{2}{\pi}}\frac{1}{\xi}\left[\frac{1}{q} - \frac{\lambda_2 q^2 + \lambda_1 q + 1}{\lambda_2 q^3 + \{\lambda_1 + \lambda_4\nu\xi^2\}q^2 + \{1 + \lambda_3\nu\xi^2\}q + \nu\xi^2}\right]. \tag{3.5}$$

Inverting (3.5) by means of the Fourier sine formula, we can write $\overline{u}(y,q)$ as

$$\overline{u}(y,q) = \frac{2U}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}\left[\frac{1}{q} - \frac{\lambda_2 q^2 + \lambda_1 q + 1}{\lambda_2 q^3 + \{\lambda_1 + \lambda_4 \nu\xi^2\}q^2 + \{1 + \lambda_3 \nu\xi^2\}q + \nu\xi^2}\right]d\xi. \tag{3.6}$$

Finally, in order to obtain the velocity field $u(y,t) = \mathcal{L}^{-1}\{\overline{u}(y,q)\}$, we apply the inverse Laplace transform to (3.6) and use (A.1) from the Appendix. As a result, we find for the velocity field, the following simple expression:

$$\begin{aligned} u(y,t) = UH(t) - \frac{2UH(t)}{\pi\lambda_2}\int_0^\infty \frac{\sin(y\xi)}{\xi} \\ \times\left[\frac{(\lambda_2 q_1^2 + \lambda_1 q_1 + 1)e^{q_1 t}}{(q_1 - q_2)(q_1 - q_3)} + \frac{(\lambda_2 q_2^2 + \lambda_1 q_2 + 1)e^{q_2 t}}{(q_2 - q_1)(q_2 - q_3)} + \frac{(\lambda_2 q_3^2 + \lambda_1 q_3 + 1)e^{q_3 t}}{(q_3 - q_1)(q_3 - q_2)}\right]d\xi, \end{aligned} \tag{3.7}$$

where

$$q_i = s_i - \frac{\lambda_1 + \lambda_4 \nu\xi^2}{3\lambda_2}, \quad i = 1,2,3, \tag{3.8}$$

are the roots of the algebraic equation $\lambda_2 q^3 + (\lambda_1 + \lambda_4 \nu\xi^2)q^2 + (1 + \lambda_3 \nu\xi^2)q + \nu\xi^2 = 0$. In above relations (see the Cardano's formulae [38]),

$$\begin{aligned} s_1 &= \sqrt[3]{-\frac{\beta_1}{2} + \sqrt{\frac{\beta_1^2}{4} + \frac{\alpha_1^3}{27}}} + \sqrt[3]{-\frac{\beta_1}{2} - \sqrt{\frac{\beta_1^2}{4} + \frac{\alpha_1^3}{27}}}, \\ s_2 &= Z\sqrt[3]{-\frac{\beta_1}{2} + \sqrt{\frac{\beta_1^2}{4} + \frac{\alpha_1^3}{27}}} + Z^2\sqrt[3]{-\frac{\beta_1}{2} - \sqrt{\frac{\beta_1^2}{4} + \frac{\alpha_1^3}{27}}}, \\ s_3 &= Z^2\sqrt[3]{-\frac{\beta_1}{2} + \sqrt{\frac{\beta_1^2}{4} + \frac{\alpha_1^3}{27}}} + Z\sqrt[3]{-\frac{\beta_1}{2} - \sqrt{\frac{\beta_1^2}{4} + \frac{\alpha_1^3}{27}}} \end{aligned} \tag{3.9}$$

are the roots of the algebraic equation $X^3 + \alpha_1 X + \beta_1 = 0$, where

$$\begin{aligned} \alpha_1 &= \frac{1 + \lambda_3 \nu\xi^2}{\lambda_2} - \frac{(\lambda_1 + \lambda_4 \nu\xi^2)^2}{3\lambda_2^2}, \\ \beta_1 &= \frac{\nu\xi^2}{\lambda_2} + \frac{2(\lambda_1 + \lambda_4 \nu\xi^2)^3}{27\lambda_2^3} - \frac{(\lambda_1 + \lambda_4 \nu\xi^2)(1 + \lambda_3 \nu\xi^2)}{3\lambda_2^2}, \\ Z &= \frac{-1 + i\sqrt{3}}{2}. \end{aligned} \tag{3.10}$$

From Routh-Hurwitz's principle [39], we get $\mathrm{Re}(q_i) < 0$ if $\lambda_1\lambda_3 - \lambda_2 + \lambda_4 > -2\sqrt{\lambda_1\lambda_3\lambda_4}$, provided $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$. The corresponding shear stress (see also (A.2))

$$\tau(y,t) = -\frac{2\mu U H(t)}{\pi\lambda_2}\int_0^\infty \cos(y\xi) \times \left[\frac{(\lambda_4 q_1^2 + \lambda_3 q_1 + 1)e^{q_1 t}}{(q_1 - q_2)(q_1 - q_3)} + \frac{(\lambda_4 q_2^2 + \lambda_3 q_2 + 1)e^{q_2 t}}{(q_2 - q_1)(q_2 - q_3)} + \frac{(\lambda_4 q_3^2 + \lambda_3 q_3 + 1)e^{q_3 t}}{(q_3 - q_1)(q_3 - q_2)}\right] d\xi, \tag{3.11}$$

is obtained in the same way from (2.4).

