Research Article

Existence of Positive Solution to the Cauchy Problem for a Fractional Diffusion Equation with a Singular Nonlinearity

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Fractional diffusion equations describe an anomalous diffusion on fractals. In this paper, by means of the successive approximation method and other analysis technique, we present a local positive solution to Cauchy problem for a fractional diffusion equation with singular nonlinearity. The fractional derivative is described in the Caputo sense.

1. Introduction

In this paper, we consider the existence of a local positive solution to the Cauchy problem for fractional diffusion equation with singular nonlinearity

$$\begin{aligned} D_t^\alpha u(t, x) &= D^2 \Delta u - u^{-\nu}, \quad x \in R^n, \quad t \in (0, T], \quad \nu > 0, \\ u(0, x) &= \phi(x) \in C_{LB}(R^n), \quad x \in R^n, \end{aligned} \quad (1.1)$$

where D_t^α is a regularized fractional derivative (the Caputo derivative) of order $0 < \alpha \leq 1$ with respect to t , defined by

$$D_t^\alpha u(t, x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\alpha} u(\tau, x) d\tau - t^{-\alpha} u(0, x) \right], \quad 0 < \alpha \leq 1, \quad (1.2)$$

$1/\Gamma(1-\alpha) \int_0^t (t-\tau)^{-\alpha} u(\tau, x) d\tau = I_t^{1-\alpha} u(t, x)$ is the Riemann-Liouville fractional integral of order $1-\alpha$ with respect to t ; D is a positive constant, $n \geq 1$,

$$\begin{aligned} C_{LB}(R^n) = \Big\{ \phi \in C(R^n) : \phi > 0 \text{ in } R^n \text{ only with} \\ \phi_{\min} = \min_{R^n} \phi > 0, \\ \text{there exist } k \geq 0 \text{ and } M > 0 \text{ only such that} \\ |x|^{-k} \phi(x) \leq \Lambda (\Lambda > 0) \text{ for } |x| \geq M \Big\}. \end{aligned} \quad (1.3)$$

Fractional diffusion equations describe an anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials); see [1, 2]. We know that there is a few papers dealing with solutions to some initial value problems for linear fractional diffusion equations, by means of the Laplace transform, Fourier transform, and so forth; see [3–6]. There are also some papers concerned with initial value problems for nonlinear fractional diffusion equations, in which the existence of solutions is considered, using some estimates and the monotone iterative method; see [7–9]. For the details for fractional diffusion equations, please see articles [10, 11] written by Kilbas and Trujillo. However, as far as we know there are few papers which consider the existence of local positive solution to the Cauchy problem (1.1). In this paper, we consider the local existence of positive solutions to singular problem (1.1).

2. Main Result

In this section, we will establish the local existence of positive solutions to the Cauchy problem (1.1). In what follows, $C = C(\cdots)$ denotes positive constants, besides the arguments inside the parenthesis, which vary from line to line.

Lemma 2.1 (see [4]). *The linear initial value problem*

$$\begin{aligned} D_t^\alpha u - D^2 \Delta u &= q(t, x), \quad t \in (0, T], \quad x \in R^n, \quad D > 0, \\ u(0, x) &= \phi(x), \quad x \in R^n, \end{aligned} \quad (2.1)$$

with a bounded continuous function ϕ (locally Hölder continuous if $n > 1$) and a bounded jointly continuous and locally Hölder continuous, in x , function q , has the following integral representation of solution:

$$\begin{aligned} u(t, x) &= \int_{R^n} Z_0(t, x - \xi; \xi) \phi(\xi) d\xi \\ &\quad + \int_0^t d\tau \int_{R^n} Y_0(t - \tau, x - \xi; \xi) q(\tau, \xi) d\xi, \end{aligned} \quad (2.2)$$

where

$$\begin{aligned}
 Z_0(t, x - y) &= \frac{\pi^{-n/2}}{D} |x - y|^{-n} H_{12}^{20} \\
 &\quad \times \left[\frac{1}{4} t^{-\alpha} |x - y|^2 \right]_{(n/2, 1), (1, 1)}^{(1, \alpha)}, \\
 Y_0(t, x - y) &= \frac{\pi^{-n/2}}{D} |x - y|^{-n} t^{\alpha-1} H_{12}^{20} \\
 &\quad \times \left[\frac{1}{4} t^{-\alpha} |x - y|^2 \right]_{(n/2, 1), (1, 1)}^{(\alpha, \alpha)}.
 \end{aligned} \tag{2.3}$$

