Research Article

Splitting Lemma for 2-Connected Graphs

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Using a splitting operation and a splitting lemma for connected graphs, Fleischner characterized connected Eulerian graphs. In this paper, we obtain a splitting lemma for 2-connected graphs and characterize 2-connected Eulerian graphs. As a consequence, we characterize connected graphic Eulerian matroids.

1. Introduction

Fleischner [1] introduced a splitting operation to characterize Eulerian graphs as follows. Let $G$ be a connected graph and $v \in V(G)$ with $d(v) \geq 3$. If $x = vv_1$ and $y = vv_2$ are two edges incident with $v$, then splitting away the pair $\{x, y\}$ of edges from the vertex $v$ results in a new graph $G_{x,y}$ obtained from $G$ by deleting the edges $x$ and $y$, and adding a new vertex $v_{x,y}$ adjacent to $v_1$ and $v_2$ (see Figure 1).

The following splitting lemma established by Fleischner [1] has been widely recognized as a useful tool in the graph theory.

Splitting Lemma 1.1 (see [1, page III-29]). Let $G$ be a connected bridgeless graph. Suppose $v \in V(G)$ such that $d(v) \geq 3$ and $x, y, z$ are the edges incident with $v$. Form the graphs $G_{x,y}$ and $G_{x,z}$ by splitting away the pairs $\{x, y\}$ and $\{x, z\}$, respectively, and assume $x$ and $z$ belong to different blocks if $v$ is a cut vertex of $G$. Then either $G_{x,y}$ or $G_{x,z}$ is connected and bridgeless.

This lemma is used to obtain the following characterization of Eulerian graphs.

Theorem 1.2 (see [1, page V-6]). A graph $G$ has an Eulerian trail $T$ if and only if $G$ can be transformed into a cycle $C$ through repeated applications of the splitting procedure on vertices of a degree exceeding 2. Moreover, the number of Eulerian trails of $G$ equals the number of different labeled cycles into which $G$ can be transformed this way.

Thus a connected graph $G$ is Eulerian if and only if there exists a sequence $G = G_0 < G_1 < \cdots < G_n$ of connected graphs such that $G_n$ is a cycle and $G_{i+1}$ is obtained from $G_i$ by applying splitting operation once.
The splitting operation may not preserve 2-connectedness of the graph. Consider the graph \( G \) of Figure 2. It is 2-connected but the graph \( G_{x,y} \) is not 2-connected for any two adjacent edges \( x \) and \( y \).

We obtain the splitting lemma for 2-connected graphs as follows.

**Theorem 1.3.** Let \( G \) be a 2-connected graph and let \( v \) be a vertex of \( G \) with \( d(v) \geq 4 \). Then either \( G_{x,y} \) is 2-connected for some pair \( \{x, y\} \) of edges incident with \( v \) or for any pair \( \{x_1, y_1\} \) of edges incident with \( v \), there is another pair \( \{x_2, y_2\} \) of adjacent edges of \( G \) such that \( (G_{x_1,y_1})_{x_2,y_2} \) is 2-connected.

The next theorem is a consequence of the above result.

**Theorem 1.4.** Let \( G \) be a 2-connected graph. Then \( G \) is Eulerian if and only if there exists a sequence of 2-connected graphs \( G = G_0 < G_1 < G_2 < \cdots < G_n \) such that \( G_n \) is a cycle and \( G_{i+1} \) is obtained from \( G_i \) by applying splitting operation once or twice for \( i = 0, 1, 2, \ldots, n-1 \).

A matroid \( M \) is Eulerian if its ground set can be partitioned into disjoint circuits, and it is connected if any pair of its elements is contained in a circuit. It is clear that an Eulerian matroid may not be connected. A matroid is graphic if it is isomorphic to the cycle matroid of a graph. For matroid concepts and terminology, we refer to Oxley [2]. Raghunathan et al. [3] generalized the splitting operation of graphs to binary matroids and characterized Eulerian matroids in terms of this operation. We characterize connected Eulerian graphic matroids.