4. Special Cases

4.1. Burgers' Fluid

Making $\lambda_4 \to 0$ into (3.7) and (3.11), we obtain the velocity field and the associated shear stress corresponding to a Burgers' fluid performing the same motion.

4.2. Oldroyd-B Fluid

Making λ_2 and $\lambda_4 = 0$ into (3.6) and following the same way as before, we get the velocity field (see also (A.3))

$$u_{\mathrm{OB}}(y,t) = UH(t) - \frac{2UH(t)}{\pi\lambda_1}\int_0^\infty \frac{\sin(y\xi)}{\xi}\left[\frac{(\lambda_1 q_8 + 1)e^{q_8 t} - (\lambda_1 q_7 + 1)e^{q_7 t}}{q_8 - q_7}\right] d\xi, \tag{4.1}$$

and the shear stress

$$\tau_{\mathrm{OB}}(y,t) = -\frac{2\mu U H(t)}{\pi\lambda_1}\int_0^\infty \cos(y\xi)\left[\frac{(\lambda_3 q_8 + 1)e^{q_8 t} - (\lambda_3 q_7 + 1)e^{q_7 t}}{q_8 - q_7}\right] d\xi, \tag{4.2}$$

corresponding to an Oldroyd-B fluid. In the above relations,

$$q_7, q_8 = \frac{-\{1 + \lambda_3 \nu \xi^2\} \pm \sqrt{\{1 + \lambda_3 \nu \xi^2\}^2 - 4\nu\lambda_1\xi^2}}{2\lambda_1} \tag{4.3}$$

and (4.1) is identical to (15) from [40].

4.3. Maxwell Fluid

Making the limit of (4.1) and (4.2) as $\lambda_3 \to 0$, we obtain the solutions

$$u_M(y,t) = UH(t) - \frac{2UH(t)}{\pi\lambda_1}\int_0^\infty \frac{\sin(y\xi)}{\xi}\left[\frac{(\lambda_1 q_{10}+1)e^{q_{10}t} - (\lambda_1 q_9 + 1)e^{q_9 t}}{q_{10}-q_9}\right]d\xi,$$
$$\tau_M(y,t) = -\frac{2\mu UH(t)}{\pi\lambda_1}\int_0^\infty \cos(y\xi)\left[\frac{e^{q_{10}t}-e^{q_9 t}}{q_{10}-q_9}\right]d\xi, \tag{4.4}$$

corresponding to a Maxwell fluid. The new roots q_9 and q_{10} are given by

$$q_9, q_{10} = \frac{-1 \pm \sqrt{1-4\nu\lambda_1\xi^2}}{2\lambda_1}. \tag{4.5}$$

4.4. Second-Grade Fluid

It is worthwhile pointing out that the similar solutions for second-grade fluids can be also obtained as limiting case of our solutions. Indeed, if we do not take into consideration the restriction $\lambda \geq \lambda_r$ and make $\lambda_1 \to 0$ into (4.1) and (4.2), we recover the expressions

$$u_{SG}(y,t) = UH(t)\left[1 - \frac{2}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi(1+\lambda_3\nu\xi^2)}\exp\left(\frac{-\nu\xi^2 t}{1+\lambda_3\nu\xi^2}\right)d\xi\right], \tag{4.6}$$

$$\tau_{SG}(y,t) = -\frac{2\mu UH(t)}{\pi}\int_0^\infty \frac{\cos(y\xi)}{1+\lambda_3\nu\xi^2}\exp\left(\frac{-\nu\xi^2 t}{1+\lambda_3\nu\xi^2}\right)d\xi, \tag{4.7}$$

corresponding to a second-grade fluid. The solution (4.6) is identical to that from [40, equation (16)] or [36, equation (14)].

4.5. Newtonian Fluid

Finally, making $\lambda_1 \to 0$ into (4.4) or $\lambda_3 \to 0$ into (4.6) and (4.7), the solutions for a Newtonian fluid

$$u_N(y,t) = UH(t)\left[1 - \frac{2}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}e^{-\nu\xi^2 t}d\xi\right], \tag{4.8}$$

$$\tau_N(y,t) = -\frac{2\mu UH(t)}{\pi}\int_0^\infty \cos(y\xi)e^{-\nu\xi^2 t}d\xi, \tag{4.9}$$