H_{12}^{20} is an H -function defined by

$$\begin{aligned}
 H_{12}^{20}(z) &= H_{12}^{20} \left[z \right]_{(n/2, 1), (1, 1)}^{(1, \alpha)} \\
 &= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(n/2 + s) \Gamma(1 + s)}{\Gamma(1 + \alpha s)} z^{-s} ds,
 \end{aligned} \tag{2.4}$$

where $\mathcal{C}$ is an infinite contour [12]. Moreover, the functions Z_0 and Y_0 are nonnegative. Z_0 satisfies the relation

$$\int_{R^n} Z_0(t, x - y) dy = 1. \tag{2.5}$$

Remark 2.2. From [4], $Y_0(t, x)$ is the Riemann-Liouville derivative of $Z_0(t, x)$ in t of the order $1 - \alpha$ (for $x \neq 0$, $Z_0(t, x) \rightarrow 0$ as $t \rightarrow 0$), so that the Riemann-Liouville derivative $D_t^{1-\alpha} Z_0(t, x)$ coincides in this case with the regularized fractional derivative $I_t^\alpha \partial Z_0(t, x) / \partial t$ and that

$$\begin{aligned}
 D_t^{1-\alpha} Z_0(t, x) &= \frac{\pi^{-n/2}}{D} |x - y|^{-n} t^{\alpha-1} H_{12}^{20} \\
 &\quad \times \left[\frac{1}{4} t^{-\alpha} |x - y|^2 \right]_{(n/2, 1), (1, 1)}^{(\alpha, \alpha)}.
 \end{aligned} \tag{2.6}$$

Therefore, together with the formulas $I_t^\alpha I_t^\beta f(t, x) = I_t^{\alpha+\beta} f(t, x)$ and $I_t^1 D_t^1 f(t, x) = f(t, x) - f(0, x)$, f is summable with respect to t , we obtain that the solution (2.2) has the following form:

$$\begin{aligned}
 u(t, x) &= \int_{R^n} Z_0(t, x - \xi; \xi) \phi(\xi) d\xi + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} d\tau \\
 &\quad \times \int_{R^n} Z_0(t - \tau, x - \xi; \xi) q(\tau, \xi) d\xi \\
 &= E^{t\Delta} \phi + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} E^{(t-\tau)\Delta} q(\tau, x) d\tau,
 \end{aligned} \tag{2.7}$$

where

$$E^{t\Delta}\phi = \frac{\pi^{-n/2}}{D} \int_{R^n} |x-y|^{-n} H_{12}^{20} \times \left[\frac{1}{4} t^{-\alpha} |x-y|^2 \right]_{(n/2, 1), (1, 1)}^{(1, \alpha)} \phi(y) dy. \quad (2.8)$$

Consequently, from the Lemma 2.1 and Remark 2.2, we see that the solution of (1.1) has the form

$$u(t, x) = E^{t\Delta}\phi - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} u^{-\nu}(s, x) ds, \quad (2.9)$$

for all $(t, x) \in [0, T] \times R^n$.

Remark 2.3. In (2.9), it follows from the definition of $E^{t\Delta}$, $t \geq 0$ and $0 \leq s \leq t$, that $E^{(t-s)\Delta}$ is well defined.

Definition 2.4. We call a function u a local positive solution of (1.1) on $R^n \times [0, T]$ if

- (i) $u \in C([0, T'] \times R^n)$ with $u_{\min} := \min_{[0, T'] \times R^n} u > 0$ for any $0 < T' \leq T$,
- (ii) $u(t, x) = E^{t\Delta}\phi - (1/\Gamma(\alpha)) \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} u^{-\nu}(s, x) ds$ for all $(t, x) \in [0, T'] \times R^n$.

The next theorem is the main result of this paper.

Theorem 2.5. Let $\phi \in C_{LB}(R^n)$. Then, (1.1) has a local positive solution v .

