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Research Article

Bifurcation Analysis and Chaos Control in Genesio System with Delayed Feedback

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We investigate the local Hopf bifurcation in Genesio system with delayed feedback control. We choose the delay as the parameter, and the occurrence of local Hopf bifurcations are verified. By using the normal form theory and the center manifold theorem, we obtain the explicit formulae for determining the stability and direction of bifurcated periodic solutions. Numerical simulations indicate that delayed feedback control plays an effective role in control of chaos.

1. Introduction

Since the pioneering work of Lorenz [1], much attention has been paid to the study of chaos. Many famous chaotic systems, such as Chen system, Chua circuit, Rössler system, have been extensively studied over the past decades. It is well known that chaos in many cases produce bad effects and therefore, in recent years, controlling chaos is always a hot topic. There are many methods in controlling chaos, among which using time-delayed controlling forces serves as a good and simple one.

In order to gain further insights on the control of chaos via time-delayed feedback, in this paper, we aim to investigate the dynamical behaviors of Genesio system with time-delayed controlling forces. Genesio system, which was proposed by Genesio and Tesi [2], is described by the following simple three-dimensional autonomous system with only one quadratic nonlinear term:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= ax + by + cz + x^2, \end{aligned} \tag{1.1}$$

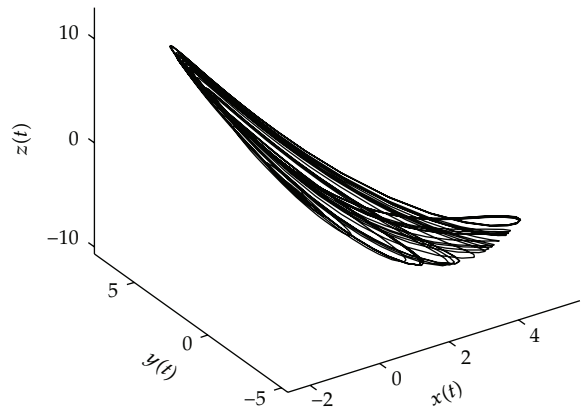

Figure 1: Genesio system's chaotic attractor.

where a, b, $c < 0$ are parameters. System (1.1) exhibits chaotic behavior when $a = -6$, $b = -2.92$, $c = -1.2$, as illustrated in Figure 1. In recent years, many researchers have studied this system from many different points of view; Park et al. [3–5] investigated synchronization of the Genesio chaotic system via backstepping approach, LMI optimization approach, and adaptive controller design. Wu et al. [6] investigated synchronization between Chen system and Genesio system. Chen and Han [7] investigated controlling and synchronization of Genesio chaotic system via nonlinear feedback control. Inspired by the control of chaos via time-delayed feedback force [8] and also following the idea of Pyragas [9], we consider the following Genesio system with delayed feedback control:

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= z(t) + M\big(y(t) - y(t-\tau)\big), \\ \dot{z}(t) &= ax(t) + by(t) + cz(t) + x^2(t), \end{aligned} \tag{1.2}$$

where $\tau > 0$ and $M \in R$.

2. Bifurcation Analysis of Genesio System with Delayed Feedback Force

It is easy to see that system (1.1) has two equilibria $E_0(0,0,0)$ and $E_1(-a,0,0)$, which are also the equilibria of system (1.2). The associated characteristic equation of system (1.2) at E_0 appears as

$$\lambda^3 - (M+c)\lambda^2 + (Mc - b)\lambda - a + \left(M\lambda^2 - Mc\lambda\right)e^{-\lambda\tau} = 0. \tag{2.1}$$

As the analysis for E_1 is similar, we here only analyze the characteristic equation at E_0. First, we introduce the following result due to Ruan and Wei [10].