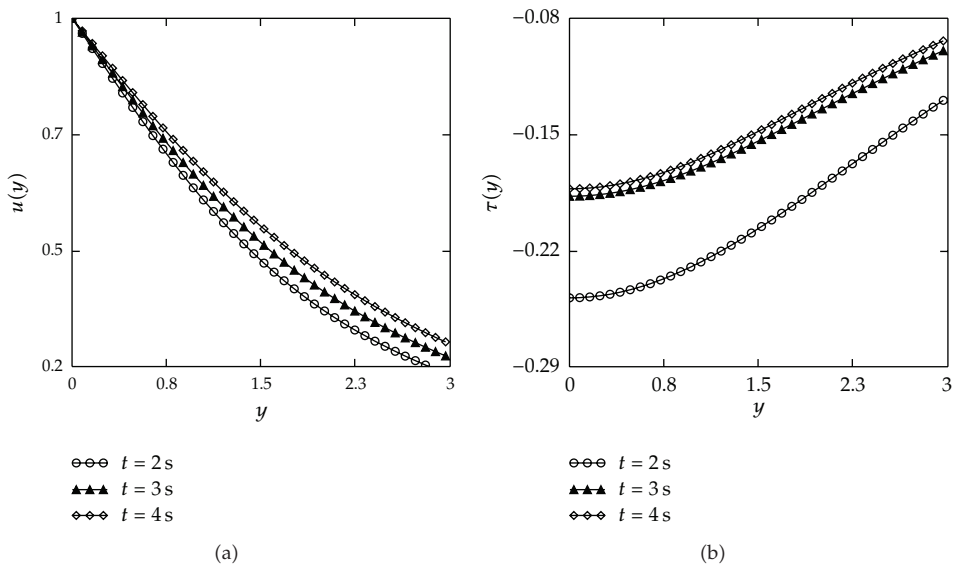

Figure 1: Profiles of the velocity field $u(y,t)$ and the shear stress and $\tau(y,t)$ given by (3.7) and (3.11), for $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda_3 = 2$, $\lambda_4 = 8$, and different values of t.

are achieved. The above equations for $u_N(y,t)$ and $\tau_N(y,t)$ can be written under classical forms

$$u_N(y,t) = U erfc\left(\frac{y}{2\sqrt{\nu t}}\right), \qquad \tau_N(y,t) = -\frac{\mu U}{\sqrt{\pi \nu t}} \exp\left(-\frac{y^2}{4\nu t}\right), \tag{4.10}$$

corresponding to the first problem of Stokes.

5. Numerical Results and Discussion

In order to reveal some relevant physical aspects of the obtained results, several graphs are sketched in this section. A series of diagrams of the velocity $u(y,t)$ and the shear stress $\tau(y,t)$ against y were performed for different situations with typical values. For example, we chose $U = 1$, $\rho = 1$, and $\nu = 1$ for simplicity, and different values for λ_1, λ_2, λ_3, and λ_4 were chosen to illustrate their effects on the fluid motion. From Figure 1, it is clear that the velocity is an increasing function with respect to t, while the shear stress in absolute value decreases with regard to t. Both are decreasing functions with respect to y. Figure 2 shows the variations of the two physical entities with respect to the kinematic viscosity ν. As it was to be expected, both the velocity and the shear stress (of course, in absolute value) are increasing functions with respect to ν.

The influence of the relaxation and retardation times λ_1 and λ_3 on the fluid motion is underlined by Figures 3 and 4. Their effects, as expected, are opposite. More exactly, both the velocity and the shear stress are decreasing functions with respect to λ_1 and increasing ones with regard to λ_3. Figures 5 and 6 show the influence of the other two material parameters on the fluid motion. From these figures, it is clear that λ_2 and λ_4 have opposite effects upon

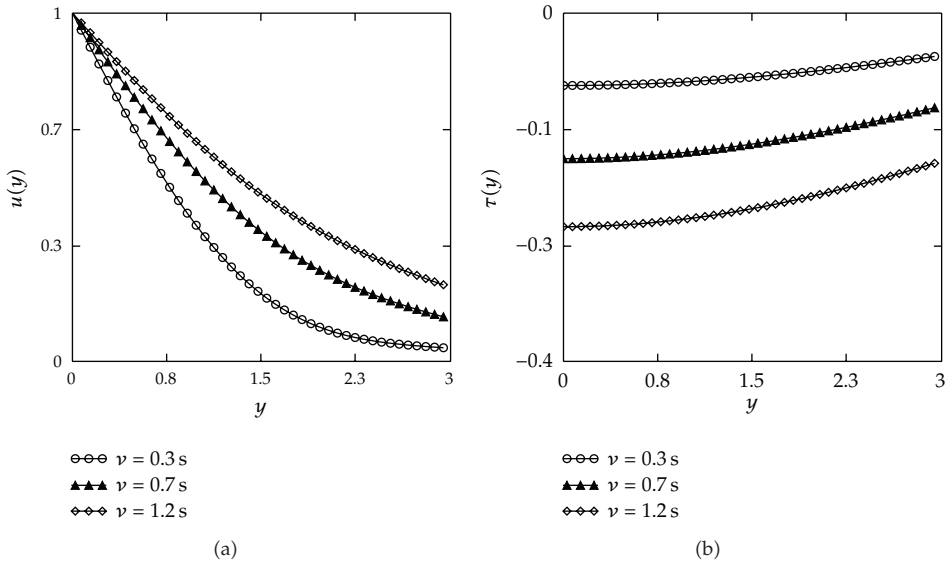

Figure 2: Profiles of the velocity field $u(y,t)$ and the shear stress and $\tau(y,t)$ given by (3.7) and (3.11), for $\lambda_1 = 3,\ \lambda_2 = 4,\ \lambda_3 = 2,\ \lambda_4 = 8,\ t = 2\,\text{s}$, and different values of ν.