Proof. Defining $\rho = \phi_{\min} > 0$, we will find $0 < t_0 \leq T$ and $\tilde{\rho} > 0$ depending on ρ , ν , and n such that (1.1) has a positive solution $v(t, x)$ satisfying

$$v(t, x) \geq \tilde{\rho} \quad \text{for } (t, x) \in [0, t_0] \times R^n. \quad (2.10)$$

Let

$$F(u) = E^{t\Delta}\phi - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} u^{-\nu}(s, x) ds. \quad (2.11)$$

For $0 < \tilde{\rho} < \rho$, which will be determined below, we construct a sequence $\{u_k\}$ as follows

$$u_k(t, x) = E^{t\Delta}\phi - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} u_{k-1}^{-\nu}(s, x) ds, \quad (2.12)$$

$k = 0, 1, 2, \dots,$

with $u_{-1}(t, x) = \tilde{\rho}$. For $0 < \tilde{\rho} < \min\{1, \rho/2\}$, by (2.5), we have

$$\begin{aligned}
 & E^{t\Delta} \phi \\
 &= \frac{\pi^{-n/2}}{D} \int_{R^n} |x-y|^{-n} H_{12}^{20} \\
 &\quad \times \left[\frac{1}{4} t^{-\alpha} |x-y|^2 \right]_{(n/2, 1), (1, 1)}^{(1, \alpha)} \phi(y) dy \\
 &\geq \frac{\pi^{-n/2} \rho}{D} \int_{R^n} |x-y|^{-n} H_{12}^{20} \\
 &\quad \times \left[\frac{1}{4} t^{-\alpha} |x-y|^2 \right]_{(n/2, 1), (1, 1)}^{(1, \alpha)} dy \\
 &= \rho \geq 2\tilde{\rho}.
 \end{aligned} \tag{2.13}$$

Also, we can obtain that

$$\begin{aligned}
 E^{t\Delta} \phi &= \frac{\pi^{-n/2}}{D} \int_{R^n} |x-y|^{-n} H_{12}^{20} \\
 &\quad \times \left[\frac{1}{4} t^{-\alpha} |x-y|^2 \right]_{(n/2, 1), (1, 1)}^{(1, \alpha)} \phi(y) dy \\
 &\leq \frac{\pi^{-n/2}}{D} \int_{R^n} |x-y|^{-n} H_{12}^{20} \left[\frac{1}{4} t^{-\alpha} |x-y|^2 \right. \\
 &\quad \times \left. \left[\frac{1}{4} t^{-\alpha} |x-y|^2 \right]_{(n/2, 1), (1, 1)}^{(1, \alpha)} C(1+|y|)^k dy \right] \\
 &= \frac{\pi^{-n/2}}{D} \int_{R^n} \eta^{-n} H_{12}^{20} \left[\frac{1}{4} |\eta|^2 \right]_{(n/2, 1), (1, 1)}^{(1, \alpha)} \\
 &\quad \times C(1+|x|+t^{\alpha/2}|\eta|)^k d\eta \\
 &\leq 2^k C(1+|x|)^k + 2^k C B t^{k\alpha/2} \\
 &\leq C(1+|x|)^k,
 \end{aligned} \tag{2.14}$$

where $B = (\pi^{-n/2}/D) \int_{R^n} \eta^{k-n} H_{12}^{20} \left[\frac{1}{4} |\eta|^2 \right]_{(n/2, 1), (1, 1)}^{(1, \alpha)} d\eta$.

On the other hand, if $0 < t \leq (\Gamma(1+\alpha)\tilde{\rho}^{1+\nu})^{1/\alpha}$, by (2.5), we have

$$\begin{aligned}
 u_0(t, x) &= E^{t\Delta} \phi - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} \tilde{\rho}^{-\nu} ds \\
 &\geq 2\tilde{\rho} - \frac{\tilde{\rho}^{-\nu}}{\Gamma(1+\alpha)} t^\alpha \geq \tilde{\rho}.
 \end{aligned} \tag{2.15}$$

Moreover, it is obvious that

$$u_0(t, x) \leq C(1+|x|)^k, \quad \text{for } 0 < t \leq T, \quad x \in R^n. \tag{2.16}$$

Hence, it holds that

$$\tilde{\rho} \leq u_0(t, x) \leq C(1 + |x|)^k \quad \text{for } (t, x) \in [0, t_0] \times R^n, \quad (2.17)$$

where $t_0 = \min\{T, (\Gamma(1 + \alpha)\tilde{\rho}^{1+\nu})^{1/\alpha}\}$. From the previous arguments, it is very easy to show that

$$\begin{aligned} \tilde{\rho} &\leq u_k(t, x) \leq C(1 + |x|)^k \\ &\text{for } (t, x) \in [0, t_0] \times R^n \text{ and all } k \geq 0. \end{aligned} \quad (2.18)$$