Lemma 2.1. *Consider the exponential polynomial*

$$\begin{aligned} P\left(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}\right) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &\quad + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \cdots \\ &\quad + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m}, \end{aligned} \tag{2.2}$$

where $\tau_i \geq 0$ $(i = 1, 2, \ldots, m)$ *and* $p_j^{(i)}$ $(i = 0, 1, \ldots, m; j = 1, 2, \ldots, n)$ *are constants. As* $(\tau_1, \tau_2, \ldots, \tau_m)$ *vary, the sum of the order of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ *on the open right half plane can change only if a zero appears on or crosses the imaginary axis.*

Denote $p = c^2 + 2b$, $q = b^2 - 2Mcb - 2Ma - 2ac$, $r = a^2$, $\Delta = p^2 - 3q = c^4 + 4bc^2 + 6(Mb + a)c + 6Ma + b^2$, $h(v) = v^3 + pv^2 + qv + r$, $v = \omega^2$, $v_1^* = (-p + \sqrt{\Delta})/3$, $v_2^* = (-p - \sqrt{\Delta})/3$, $\tau_k^{(j)} = (1/\omega_k)\{\cos^{-1}((M\omega_k^2 + Mc^2 - bc - a)/M(\omega_k^2 + c^2)) + 2j\pi\}$, $\tau_0 = \min_{k\in\{1,2,3\}}\{\tau_k^{(0)}\}$. Following the detailed analysis in [8], we have the following results.

Lemma 2.2. (i) *If* $\Delta \leq 0$, *then all roots with positive real parts of* (2.1) *when* $\tau > 0$ *has the same sum to those of* (2.1) *when* $\tau = 0$.

(ii) *If* $\Delta > 0$, $v_1^* > 0$, $h(v_1^*) \leq 0$, *then all roots with positive of* (2.1) *when* $\tau \in [0, \tau_0]$ *has the same sum to those of* (2.1) *when* $\tau = 0$.

Lemma 2.3. *Suppose that* $h'(v_k) \neq 0$, *then* $\mathrm{d}(\mathrm{Re}\lambda(\tau_k^j))/\mathrm{d}\tau \neq 0$, *and* $\mathrm{sign}\{\mathrm{d}(\mathrm{Re}\lambda(\tau_k^j))/\mathrm{d}\tau\} = \mathrm{sign}\{h'(v_k)\}$.

Proof. Substituting $\lambda(\tau)$ into (2.1) and taking the derivative with respect to τ, we can easily calculate that

$$\left[\frac{\mathrm{d}\left(\mathrm{Re}\,\lambda\left(\tau_k^j\right)\right)}{\mathrm{d}\tau}\right]^{-1} = \frac{3v_k^2 + 2pv_k + q}{M^2\omega_k^2\left(\omega_k^2 + c^2\right)} = \frac{h'(v_k)}{M^2\omega_k^2\left(\omega_k^2 + c^2\right)}, \tag{2.3}$$

thus the results hold. □

Theorem 2.4. (i) *If* $\Delta \leq 0$, *then* (2.1) *has two roots with positive real parts for all* $\tau > 0$.

(ii) *If* $\Delta > 0$, $v_1^* > 0$, $h(v_1^*) \leq 0$, *then* (2.1) *has two roots with positive real parts for* $0 \leq \tau < \tau_0$.

(iii) *If* $\Delta > 0$, $v_1^* > 0$, $h(v_1^*) \leq 0$ *and* $h'(v_k) \neq 0$, *then system* (1.2) *exhibits the Hopf bifurcation at the equilibrium* E_0 *for* $\tau = \tau_k^{(j)}$.