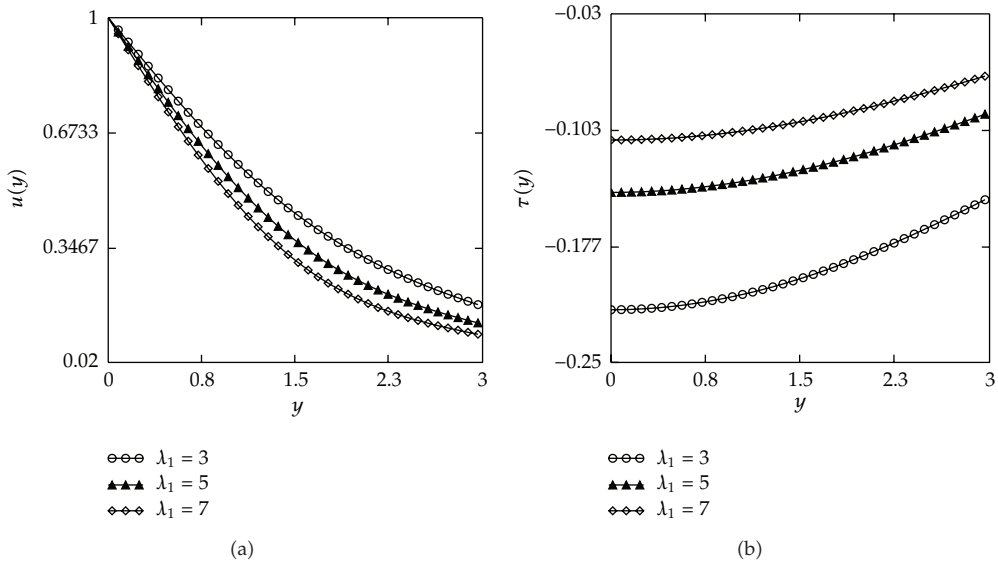

Figure 3: Profiles of the velocity field $u(y,t)$ and the shear stress and $\tau(y,t)$ given by (3.7) and (3.11), for $\lambda_2 = 4,\ \lambda_3 = 2,\ \lambda_4 = 8,\ t = 2\,\text{s}$, and different values of λ_1.

velocity on the whole domain and shear stress on a part only. More exactly, the velocity of the fluid is everywhere a decreasing function with respect to λ_2 and an increasing one with regard to λ_4 exist. The shear stress is a decreasing function of λ_2 on the whole domain and of λ_4 near the plate. The effects of λ_1 and λ_2 on the fluid motion and of λ_3 and λ_4 upon velocity are qualitatively the same.

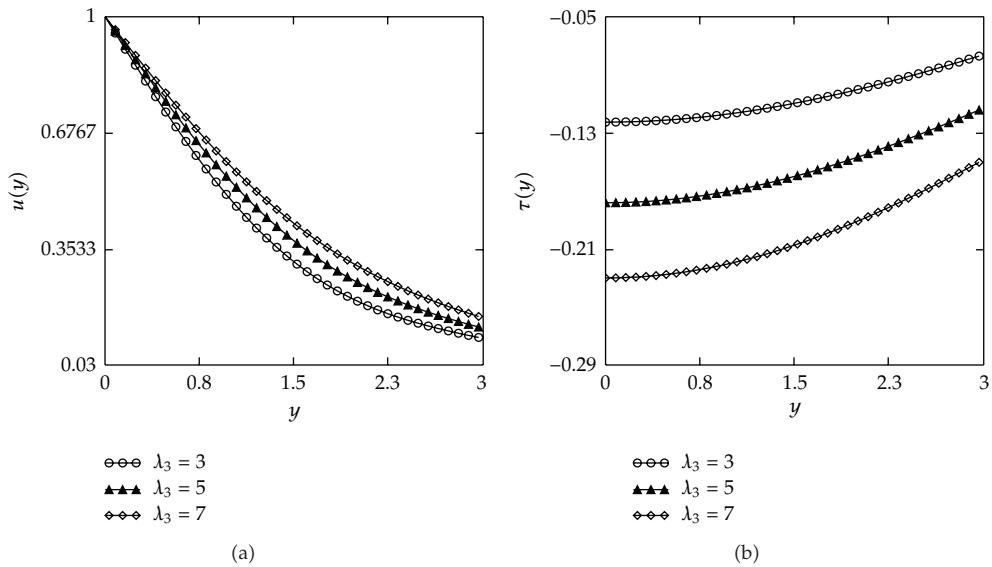

Figure 4: Profiles of the velocity field $u(y,t)$ and the shear stress and $\tau(y,t)$ given by (3.7) and (3.11), for $\lambda_1 = 8,\ \lambda_2 = 2,\ \lambda_4 = 8,\ t = 2\,\text{s}$, and different values of λ_3.

