Research Article

Existence of Alternate Steady States in a Phosphorous Cycling Model

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We analyze the positive solutions to the steady-state reaction diffusion equation with Dirichlet boundary conditions of the form: $-\Delta u = \lambda[K - u + c(u^4/(1 + u^4))]$, $x \in \Omega$, $u = 0$, $x \in \partial\Omega$. Here, $\Delta u = \text{div}(\nabla u)$ is the Laplacian of u , $1/\lambda$ is the diffusion coefficient, K and c are positive constants, and $\Omega \subset \mathbb{R}^N$ is a smooth bounded region with $\partial\Omega$ in C^2 . This model describes the steady states of phosphorus cycling in stratified lakes. Also, it describes the colonization of barren soils in drylands by vegetation. In this paper, we discuss the existence of multiple positive solutions leading to the occurrence of an S-shaped bifurcation curve. We prove our results by the method of subsuper solutions.

1. Introduction

The nonlinear boundary value problem

$$\begin{aligned} -\Delta u &= \lambda \left[K - u + c \frac{u^4}{1 + u^4} \right] =: \lambda f(u), & x \in \Omega, \\ u &= 0, & x \in \partial\Omega, \end{aligned} \quad (1.1)$$

where $\Delta u = \text{div}(\nabla u)$ is the Laplacian of u , $1/\lambda$ is the diffusion coefficient, K and c are positive constants, and $\Omega \subset \mathbb{R}^N$ is a smooth bounded region with $\partial\Omega$ in C^2 . This model describes phosphorus cycling in stratified lakes (see [1]). In particular, it illustrates the decrease in the amount of phosphorus in the epilimnion (upper layer) and the rapid recycling

Figure 1: Restricted solution space for the falling zero problem.

that occurs when the hypolimnion (lower layer) is depleted of oxygen. Here, u is the mass or concentration of phosphorous (P) in the water column, and K is the rate of P input from the watershed. The rate of recycling of P is given by $cu^4/(1 + u^4)$, where c is the maximum recycling rate. The assumption here is that the recycling is primarily from the sediments. The same equation has also been used to describe plant colonization of barren soils in drylands (see [2]). In this case, u is the amount of barren soil, and $cu^4/(1 + u^4)$ represents erosion by wind and runoff.

It is known that when $f(u) > 0; [0, \infty)$ with $\lim_{u \rightarrow \infty} (f(u)/u) = 0$, a positive solution exists for all $\lambda > 0$, and this solution is unique if $u/f(u)$ is increasing. The existence of multiple positive solutions to such problems has also been studied extensively (see [3–6]). On the other hand, proving multiplicity results for nonlinearities with a falling zero (say at $r_0 > 0$) is very challenging and often remains an open problem (see [7] and example (iv) in [4]). For such problems, the solution space is restricted as $\|u\|_\infty < r_0$ (see Figure 1). Our model falls in this category.

Instead of working with the particular reaction term in (1.1), we will prove our results for a class of functions f which satisfy the following hypothesis:

$$(H1) \quad f \in C^2([0, \infty)), \quad f(u) > 0 \text{ on } [0, r_0) \text{ and } f(u) < 0 \text{ for } u > r_0.$$

To state our multiplicity result, for $0 < a < b$, let

$$Q(a, b, \Omega) := \frac{(b/f(b))((N+1)/N)^{N+1}(N^2/R^2)}{\min\{a/\|e_\Omega\|_\infty f^*(a), 2NM/f(b)R^2\}}, \quad (1.2)$$

where $B_R = B(0, R)$ is the largest inscribed ball on Ω , $f^*(s) = \max_{t \in [0, s]} f(t)$ (see Figure 2) and e_Ω is the unique positive solution of $-\Delta e = 1$ in Ω , $e = 0$ on $\partial\Omega$.

Now, we establish the following result.

Theorem 1.1. *Let $m, M \in (0, r_0)$ be such that f is nondecreasing in (m, M) . Assume that there exists $b \in [m, M]$ and $a \in (0, b)$ such that $Q(a, b, \Omega) < 1$, then (1.1) has three positive solutions for $\lambda \in ((b/f(b))((N+1)/N)^{N+1}(N^2/R^2), \min\{a/\|e_\Omega\|_\infty f^*(a), 2NM/f(b)R^2\})$.*

Figure 2: Definition of B_R and f^* .