3. Some Properties of the Hopf Bifurcation

In this section, we apply the normal form method and the center manifold theorem developed by Hassard et al. in [11] to study some properties of bifurcated periodic solutions. Without loss of generality, let (x_*, y_*, z_*) be the equilibrium point of system (1.2). For the sake of

convenience, we rescale the time variable $t = \tau t$ and let $\tau = \tau_k + \mu$, $x_1 = x - x_*$, $x_2 = y - y_*$, $x_3 = z - z_*$, then system (1.2) can be replaced by the following system:

$$\dot{x}(t) = L_\mu(x_t) + f(\mu, x_t), \tag{3.1}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T \in R^3$, and for $\phi = (\phi_1, \phi_2, \phi_3)^T \in C$, L_μ and f are, respectively, given as

$$L_\mu(\phi) = (\tau_k + \mu)\begin{pmatrix} 0 & 1 & 0 \\ 0 & M & 1 \\ a + 2x_* & b & c \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + (\tau_k + \mu)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -M & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix}, \tag{3.2}$$

$$f(\tau, \phi) = (\tau_k + \mu)\begin{pmatrix} 0 \\ 0 \\ \phi_3^2(0) \end{pmatrix}. \tag{3.3}$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu(\phi) = \int_{-1}^{0} \mathrm{d}\eta(\theta, 0)\phi(\theta), \quad \phi \in C. \tag{3.4}$$

In fact, the above equation holds if we choose

$$\eta(\theta, \mu) = (\tau_k + \mu)\begin{pmatrix} 0 & 1 & 0 \\ 0 & M & 1 \\ a + 2x_* & b & c \end{pmatrix}\delta(\theta) - (\tau_k + \mu)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -M & 0 \\ 0 & 0 & 0 \end{pmatrix}\delta(\theta + 1), \tag{3.5}$$

where δ is Durac function. For $\phi \in C^1([-\tau, 0], R)$, let

$$\begin{aligned} A(\mu)\phi &= \begin{cases} \dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-\tau}^{0} \mathrm{d}\eta(\theta, \mu)\phi(\theta), & \theta = 0, \end{cases} \\ R(\mu)\phi &= \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases} \end{aligned} \tag{3.6}$$

Then (1.2) can be rewritten in the following form:

$$\dot{x}(t) = A(\mu)x_t + R(\mu)x_t. \tag{3.7}$$

For $\psi \in C[0,1]$, we consider the adjoint operator A^* of A defined by

$$A^*(\mu)\psi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & 0 < s \le 1, \\ \displaystyle\int_{-\tau}^{0} \mathrm{d}\eta^T(t,0)\psi(-t), & s = 0. \end{cases} \tag{3.8}$$

For $\phi \in C[-1,0]$ and $\psi \in C[0,1]$, we define the bilinear inner product form as

$$\langle \psi, \phi \rangle = \overline{\psi}(0)\phi(0) - \int_{\theta=-\tau}^{0} \int_{s=0}^{\theta} \overline{\psi}(s-\theta)\mathrm{d}\eta(\theta)\phi(s)\mathrm{d}s. \tag{3.9}$$

Suppose that $q(\theta) = (1,\alpha,\beta)^T e^{\mathrm{i}\theta\omega_k\tau_k}(-1 \le \theta \le 0)$ is the eigenvectors of $A(0)$ with respect to $\mathrm{i}\omega_k\tau_k$, then $A(0)q(\theta) = \mathrm{i}\omega_k\tau_k q(\theta)$. By the definition of A and (3.2), (3.4), and (3.5) we have

$$\tau_k \begin{pmatrix} \mathrm{i}\omega_k & -1 & 0 \\ 0 & \mathrm{i}\omega_k - M + Me^{-\mathrm{i}\omega_k\tau_k} & -1 \\ -a-2x_* & -b & \mathrm{i}\omega_k - c \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.10}$$

Hence

$$q(s) = (1,\alpha,\beta)^T = \left(1, \mathrm{i}\omega_k, \frac{a+2x_* + \mathrm{i}b\omega_k}{\mathrm{i}\omega_k - c}\right)^T e^{\mathrm{i}\theta\omega_k\tau_k}. \tag{3.11}$$