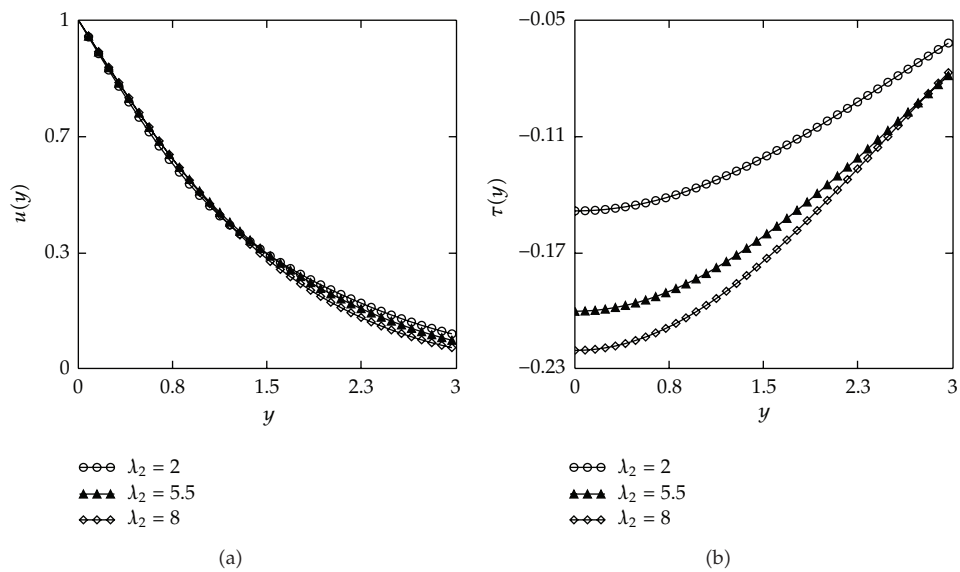

Figure 5: Profiles of the velocity field $u(y,t)$ and the shear stress and $\tau(y,t)$ given by (3.7) and (3.11), for $\lambda_1 = 5.5,\ \lambda_3 = 2,\ \lambda_4 = 6,\ t = 2\,\text{s}$, and different values of λ_2.

Finally, for comparison, the profiles of the velocity $u(y,t)$ and the shear stress $\tau(y,t)$ corresponding to three models (Newtonian, Oldroyd-B, and generalized Burgers') are together depicted in Figure 7 for the same values of t and the common material constants. It is clearly seen from these figures that the Newtonian fluid is the swiftest and the generalized Burgers' fluid is the slowest. Furthermore, the non-Newtonian effects disappear in time and the behavior of Oldroyd-B and generalized Burgers' fluids, as it results from Figure 8, can

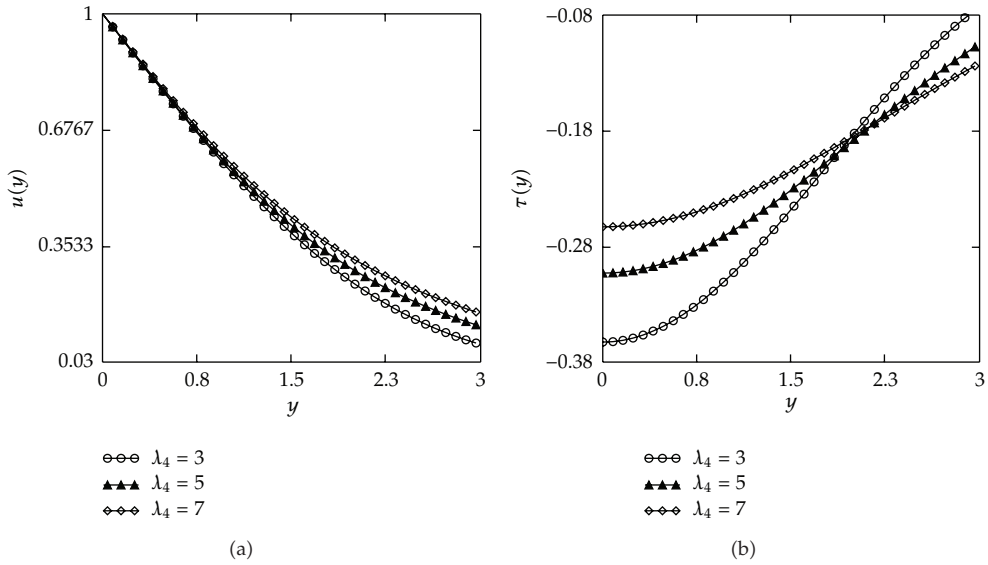

Figure 6: Profiles of the velocity field $u(y,t)$ and the shear stress and $\tau(y,t)$ given by (3.7) and (3.11), for $\lambda_1 = 3,\ \lambda_2 = 4,\ \lambda_3 = 2,\ t = 2\,\text{s}$, and different values of λ_4.

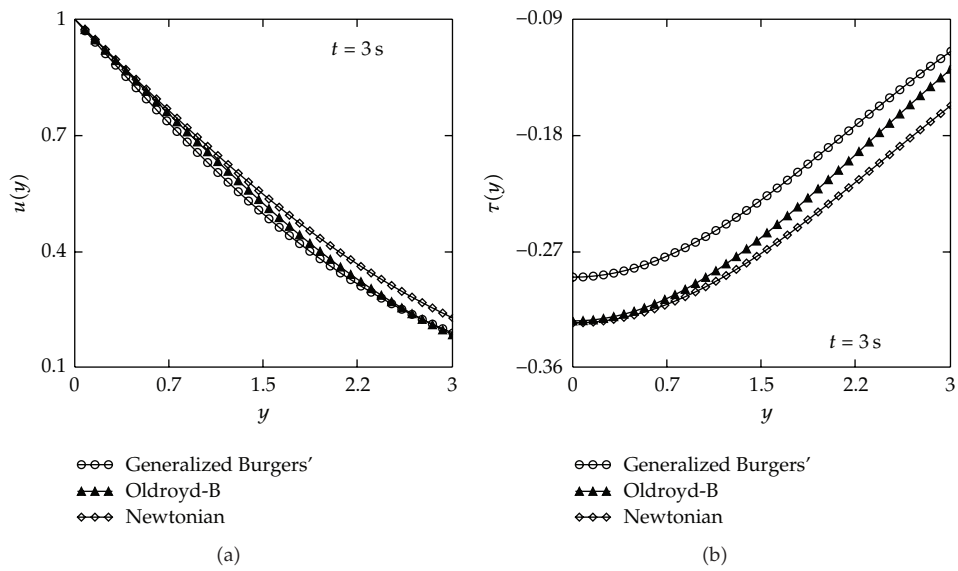

Figure 7: Profiles of the velocity $u(y,t)$ and the shear stress $\tau(y,t)$ for generalized Burgers', Oldroyd-B and Newtonian fluids, for $\lambda_1 = 4,\ \lambda_2 = 2,\ \lambda_3 = 3,\ \lambda_4 = 3$, and $t = 3\,\text{s}$.