In Section 2, we prove Theorems 1.3 and 1.4. The matroid extension is considered in Section 3.

### 2. Eulerian 2-Connected Graphs

A block of a connected graph \( G \) is a pendant block if it contains exactly one cut vertex of \( G \). For an edge \( e \in E(G) \), we denote the set of end vertices of \( e \) by \( V(e) \). For a vertex \( v \) of \( G \), let \( E_G(v) \) denote the set of edges of \( G \) which are incident with \( v \), that is, \( E_G(v) = \{ e \in E(G) \mid v \in V(e) \} \).

Raghunathan et al. [3] characterized the circuits of the graph \( G_{x,y} \) in terms of circuits of \( G \) as follows.

**Lemma 2.1** (see [3]). Let \( G \) be a graph and let \( \{x, y\} \) be a pair of adjacent edges of \( G \). Then a subset \( C \) of edges of the graph \( G_{x,y} \) is a circuit in \( G_{x,y} \) if and only if \( C \) satisfies one of the following conditions:

(i) \( C \) is a circuit in \( G \) containing \( x \) and \( y \);

(ii) \( C \) is a circuit in \( G \) containing neither \( x \) nor \( y \);

(iii) \( C = C_1 \cup C_2 \), where \( C_1 \) and \( C_2 \) are edge disjoint circuits of \( G \) with \( x \in C_1 \), \( y \in C_2 \), and \( C_1 \cup C_2 \) does not contain a circuit in \( G \) satisfying either (i) or (ii) above.
Lemma 2.2. Let $G$ be a 2-connected graph and $v$ be a vertex of $G$ with $d(v) \geq 3$ and $\{x, y\} \subseteq E_G(v)$ such that the graph $G_{x,y}$ is not 2-connected. Then $G_{x,y}$ is connected and has exactly two pendant blocks. Further, one pendant block contains $\{x, y\}$, and the other pendant block contains $E_G(v) - \{x, y\}$.

Proof. The proof is straightforward (see Figure 3).

Lemma 2.3. Let $G$ be a 2-connected graph and let $v$ be a vertex of $G$ with $d(v) \geq 4$ such that $G_{x,y}$ is not 2-connected for all $\{x, y\} \subseteq E_G(v)$. Then, for a given $\{x, y\} \subseteq E_G(v)$, the graph $G_{x,y}$ is connected and has one cut vertex and two blocks.

Proof. Let $\{x, y\}$ be a pair of edges incident with $v$. By Lemma 2.2, $G_{x,y}$ is connected and has exactly two pendant blocks, say $B_1$ and $B_2$. We may assume that $B_1$ contains $\{x, y\}$ and $B_2$ contains $E_G(v) - \{x, y\}$. As $d(v) \geq 4$, we can choose two edges $z, w$ from $E_G(v) - \{x, y\}$. Let $P_1$ and $P_2$ be paths in $G_{x,y}$ from $v$ to $v_{x,y}$ with $\{x, z\} \subseteq E(P_1)$, $\{y, w\} \subseteq E(P_1)$, and $\{x, z\} \cap E(P_2) = \emptyset$. Each of $P_1$ and $P_2$ corresponds to a cycle in $G$. By Lemma 2.1, these cycles are preserved in the graph $G_{y,w}$. Therefore $P_1$ is contained in a block of $G_{y,w}$ for $i = 1, 2$. By Lemma 2.2, $G_{y,w}$ has two pendant blocks one containing edges $y, w$ and the other containing $E_G(v) - \{y, w\}$. Hence $P_1$ and $P_2$ belong to different pendant blocks of $G_{y,w}$. Hence $P_1$ and $P_2$ share at most one vertex of $G_{y,w}$. However, $P_1$ and $P_2$ share all cut vertices of $G_{x,y}$. This implies that $G_{x,y}$ has exactly one cut vertex. Therefore, by Lemma 2.2, $G_{x,y}$ is connected and has exactly two blocks.