Similarly, let $q^*(s) = B(1,\alpha^*,\beta^*)e^{\mathrm{i}s\omega_k\tau_k}(0 \le s \le 1)$ be the eigenvector of A^* with respect to $-\mathrm{i}\omega_k\tau_k$, by the definition of A^* and (3.2), (3.4), and (3.5) we can obtain

$$q^*(s) = B(1,\alpha^*,\beta^*)e^{\mathrm{i}s\omega_k\tau_k} = \frac{1}{1 + \overline{\alpha}\alpha^* + \overline{\beta}\beta^* - M\overline{\alpha}\alpha^*\tau_k e^{-\mathrm{i}\omega_k\tau_k}}\left(1, \frac{\mathrm{i}\omega_k(\mathrm{i}\omega_k + c)}{a+2x_*}, \frac{-\mathrm{i}\omega_k}{a+2x_*}\right). \tag{3.12}$$

Furthermore, $\langle q^*, q\rangle = 1$, $\langle q^*, \overline{q}\rangle = 0$.

Let $z(t) = \langle q^*, x_t\rangle$, where x_t is the solution of (3.7) when $\mu = 0$. We denote $w(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}$, then

$$\dot{z}(t) = \mathrm{i}\omega_k\tau_k z(t) + \overline{q^*}(0)f\left(\mu_0, w(z,\overline{z}) + 2\,\mathrm{Re}\{zq(0)\}\right) = \mathrm{i}\omega_k\tau_k z(t) + \overline{q^*}(0)f_0(z,\overline{z}). \tag{3.13}$$

We Rewrite (3.13) in the following form:

$$\dot{z}(t) = \mathrm{i}\omega_k\tau_k z(t) + g(z,\overline{z}), \tag{3.14}$$

where

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{3.15}$$

Noticing that

$$w(z,\overline{z}) = w_{20}\frac{z^2}{2} + w_{11}z\overline{z} + w_{02}\frac{\overline{z}^2}{2} + \cdots, \tag{3.16}$$

we have

$$\dot{w} = \begin{cases} Aw - 2\operatorname{Re}\left\{\overline{q^*}(0)f_0 q(\theta)\right\}, & -1 \le \theta < 0, \\ Aw - 2\operatorname{Re}\left\{\overline{q^*}(0)f_0 q(\theta)\right\} + f_0, & \theta = 0. \end{cases} \tag{3.17}$$

Define

$$\dot{w} = Aw + H(z,\overline{z},\theta), \tag{3.18}$$

with

$$H(z,\overline{z},\theta) = H_{20}\frac{z^2}{2} + H_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots. \tag{3.19}$$

On the other hand,

$$\dot{w} = w_z\dot{z} + w_{\overline{z}}\dot{\overline{z}} = Aw + H(z,\overline{z},\theta). \tag{3.20}$$

Expanding the above series and comparing the corresponding coefficients, we obtain

$$\begin{gathered} (A - 2i\omega_k\tau_k)w_{20}(\theta) = -H_{20}(\theta) \\ Aw_{11}(\theta) = -H_{11}(\theta) \\ \vdots \end{gathered} \tag{3.21}$$

While

$$\begin{gathered} x_t(\theta) = w(z,\overline{z})(\theta) + zq(\theta) + \overline{zq}(\theta), \\ g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots = \overline{q}^*(0)f_0. \end{gathered} \tag{3.22}$$

Let $x_t(\theta) = (x_t^{(1)}(\theta), x_t^{(2)}(\theta), x_t^{(3)}(\theta))$, then we have

$$\begin{aligned} x_t^{(1)}(0) &= z + \overline{z} + w_{20}^{(1)}(0)\frac{z^2}{2} + w_{11}^{(1)}(0)z\overline{z} + w_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + O\left(|(z,\overline{z})|^3\right), \\ x_t^{(2)}(0) &= \alpha z + \overline{\alpha z} + w_{20}^{(2)}(0)\frac{z^2}{2} + w_{11}^{(2)}(0)z\overline{z} + w_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + O\left(|(z,\overline{z})|^3\right), \\ x_t^{(3)}(0) &= \beta z + \overline{\beta}\overline{z} + w_{20}^{(3)}(0)\frac{z^2}{2} + w_{11}^{(3)}(0)z\overline{z} + w_{02}^{(3)}(0)\frac{\overline{z}^2}{2} + O\left(|(z,\overline{z})|^3\right). \end{aligned} \tag{3.23}$$