be well enough approximated by that of the Newtonian fluids. From the expressions (4.6) and (4.8) of the velocity field $u(y,t)$, it results that the required time to reach the steady state is lower for Newtonian fluids in comparison with second-grade fluids. A comparison with other types of fluids has been also realized by graphical illustrations.

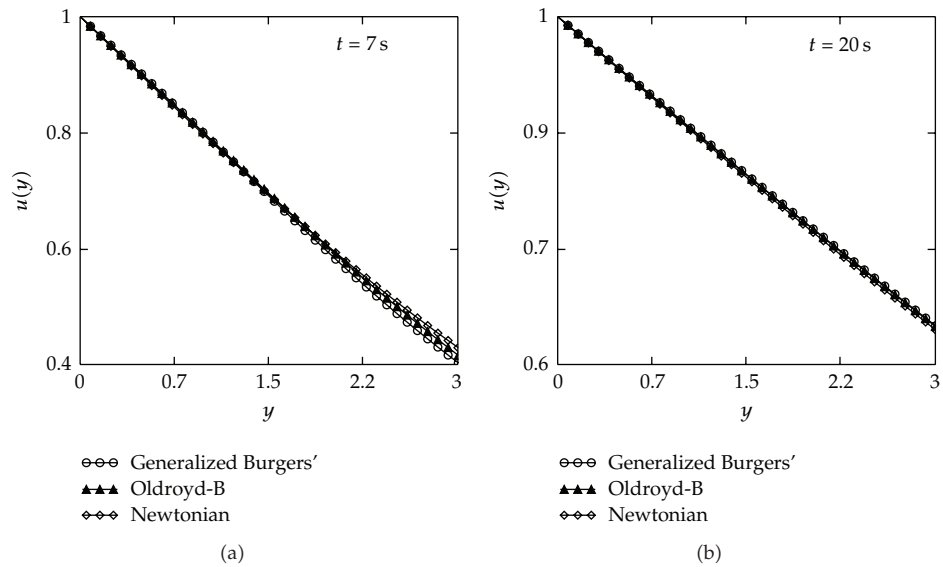

Figure 8: Profiles of the velocity $u(y,t)$ for generalized Burgers', Oldroyd-B, and Newtonian fluids, for $\lambda_1 = 4,\ \lambda_2 = 2,\ \lambda_3 = 3,\ \lambda_4 = 3$, and $t = 7$ s and $t = 20$ s.

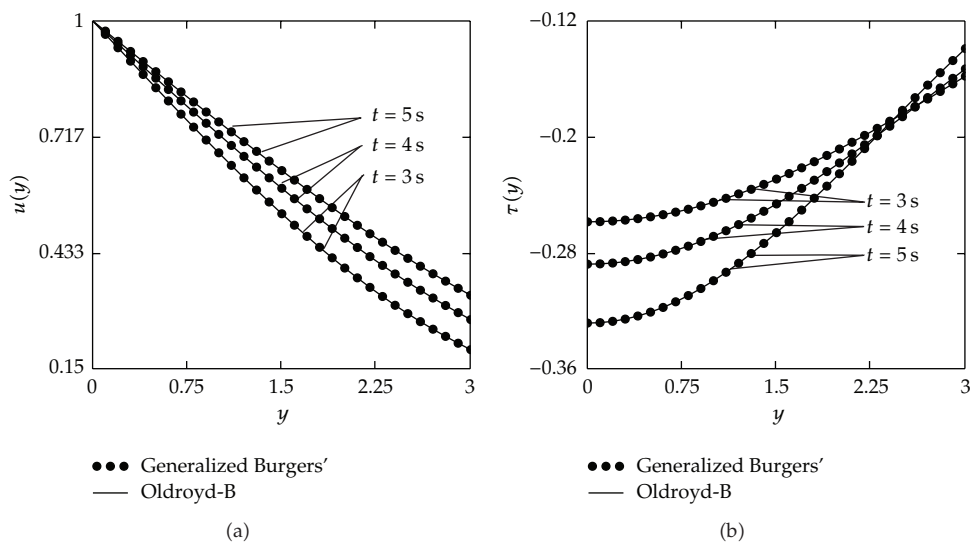

Figure 9: Profiles of the velocity $u(y,t)$ and the shear stress $\tau(y,t)$ for generalized Burgers' and Oldroyd-B fluids, for $\lambda_1 = 3,\ \lambda_2 = 0.00001,\ \lambda_3 = 2.5,\ \lambda_4 = 0.00001$, and different values of t.