Lemma 2.4. Let $G$ and $v$ be as stated in Lemma 2.3. Then there exists a vertex $u$ in $G$ such that $u$ is the cut vertex of $G_{x,y}$ for all $\{x, y\} \subseteq E_G(v)$.

Proof. Let $\{x, y\} \subseteq E_G(v)$. By Lemma 2.3, $G_{x,y}$ is connected and has one cut vertex, say $u$. Let $\{z, w\} \subseteq E_G(v)$. Then, by Lemma 2.3, $G_{z,w}$ is also connected and has two blocks and one cut vertex. It suffices to prove that $u$ is the cut vertex of $G_{z,w}$. If $\{x, y\} = \{z, w\}$, then there is nothing to prove. Suppose $\{x, y\} \neq \{z, w\}$. We may assume that $x \notin \{z, w\}$ and $y \notin \{z, w\}$.

By Lemma 2.3, $G_{z,w}$ is connected and has two blocks, say $B_1$ and $B_2$. By Lemma 2.2, we may assume that $B_1$ contains $\{x, w\}$ and $B_2$ contains $E_G(v) - \{x, w\}$. Let $e$ be an edge of $G$ incident with $v$ such that $e \notin \{x, y, w\}$. Let $P_1$ and $P_2$ be paths in $G_{x,y}$ from the vertex $v_{x,y}$ to $v$ such that $\{x, w\} \subseteq E(P_1)$, $\{y, e\} \subseteq E(P_2)$, and $\{x, w\} \cap E(P_2) = \emptyset$. Then each of $P_1$ and $P_2$ contains all cut vertices of $G_{x,y}$. Therefore $u$ is a common vertex of $P_1$ and $P_2$. Further, each of $P_1$ and $P_2$ corresponds to a cycle in $G$. By Lemma 2.1, these cycles are preserved in the graph.
\begin{figure}
\centering
\begin{tikzpicture}
\node[vertex] (v) at (0,0) {$v$};
\node[vertex] (x) at (-1,-1) {$x$};
\node[vertex] (y) at (1,-1) {$y$};
\node[vertex] (g) at (-2,-2) {$G$};
\node[vertex] (gx) at (2,-2) {$G_{x,y}$};
\draw (v) -- (x);
\draw (v) -- (y);
\draw (g) -- (v);
\draw (gx) -- (x);
\draw (gx) -- (y);
\end{tikzpicture}
\caption{Figure 3}
\end{figure}
connected graph by $G$

In this section, we extend Theorem 1.4 to connected Eulerian matroids. Raghunathan et al.

3. Eulerian 2-Connected Matroids

Proof of Theorem 1.4. Let $G = G_0$ be an Eulerian 2-connected graph. Suppose $G$ is not a cycle. Then $G$ has a vertex $v$ of a degree of at least 4. By Theorem 1.3, we get a pair $\{x_1, y_1\}$ of edges incident with $v$ such that either $G_{x_1,y_1}$ is 2-connected or $(G_{x_1,y_1})_{x_1,y_2}$ is 2-connected for some pair $\{x_2, y_2\}$ of edges of $G$ having a common vertex other than $v$. Denote this new 2-connected graph by $G_1$. If $G_1 = G_{x_1,y_1}$, then $d_{G_1}(v) = d_G(v) - 2$ and $d_{G_1}(w) = d_G(w)$ for any $w \in V(G) - \{v\}$. If $G_1 = (G_{x_1,y_1})_{x_2,y_2}$, then $d_{G_1}(v) = d_G(v) - 2$, $d_{G_1}(u) = d_G(u) - 2$, and $d_{G_1}(w) = d_G(w)$ for any $w \in V(G) - \{u,v\}$, where $u$ is the common vertex of $x_2$ and $y_2$ other than $v$. Further, the new vertices of $G_1$ that are created in the splitting procedure have degree two. Obviously, $G_1$ is Eulerian. If $G_1$ is not a cycle, then we obtain a 2-connected Eulerian graph $G_2$ from $G_1$ by applying splitting operation once (or twice) which results in reducing the degree of a vertex (or two vertices) of $G_1$ by 2. By repeating the same procedure and through a sequence of once or twice splitting operations performed in such a way that at each step the resulting graph is still 2-connected one finally arrives at a cycle which corresponds to an Eulerian trail of $G$. The converse is obvious.