Therefore we have

$$\begin{aligned} g(z,\overline{z}) &= \overline{B}\tau_k\left(1,\overline{\alpha^*},\overline{\beta^*}\right)\begin{pmatrix} 0 \\ 0 \\ \left(x_t^{(1)}(0)\right)^2 \end{pmatrix} \\ &= \overline{B}\tau_k\overline{\beta^*}\left[z+\overline{z}+w_{20}^{(1)}(0)\frac{z^2}{2}+w_{11}^{(1)}(0)z\overline{z}+w_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+O\left(|(z,\overline{z})|^3\right)\right]^2. \end{aligned} \tag{3.24}$$

Comparing the corresponding coefficients, we have

$$\begin{aligned} g_{20} &= g_{11} = g_{02} = 2\overline{B}\tau_k\overline{\beta^*}, \\ g_{21} &= 2\overline{B}\tau_k\overline{\beta^*}\left[w_{20}^{(1)}(0)+2w_{11}^{(1)}(0)\right]. \end{aligned} \tag{3.25}$$

In what follows we will need to compute $w_{11}(\theta)$ and $w_{20}(\theta)$. Firstly we compute $w_{11}(\theta)$, $w_{20}(\theta)$ when $\theta \in [-1,0)$. It follows from (3.18) that

$$H(z,\overline{z},\theta) = -2\,\mathrm{Re}\left\{\overline{q}^*(0)f_0q(\theta)\right\} = -gq(\theta)-\overline{gq}(\theta). \tag{3.26}$$

Substituting the above equation into (3.21) and comparing the corresponding coefficients yields

$$H_{20}(\theta) = -g_{20}q(\theta)-\overline{g}_{02}\overline{q}(\theta), \tag{3.27}$$

$$H_{11}(\theta) = -g_{11}q(\theta)-\overline{g}_{11}\overline{q}(\theta). \tag{3.28}$$

By (3.21), (3.28), and the definition of A we have

$$\dot{w}_{20}(\theta) = 2\mathrm{i}\omega_k\tau_kw_{20}(\theta)+g_{20}q(\theta)+\overline{g}_{02}\overline{q}(\theta). \tag{3.29}$$

Hence

$$w_{20}(\theta) = \frac{\mathrm{i}g_{20}}{\omega_k\tau_k}q(0)e^{\mathrm{i}\omega_k\tau_k\theta}+\frac{\mathrm{i}\overline{g}_{02}}{3\omega_k\tau_k}\overline{q}(0)e^{-\mathrm{i}\omega_k\tau_k\theta}+E_1e^{2\mathrm{i}\omega_k\tau_k\theta}. \tag{3.30}$$

Similarly we have

$$w_{11}(\theta) = \frac{-\mathrm{i}g_{11}}{\omega_k\tau_k}q(0)e^{\mathrm{i}\omega_k\tau_k\theta}+\frac{\mathrm{i}\overline{g}_{11}}{\omega_k\tau_k}\overline{q}(0)e^{-\mathrm{i}\omega_k\tau_k\theta}+E_2. \tag{3.31}$$

In what follows, we will seek appropriate E_1 and E_2 in (3.30) and (3.31). When $\theta = 0$,

$$H(z,\overline{z},\theta) = -2\,\mathrm{Re}\left\{\overline{q^*}(0)f_0q(\theta)\right\}+f_0 = -gq(0)-\overline{gq}(0)+f_0 \tag{3.32}$$

with

$$f_0 = f_{0,z^2}\frac{z^2}{2}+f_{0,\,z\overline{z}}z\overline{z}+f_{0,\,\overline{z}^2}\frac{\overline{z}^2}{2}+\cdots. \tag{3.33}$$