6. Concluding Remarks

In this paper, the velocity field $u(y,t)$ and the adequate shear stress $\tau(y,t)$ corresponding to the first problem of Stokes for generalized Burgers' fluids are determined using the Fourier sine and Laplace transforms. The solutions that have been obtained are presented under integral form in terms of the elementary functions $\sin(\bullet)$, $\cos(\bullet)$, and $\exp(\bullet)$ and satisfy all imposed initial and boundary conditions. They are written as a sum of steady and transient

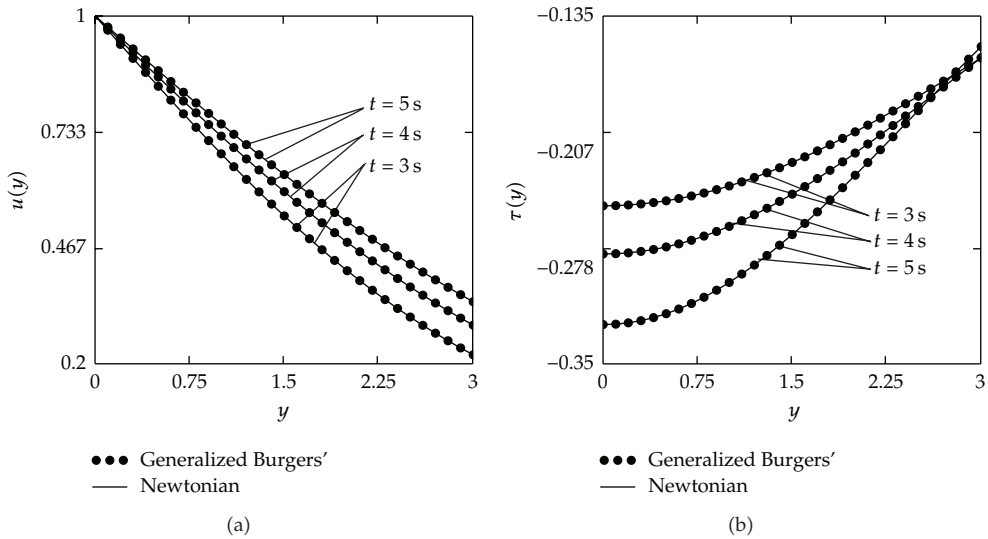

Figure 10: Profiles of the velocity $u(y,t)$ and the shear stress $\tau(y,t)$ for generalized Burgers' and Newtonian fluids, for $\lambda_1 = 0.00001$, $\lambda_2 = 0.00001$, $\lambda_3 = 0.00001$, $\lambda_4 = 0.00001$, and different values of t.

solutions and can be easily reduced to give the similar solutions for Burgers' fluids. The steady solutions

$$u_S(y) = u_S(y,\infty) = U, \qquad \tau_S(y) = \tau(y,\infty) = 0 \tag{6.1}$$

are the same for both types of fluids if the conditions $\lambda_1\lambda_3-\lambda_2+\lambda_4 > -2\sqrt{\lambda_1\lambda_3\lambda_4}$ and $\lambda_1\lambda_3-\lambda_2 > 0$ are satisfied. Furthermore, they are also identical to the steady solutions corresponding to Oldroyd-B, Maxwell, and second-grade and Newtonian fluids. The required time to reach the steady-state can be easily determined by graphical illustrations. It depends of the material constants and differs from a fluid to another one.

The general solutions (3.7) and (3.11) presented in the simplest forms, and their correctness has been graphically verified by comparison with the known solutions for Oldroyd-B and Newtonian fluids. More exactly, from Figures 9 and 10, it clearly results that for small values of the material constants λ_2 and λ_4 or λ_1, λ_2, λ_3, and λ_4, as expected, the diagrams of these solutions are almost identical to those corresponding to Oldroyd-B and Newtonian fluids, respectively. Finally, in order to bring light on some relevant physical aspects of the obtained results, the influence of the material parameters on the fluid motion is underlined by graphical illustrations. A comparison between the Newtonian, Oldrotd-B, and generalized Burgers' fluid is also realized. The main outcomes of this study are as follows.

(i) The general solutions (3.7) and (3.11) are presented under simple forms as a sum of steady and transient solutions. They have been immediately particularized to give the similar solutions for Burgers' fluids.

(ii) As a check of our calculi, we showed that for small values of the material constants λ_2 and λ_4 or λ_1, λ_2, λ_3, and λ_4 the diagrams of these solutions as it was to be

expected, are almost identical to those corresponding to Oldroyd-B and Newtonian fluids.

(iii) The velocity $u(y,t)$ and the shear stress $\tau(y,t)$ (in absolute value) are increasing functions with respect to ν.

(iv) The relaxation and retardation times, λ_1 and λ_3, as expected, have opposite effects on the fluid motion. Both the velocity and the shear stress (in absolute value) are decreasing functions with respect to λ_1 and increasing ones with regard to λ_3.

(v) The other two material constants λ_2 and λ_4 have opposite effects on the velocity on the whole flow domain. Their effect on the shear stress is qualitatively the same near the plate and different in rest. Roughly speaking, the effects of λ_2 and λ_4 on the fluid velocity are qualitatively the same as those of λ_1 and λ_3.

(vi) The Newtonian fluid is the swiftest, and the generalized Burgers' fluid is the slowest. The non-Newtonian effects disappear in time, and the required time to reach the steady-state is the lowest for Newtonian fluid.