We will use the method of subsupersolutions to prove our results. By a subsolution (supersolution) of (1.1), we mean a function $\psi \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ such that $\psi = 0$ on $\partial\Omega$ and

$$\int_{\Omega} \nabla \psi \cdot \nabla q \leq (\geq) \int_{\Omega} \lambda f(\psi) q, \quad (1.3)$$

for every $q \in \{\eta \in C_0^\infty(\Omega) : \eta \geq 0 \text{ in } \Omega\}$. Then, the following lemma holds.

Lemma 1.2. *Let ψ be a subsolution of (1.1), and let ϕ be a supersolution of (1.1) such that $\psi \leq \phi$, then, (1.1) has a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $\psi \leq u \leq \phi$.*

To establish our main multiplicity result (Theorem 1.1), we use the following very useful result discussed in [8, 9].

Lemma 1.3. *Suppose that there exists a subsolution ψ_1 , a strict supersolution Z_1 , a strict subsolution ψ_2 , and a supersolution Z_2 for (1.1) such that $\psi_1 < Z_1 < Z_2$, $\psi_1 < \psi_2 < Z_2$, and let $\psi_2 \not\leq Z_1$, then, (1.1) has at least three distinct solutions u_1, u_2 , and u_3 such that $\psi_1 \leq u_1 < u_2 < u_3 \leq Z_2$.*

Note here that by $\psi_1 < \psi_2$ we mean that $\psi_1 \leq \psi_2$ and $\psi_1 \neq \psi_2$.

We prove Theorem 1.1 in Section 2. The proof of Theorem 1.1 is motivated by the arguments in [7] where the authors establish a multiplicity result for a model used to describe a logistically growing species with grazing. In Section 3, we analyze in detail the phosphorus cycling model when $f(u) = K - u + c(u^4/(1 + u^4))$ has a unique positive zero r_0 . This will be the case when $K > K_0 := (3/4)\sqrt[4]{3/5} - (1/4)\sqrt[4]{3/5}^5$ and $c \gg 1$. We will prove that an S-shaped bifurcation curve occurs when $c \gg 1$ and $K_0 < K < 9c/16$. This analysis turned out to be quite nontrivial and challenging. This study is motivated by the results in [10, 11] where such a multiplicity result for the case $N = 1$ was discussed. Here, we extend this study for the higher dimension case. We also obtained more detailed analytical and computational results for the case $N = 1$, which are presented in the appendix.

Figure 3: Graph of $\tilde{f}(u)$.

2. Proof of Theorem 1.1

To establish the multiplicity result, we have to construct a subsolution φ_1 , a strict supersolution Z_1 , a strict subsolution φ_2 , and a supersolution Z_2 for (1.1) such that $\varphi_1 < Z_1 < Z_2$, $\varphi_1 < \varphi_2 < Z_2$, and $\varphi_2 \not\leq Z_1$. Clearly, $\varphi_1 = 0$ is a strict subsolution since $f(0) > 0$. For the large supersolution, choose $Z_2 = M(\lambda)e_\Omega$ where $M(\lambda) > \lambda \max_{t \in [0, r_0]} f(t)$. Then, $-\Delta Z_2 = M(\lambda) \geq \lambda f(Z_2)$ making Z_2 a positive supersolution.

Now for the smaller strict supersolution, define $Z_1 = ae_\Omega / \|e_\Omega\|_\infty$. Since $\lambda < a / \|e_\Omega\|_\infty f^*(a)$, $-\Delta Z_1 = a / \|e_\Omega\|_\infty > \lambda f^*(a) \geq \lambda f^*(ae_\Omega / \|e_\Omega\|_\infty) \geq \lambda f(ae_\Omega / \|e_\Omega\|_\infty) = \lambda f(Z_1)$ in Ω . Here, $f^*(s) = \max_{t \in [0, s]} f(t)$. Hence, Z_1 is a strict supersolution.

We will now construct the strict subsolution φ_2 . Let

$$\tilde{f}(u) = \begin{cases} \hat{f}(u), & u < m, \\ f(u), & u \geq m, \end{cases} \quad (2.1)$$

where $\hat{f}(u)$ is defined so that the function $\tilde{f}(u)$ is strictly increasing on $(0, M)$ and $\tilde{f}(u) \leq f(u)$ (see Figure 3).