Comparing the coefficients in (3.18) we have

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + f_{0,z^2}, \\ H_{11}(0) &= -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + f_{0,z\overline{z}}. \end{aligned} \tag{3.34}$$

By (3.21) and the definition of A we have

$$\int_{-1}^{0} \mathrm{d}\eta(\theta)w_{20}(\theta) = 2\mathrm{i}\omega_k\tau_k w_{20}(\theta) + g_{20}q(0) + \overline{g}_{02}\overline{q}(0) + \begin{pmatrix} 0 \\ 0 \\ 2\tau_k \end{pmatrix}. \tag{3.35}$$

$$\int_{-1}^{0} \mathrm{d}\eta(\theta)w_{11}(\theta) = g_{11}q(0) + \overline{g}_{11}\overline{q}(0) + \begin{pmatrix} 0 \\ 0 \\ 2\tau_k \end{pmatrix}. \tag{3.36}$$

Substituting (3.30) into (3.36) and noticing that

$$\begin{aligned} \mathrm{i}\omega_k\tau_k q(0) &= \left(\int_{-1}^{0} e^{\mathrm{i}\omega_k\tau_k\theta}\mathrm{d}\eta(\theta)\right)q(0), \\ -\mathrm{i}\omega_k\tau_k\overline{q}(0) &= \left(\int_{-1}^{0} e^{-\mathrm{i}\omega_k\tau_k\theta}\mathrm{d}\eta(\theta)\right)\overline{q}(0), \end{aligned} \tag{3.37}$$

we have

$$\left(2\mathrm{i}\omega_k\tau_k I - \int_{-1}^{0} e^{2\mathrm{i}\omega_k\tau_k\theta}\mathrm{d}\eta(\theta)\right)E_1 = f_{0,z^2}, \tag{3.38}$$

namely,

$$\begin{pmatrix} 2\mathrm{i}\omega_k & -1 & 0 \\ 0 & 2\mathrm{i}\omega_k - M + Me^{-2\omega_k\tau_k} & -1 \\ -a - 2x_* & -b & 2\mathrm{i}\omega_k - c \end{pmatrix} E_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}. \tag{3.39}$$

Thus

$$E_1^{(1)} = \frac{2}{R}, \qquad E_1^{(2)} = \frac{4\mathrm{i}\omega_k}{R}, \qquad E_1^{(3)} = \frac{4\mathrm{i}\omega_k\left(2\mathrm{i}\omega_k - M + Me^{-2\omega_k\tau_k}\right)}{R}, \tag{3.40}$$

where

$$R = \begin{vmatrix} 2\mathrm{i}\omega_k & -1 & 0 \\ 0 & 2\mathrm{i}\omega_k - M + Me^{-2\omega_k\tau_k} & -1 \\ -a - 2x_* & -b & 2\mathrm{i}\omega_k - c \end{vmatrix}. \tag{3.41}$$

Following the similar analysis, we also have

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -a-2x_* & -b & -c \end{pmatrix} E_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \tag{3.42}$$

hence

$$E_2^{(1)} = \frac{2}{S}, \qquad E_2^{(2)} = 0, \qquad E_2^{(3)} = 0, \tag{3.43}$$

where

$$S = \begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -a-2x_* & -b & -c \end{vmatrix}. \tag{3.44}$$

Thus the following values can be computed:

$$\begin{gathered} c_1(0) = \frac{\mathrm{i}}{2\omega_k\tau_k}\left[g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right] + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_k)\}}, \\ \chi_2 = -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_k)\}}{\omega_k\tau_k}, \\ \beta_2 = 2\operatorname{Re}\{c_1(0)\}. \end{gathered} \tag{3.45}$$

It is well known in [11] that μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0(< 0)$, then the Hopf bifurcation is supercritical(subcritical) and the bifurcated periodic solution exists if $\tau > \tau_k(\tau < \tau_k)$; χ_2 determines the period of the bifurcated periodic solution: if $\chi_2 > 0(< 0)$, then the period increase(decrease); β_2 determines the stability of the Hopf bifurcation: if $\beta_2 < 0(> 0)$, then the bifurcated periodic solution is stable(unstable).