Appendix

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{\lambda_2 q^2+\lambda_1 q+1}{\lambda_2 q^3+\{\lambda_1+\lambda_4\nu\xi^2\}q^2+\{1+\lambda_3\nu\xi^2\}q+\nu\xi^2}\right\}\\
&\quad=\frac{1}{\lambda_2}\mathcal{L}^{-1}\left\{\frac{\lambda_2 q^2+\lambda_1 q+1}{(q-q_1)(q-q_2)(q-q_3)}\right\}\\
&\quad=\frac{1}{\lambda_2}\mathcal{L}^{-1}\left\{\frac{\lambda_2 q_1^2+\lambda_1 q_1+1}{(q_1-q_2)(q_1-q_3)(q-q_1)}+\frac{\lambda_2 q_2^2+\lambda_1 q_2+1}{(q_2-q_1)(q_2-q_3)(q-q_2)}\right.\\
&\qquad\qquad\left.+\frac{\lambda_2 q_2^2+\lambda_1 q_2+1}{(q_3-q_1)(q_3-q_2)(q-q_3)}\right\}\\
&\quad=\frac{1}{\lambda_2}\left[\frac{(\lambda_2 q_1^2+\lambda_1 q_1+1)e^{q_1 t}}{(q_1-q_2)(q_1-q_3)}+\frac{(\lambda_2 q_2^2+\lambda_1 q_2+1)e^{q_2 t}}{(q_2-q_1)(q_2-q_3)}+\frac{(\lambda_2 q_3^2+\lambda_1 q_3+1)e^{q_3 t}}{(q_3-q_1)(q_3-q_2)}\right],\\
&\int_0^\infty \frac{\sin(y\xi)}{\xi}d\xi=\frac{\pi}{2},\quad y>0,
\end{aligned}\tag{A.1}$$

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{\lambda_4 q^2+\lambda_3 q+1}{\lambda_2 q^3+\{\lambda_1+\lambda_4\nu\xi^2\}q^2+\{1+\lambda_3\nu\xi^2\}q+\nu\xi^2}\right\}\\
&\quad=\frac{1}{\lambda_2}\mathcal{L}^{-1}\left\{\frac{\lambda_2 q^2+\lambda_1 q+1}{(q-q_1)(q-q_2)(q-q_3)}\right\}
\end{aligned}$$

$$= \frac{1}{\lambda_2}\mathcal{L}^{-1}\left\{\frac{\lambda_4 q_1^2+\lambda_3 q_1+1}{(q_1-q_2)(q_1-q_3)(q-q_1)}+\frac{\lambda_4 q_2^2+\lambda_3 q_2+1}{(q_2-q_1)(q_2-q_3)(q-q_2)}\right.$$
$$\left.+\frac{\lambda_4 q_2^2+\lambda_3 q_2+1}{(q_3-q_1)(q_3-q_2)(q-q_3)}\right\}$$
$$= \frac{1}{\lambda_2}\left[\frac{(\lambda_4 q_1^2+\lambda_3 q_1+1)e^{q_1 t}}{(q_1-q_2)(q_1-q_3)}+\frac{(\lambda_4 q_2^2+\lambda_3 q_2+1)e^{q_2 t}}{(q_2-q_1)(q_2-q_3)}+\frac{(\lambda_4 q_3^2+\lambda_3 q_3+1)e^{q_3 t}}{(q_3-q_1)(q_3-q_2)}\right], \tag{A.2}$$

$$\mathcal{L}^{-1}\left\{\frac{\lambda_1 q+1}{\lambda_1 q^2+\{1+\lambda_3\nu\xi^2\}q+\nu\xi^2}\right\}$$
$$= \frac{1}{\lambda_1}\mathcal{L}^{-1}\left\{\frac{\lambda_1 q+1}{(q-q_7)(q-q_8)}\right\}$$
$$= \frac{1}{\lambda_1}\mathcal{L}^{-1}\left\{\frac{\lambda_1 q+1}{(q_7-q_8)(q-q_7)}+\frac{\lambda_1 q+1}{(q_8-q_7)(q-q_8)}\right\}$$
$$= \frac{1}{\lambda_1}\left[\frac{(\lambda_1 q+1)e^{q_7 t}}{(q_7-q_8)}+\frac{(\lambda_1 q+1)e^{q_8 t}}{(q_8-q_7)}\right],$$
$$\mathcal{L}^{-1}\left\{\frac{\lambda_3 q+1}{\lambda_1 q^2+\{1+\lambda_3\nu\xi^2\}q+\nu\xi^2}\right\} \tag{A.3}$$
$$= \frac{1}{\lambda_1}\mathcal{L}^{-1}\left\{\frac{\lambda_3 q+1}{(q-q_7)(q-q_8)}\right\}$$
$$= \frac{1}{\lambda_1}\mathcal{L}^{-1}\left\{\frac{\lambda_3 q+1}{(q_7-q_8)(q-q_7)}+\frac{\lambda_3 q+1}{(q_8-q_7)(q-q_8)}\right\}$$
$$= \frac{1}{\lambda_1}\left[\frac{(\lambda_3 q+1)e^{q_7 t}}{(q_7-q_8)}+\frac{(\lambda_3 q+1)e^{q_8 t}}{(q_8-q_7)}\right].$$

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