Now, we show that

$$|u_k - u_{k-1}| \leq \nu^{k-1} \left(\frac{2}{\tilde{\rho}^{\nu+1}} \right)^k \frac{1}{\Gamma(1 + k\alpha)} t^{k\alpha}, \quad \forall k \geq 1. \quad (2.19)$$

In fact, we have that

$$\begin{aligned} |u_1 - u_0| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} |u_0^{-\nu} - \tilde{\rho}^{-\nu}| ds \\ &\leq \frac{2}{\tilde{\rho}^\nu} \frac{1}{\Gamma(1 + \alpha)} t^\alpha \leq \frac{2\nu^0}{\tilde{\rho}^{\nu+1}} \frac{1}{\Gamma(1 + \alpha)} t^\alpha. \end{aligned} \quad (2.20)$$

If we assume that (2.19) holds for $k = m-1$ ($m > 2$), next, we will show that (2.19) is valid for $k = m$. Indeed, since

$$\begin{aligned} u_{m-1}^{-\nu} - u_{m-2}^{-\nu} &= -\nu \xi^{-(\nu+1)} (u_{m-1} - u_{m-2}) \\ &= -C(t, x) (u_{m-1} - u_{m-2}), \end{aligned} \quad (2.21)$$

where $\xi = su_{m-1} + (1-s)u_{m-2}$ with $s \in (0, 1)$, it follows from (2.18) that $C(t, x) \leq \nu \tilde{\rho}^{-(\nu+1)}$.

Hence, we can obtain that

$$\begin{aligned} |u_m - u_{m-1}| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} |u_{m-1}^{-\nu} - u_{m-2}^{-\nu}| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} C(t, x) |u_{m-1} - u_{m-2}| ds \\ &\leq \frac{\nu}{\tilde{\rho}^{\nu+1}} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} |u_{m-1} - u_{m-2}| ds \\ &\leq \frac{\nu}{\tilde{\rho}^{\nu+1}} \nu^{m-2} \left(\frac{2}{\tilde{\rho}^{\nu+1}} \right)^{m-1} \frac{1}{\Gamma(1 + (m-1)\alpha)} \frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} s^{(m-1)\alpha} ds \end{aligned}$$

$$\begin{aligned}
&= \nu^{m-1} \frac{2^{m-1}}{(\tilde{\rho}^{\nu+1})^m} \frac{\Gamma(1+(m-1)\alpha)}{\Gamma(1+(m-1)\alpha)\Gamma(1+m\alpha)} t^{m\alpha} \\
&\leq \nu^{m-1} \left(\frac{2}{\tilde{\rho}^{\nu+1}} \right)^m \frac{1}{\Gamma(1+m\alpha)} t^{m\alpha}.
\end{aligned} \tag{2.22}$$

Therefore, (2.19) is valid for all $k = 1, 2, \dots$. Thus, we see that, for $0 < t \leq t_0$, $x \in R^n$, it holds that

$$|u_k - u_{k-1}| \leq \nu^{k-1} \left(\frac{2}{\tilde{\rho}^{\nu+1}} \right)^k \frac{1}{\Gamma(1+k\alpha)} t_0^{k\alpha}. \tag{2.23}$$

Define $\zeta_k = \sum_{j=1}^k (u_j - u_{j-1})$. Since $|\zeta_k| \leq \sum_{j=1}^k \nu^{j-1} (2/\tilde{\rho}^{\nu+1})^j (t_0^{j\alpha}/\Gamma(1+j\alpha))$, we can claim that $\zeta_k \rightarrow \zeta$ uniformly in $[0, t_0] \times R^n$ as $k \rightarrow \infty$ and $\zeta \in C([0, t_0] \times R^n)$. In fact, from the Stirling formula, we have that

$$\Gamma(1+n\alpha) = \sqrt{2\pi n\alpha} \left(\frac{n\alpha}{e} \right)^{n\alpha} e^{\theta/12n\alpha}, \tag{2.24}$$

where $0 < \theta < 1$. We let $\sum_{j=1}^{\infty} \nu^{j-1} (2/\tilde{\rho}^{\nu+1})^j (t_0^{j\alpha}/\Gamma(1+j\alpha)) = \sum_{j=1}^{\infty} x_j$; thus, we have that