4. Numerical Simulations

In this section, we apply the analysis results in the previous sections to Genesio chaotic system with the aim to realize the control of chaos. We consider the following system:

$$\begin{gathered} \dot{x}(t) = y(t), \\ \dot{y}(t) = z(t) + M\big(y(t) - y(t-\tau)\big), \\ \dot{z}(t) = -6x(t) - 2.92y(t) - 1.2z(t) + x^2(t). \end{gathered} \tag{4.1}$$

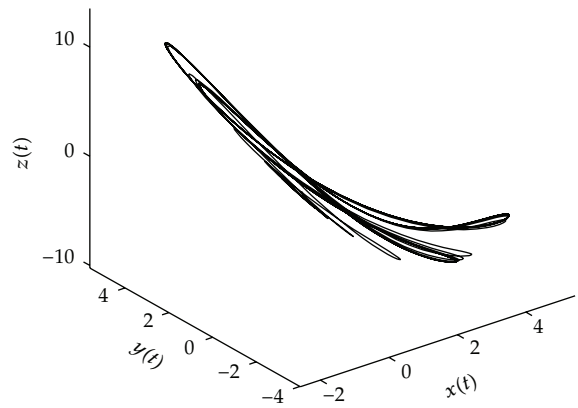

Figure 2: Phase diagram for system (4.3) with $\tau = 0.1$ and initial value (0.1, 0.1, 0.2).

Obviously, system (4.1) has two equilibria $E_0(0,0,0)$ and $E_1(6,0,0)$. In what follows we analyze the case of E_0 only, the analysis for E_1 is similar. The corresponding characteristic equation of system (4.1) at E_0 appears as

$$\lambda^3 - (M - 1.2)\lambda^2 + (-1.2M + 2.92)\lambda + 6 + \left(M\lambda^2 + 1.2M\lambda\right)e^{-\lambda\tau} = 0. \tag{4.2}$$

Hence we have $p = 4.4$, $q = -5.8736 + 4.992M$, $r = 36$, $\Delta = 36.9808 - 14.976M$, $h(v) = v^3 + pv^2 + qv + r$, $v = \omega^2$, $v_1^* = (-p + \sqrt{\Delta})/3 = (1/3)(4.4 + \sqrt{36.9808 - 14.976M})$, $v_2^* = (1/3)(4.4 - \sqrt{36.9808 - 14.976M})$, $\tau_k^{(j)} = (1/\omega_k)\{\cos^{-1}((M\omega_k^2 + Mc^2 - bc - a)/(M\omega_k^2 + Mc^2)) + 2j\pi\}$, $\tau_0 = \min_{k\in\{1,2,3\}}\{\tau_k^{(0)}\}$. By Theorem 2.4, when $\Delta = 36.9808 - 14.976M \leq 0$, that is, $M \geq 2.46934$, (4.2) has two roots with positive real parts for all $\tau > 0$. In order to realize the control of chaos, we will consider $M < 2.46934$. We take $M = -8$ as a special case. In this case, system (4.1) takes the form of

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= z(t) - 8y(t) + 8y(t-\tau), \\ \dot{z}(t) &= -6x(t) - 2.92y(t) - 1.2z(t) + x^2(t). \end{aligned} \tag{4.3}$$