Now, we prove Theorem 1.4. Let $d_H(v)$ denotes the degree of a vertex $v$ in a graph $H$.

3. Eulerian 2-Connected Matroids

In this section, we extend Theorem 1.4 to connected Eulerian matroids. Raghunathan et al. [3] generalized the splitting operation of graphs to binary matroids and characterized Eulerian matroids in terms of this operation. In this section, we characterize connected Eulerian graphic matroids.

Definition 3.1 (see [3]). Let $M = M[A]$ be a binary matroid and suppose $x, y \in E(M)$. Let $A_{x,y}$ be the matrix obtained from $A$ by adjoining the row that is zero everywhere except for the entries of 1 in the columns labeled by $x$ and $y$. The splitting matroid $M_{x,y}$ is defined to be the vector matroid of the matrix $A_{x,y}$. The transition from $M$ to $M_{x,y}$ is called a splitting operation. The splitting operation for binary matroids is also studied in [3–6].
We need the following three results.

**Lemma 3.2** (see [3]). If $M(G)$ denotes the circuit matroid of a graph $G$, then for a pair $\{x, y\}$ of adjacent edges in a graph $G$, $M(G_{x,y}) = (M(G))_{x,y}$.

**Lemma 3.3** (see [3]). Let $M$ be a binary matroid and $x, y \in E(M)$. Then $M$ is Eulerian if and only if $M_{x,y}$ is Eulerian.

**Theorem 3.4** (see [2, page 127]). Let $G$ be a loopless graph without isolated vertices. If $G$ has at least three vertices, then $M(G)$ is a connected matroid if and only if $G$ is a 2-connected graph.

We obtain the following characterization of connected Eulerian graphic matroids.

**Theorem 3.5.** Let $M$ be a connected graphic matroid. Then $M$ is Eulerian if and only if it can be transformed into a circuit $C$ through a sequence $M = M_0 < M_1 < \cdots < M_n = C$ of connected graphic matroids such that $M_{i+1}$ is obtained from $M_i$ by applying splitting operation once or twice.

**Proof.** Let $M$ be a connected graphic matroid. Then $M$ is isomorphic to a cycle matroid $M(G)$ of some graph $G$. In view of Theorem 3.4, we may assume that $G$ is 2-connected. Suppose $M$ is Eulerian. Then the graph $G$ is Eulerian. By Theorem 1.4, there is a sequence of 2-connected graphs $G = G_0 < G_1 < G_2 < \cdots < G_n$ such that $G_n$ is a cycle, and $G_{i+1}$ is obtained from $G_i$ by applying splitting operation once or twice for $i = 0, 1, 2, \ldots, n-1$. Let $M_i = M(G_i)$ for $i = 0, 1, \ldots, n$. By Theorem 3.4, each $M_i$ is connected. It follows from Lemma 3.2 that if $G_{i+1}$ is obtained from $G_i$ by applying splitting operation once or twice then $M_{i+1}$ is obtained from $M_i$ by applying splitting operation once or twice, respectively. Further, by Lemma 3.3, $M_i$ is Eulerian for each $i = 1, 2, \ldots, n$.

Conversely, suppose there exists a sequence of connected graphic matroids $M = M_0 < M_1 < \cdots < M_n = C$, where $M_{i+1}$ is obtained from $M_i$ by applying splitting operation once or twice for $i = 0, 1, 2, \ldots, n-1$. Since $M_n = C$ is Eulerian, by Lemma 3.3, $M_{n-1}$ is Eulerian. By repeated applications of Lemma 3.3, we see that $M_i$ is Eulerian for each $i$. Thus $M = M_0$ is Eulerian. \qed

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