$$\begin{aligned}
\frac{x_{j+1}}{x_j} &= \frac{2\nu t_0^\alpha \Gamma(1+j\alpha)}{\tilde{\rho}^{\nu+1} \Gamma(1+(j+1)\alpha)} \\
&= \frac{2\nu t_0^\alpha}{\tilde{\rho}^{\nu+1}} \frac{\sqrt{2\pi j\alpha} (j\alpha/e)^{j\alpha} e^{\theta/12j\alpha}}{\sqrt{2\pi (j+1)\alpha} ((j+1)\alpha/e)^{(j+1)\alpha} e^{\theta/12(j+1)\alpha}} \\
&\leq \frac{2\nu t_0^\alpha}{\tilde{\rho}^{\nu+1}} \frac{\sqrt{2\pi (j+1)\alpha} ((j+1)\alpha/e)^{j\alpha} e^{\theta/12j\alpha}}{\sqrt{2\pi (j+1)\alpha} ((j+1)\alpha/e)^{(j+1)\alpha} e^{\theta/12(j+1)\alpha}} \\
&= \frac{2\nu t_0^\alpha}{\tilde{\rho}^{\nu+1}} e^{\theta/12j(j+1)\alpha} \left(\frac{(j+1)\alpha}{e} \right)^{-\alpha}, \\
0 \leq \lim_{j \rightarrow \infty} \frac{x_{j+1}}{x_j} &\leq \lim_{j \rightarrow \infty} \frac{2\nu t_0^\alpha}{\tilde{\rho}^{\nu+1}} e^{\theta/12j(j+1)\alpha} \left(\frac{(j+1)\alpha}{e} \right)^{-\alpha} = 0,
\end{aligned} \tag{2.25}$$

that is,

$$\lim_{j \rightarrow \infty} \frac{x_{j+1}}{x_j} = 0; \tag{2.26}$$

then, by the convergence principle of D'Alembert, we can claim that $\zeta_k \rightarrow \zeta$ uniformly in $[0, t_0] \times R^n$ as $k \rightarrow \infty$ and $\zeta \in C([0, t_0] \times R^n)$. Moreover, we can obtain that $u_k = u_0 + \zeta_k \rightarrow$

$u_0 + \zeta := v(t, x)$ uniformly in $[0, t_0] \times R^n$, as $k \rightarrow \infty$, and $v(t, x) \in C([0, t_0] \times R^n)$. Combining with (2.18), we have

$$v(t, x) \geq \tilde{\rho} \quad \text{in } [0, t_0] \times R^n, \quad (2.27)$$

and, letting $k \rightarrow \infty$ in (2.12), we have

$$v(t, x) = E^{t\Delta} \phi - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} v^{-\nu}(s, x) ds. \quad (2.28)$$

This mean that $v(t, x)$ is a local positive to the Cauchy problem (1.1). Thus, the proof is complete. $\square$

Remark 2.6. Under the assumption that

$$\phi_{\min} = \rho > 2 \left(\frac{T^\alpha}{\Gamma(1+\alpha)} \right)^{1/(1+\nu)}, \quad (2.29)$$

by a similar proof to that of Theorem 2.5, we can claim that problem (1.1) has positive solution on the interval $[0, T] \times R^n$. In fact, we may let

$$\left(\frac{T^\alpha}{\Gamma(1+\alpha)} \right)^{1/(1+\nu)} < \tilde{\rho} < \left\{ 1, \frac{\rho}{2} \right\}; \quad (2.30)$$

then we can obtain that

$$\begin{aligned} u_0(t, x) &= E^{t\Delta} \phi - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E^{(t-s)\Delta} \tilde{\rho}^{-\nu} ds \\ &\geq 2\tilde{\rho} - \frac{\tilde{\rho} \tilde{\rho}^{-1-\nu}}{\Gamma(1+\alpha)} t^\alpha \\ &\geq 2\tilde{\rho} - \frac{\tilde{\rho} \tilde{\rho}^{-1-\nu}}{\Gamma(1+\alpha)} T^\alpha \\ &\geq 2\tilde{\rho} - \tilde{\rho} = \tilde{\rho}. \end{aligned} \quad (2.31)$$

The remaining proofs may be finished by the same way as the Theorem 2.5.

3. Conclusion

In this paper, we obtain the existence of a local (or nonlocal) positive of problem (1.1), that is, our main results: Theorem 2.5 and Remark 2.2. It is well known that fractional-order models are more accurate than integer-order models, that is, there are more degrees of freedom in the fractional-order models. Problem (1.1) may appear in several applications in mechanics and physics and in particular can be used to model the electrostatic micro-electromechanic System devices, in which the derivatived $u_t(t, x)$ should be replace by $D_t^\alpha u(t, x)$, $0 < \alpha \leq 1$.

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