Thus we can compute $\Delta = 156.789$, $v_1^* \doteq 5.64051$, $h(v_1^*) \doteq -182.922$, $v_1 \doteq 4.30249$, $v_2 \doteq 0.873751$, $\omega_1 \doteq 2.07424$, $\omega_2 \doteq 0.934746$, $h'(v_1) \doteq -28.1373$, $h'(v_2) \doteq -51.2083$, $\tau_1^{(0)} \doteq 0.632012$, $\tau_2^{(0)} \doteq 1.85965$,$\tau_0 \doteq 0.632012$. Therefore, using the results in the previous sections, we have the following conclusions: when the delay $\tau = 0.1 < 0.632012$, the attractor still exists, see Figure 2; when the delay $\tau = 0.632$, Hopf bifurcation occurs, see Figure 3. Moreover, $\mu_2 > 0$, $\beta_2 < 0$, the bifurcating periodic solutions are orbitally asymptotically stable; when the delay $\tau = 1.2 \in [0.632012, 1.85965]$, the steady state S_0 is locally stable, see Figure 4; when the delay $\tau = 3.2 > 1.85965$, the steady state S_0 is unstable, see Figure 5. Numerical results indicate that as the delay sets in an interval, the chaotic behaviors really disappear. Therefore the parameter τ works well in control of chaos.

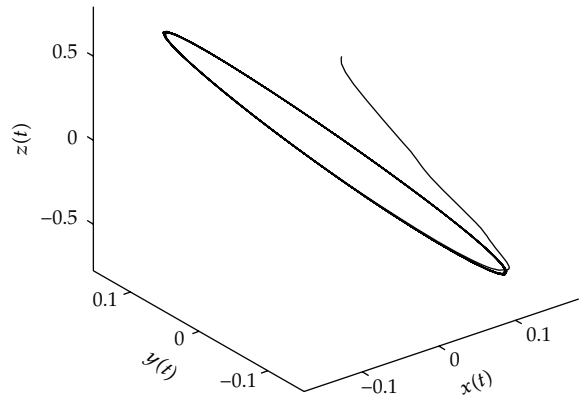

Figure 3: Phase diagram for system (4.3) with $\tau = 0.632$ and initial value (0.1, 0.1, 0.2).

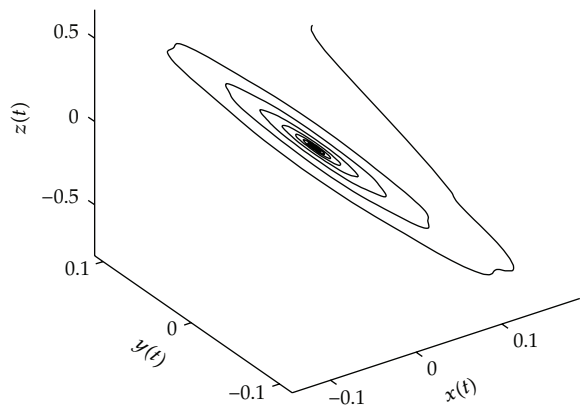

Figure 4: Phase diagram for system (4.3) with $\tau = 1.2$ and initial value (0.1, 0.1, 0.2).

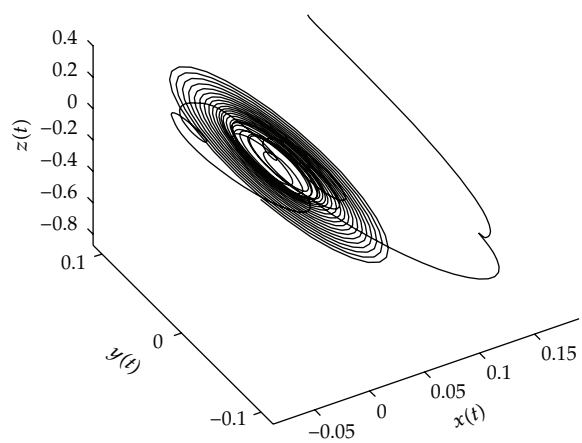

Figure 5: Phase diagram for system (4.3) with $\tau = 3.2$ and initial value (0.1, 0.1, 0.2).

5. Concluding Remarks

In this paper we have introduced time-delayed feedback as a simple and powerful controlling force to realize control of chaos of Genesio system. Regarding the delay as the parameter, we have investigated the dynamics of Genesio system with delayed feedback. To show the effectiveness of the theoretical analysis, numerical simulations have been presented. Numerical results indicate that the delay works well in control of chaos.

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