Let

$$\rho(r) = \begin{cases} 1, & r \in [0, \epsilon], \\ 1 - \left[1 - \left(\frac{R-r}{R-\epsilon} \right)^\beta \right]^\alpha, & r \in (\epsilon, R], \quad \alpha, \beta > 1. \end{cases} \quad (2.2)$$

Note that

$$\rho'(r) = \begin{cases} 0, & r \in [0, \epsilon], \\ -\alpha\beta \left[1 - \left(\frac{R-r}{R-\epsilon} \right)^\beta \right]^{\alpha-1} \left(\frac{R-r}{R-\epsilon} \right)^{\beta-1}, & r \in (\epsilon, R], \quad \alpha, \beta > 1 \end{cases} \quad (2.3)$$

and $|\rho'(r)| < \alpha\beta/(R - \epsilon)$. Now define $w(r) := b\rho(r)$ and

$$\psi_2(x) = \begin{cases} \tilde{\psi}_2, & x \in B_R, \\ 0, & x \in \Omega - B_R, \end{cases} \quad (2.4)$$

where $\tilde{\psi}_2$ is the solution of

$$\begin{aligned} -\tilde{\psi}_2''(r) - \frac{N-1}{r}\tilde{\psi}_2'(r) &= \lambda\tilde{f}(w(r)), \quad r \in (0, R), \\ \tilde{\psi}_2'(0) &= 0 = \tilde{\psi}_2(R), \end{aligned} \quad (2.5)$$

and B_R is the largest inscribed ball in Ω . Then, $\psi_2 \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ and $\psi_2 = 0$ on $\partial\Omega$. We will now establish that $\tilde{\psi}_2(r) \in (w(r), M]$ on $[0, R)$. Then, $-\Delta\psi_2 = \lambda\tilde{f}(w(r)) < \lambda\tilde{f}(\tilde{\psi}_2(r)) \leq \lambda f(\psi_2(r))$ on $[0, R)$, while outside B_R , we have $-\Delta\psi_2 = 0 = \lambda f(0) = \lambda f(\psi_2)$, and hence, ψ_2 will be a strict subsolution.

First, we will show that $\tilde{\psi}_2(r) \leq M$. Now

$$\begin{aligned} \left(r^{N-1}\tilde{\psi}_2'(r)\right)' &= -\lambda r^{N-1}\tilde{f}(w(r)), \\ \tilde{\psi}_2'(r) &= \frac{-\lambda}{r^{N-1}} \int_0^r s^{N-1}\tilde{f}(w(s))ds, \\ \tilde{\psi}_2(t) - \tilde{\psi}_2(0) &= - \int_0^t \frac{\lambda}{r^{N-1}} \left\{ \int_0^r s^{N-1}\tilde{f}(w(s))ds \right\} dr. \end{aligned} \quad (2.6)$$

But $\tilde{\psi}_2(R) = 0$. Hence, we get

$$\begin{aligned} \tilde{\psi}_2(0) &= \int_0^R \frac{\lambda}{r^{N-1}} \left\{ \int_0^r s^{N-1}\tilde{f}(w(s))ds \right\} dr \\ &\leq \frac{\lambda\tilde{f}(b)}{N} \int_0^R r ds \\ &= \frac{\lambda f(b)}{2N} R^2 \quad \left(\text{since } b \geq m, \tilde{f}(s) = f(s) \text{ for } s \geq m \right). \end{aligned} \quad (2.7)$$

But $\lambda < 2NM/f(b)R^2$. Hence, $\|\tilde{\psi}_2\|_\infty = \tilde{\psi}_2(0) < M$.

Next, to establish $\tilde{\psi}_2 > w$ on $[0, R]$, we will show that $\tilde{\psi}'_2 < w' \leq 0$ on $[0, R]$. This will be sufficient, since $\tilde{\psi}_2(R) = w(R) = 0$. Now $w' = 0$ and $\tilde{\psi}'_2 < 0$ in the interval $[0, \epsilon]$, and hence, $\tilde{\psi}'_2 < w' \leq 0$ in that interval. For $r > \epsilon$, we have

$$\begin{aligned}
 -\tilde{\psi}'_2(r) &= \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} \tilde{f}(w(s)) ds \\
 &\geq \frac{\lambda}{r^{N-1}} \int_0^\epsilon s^{N-1} \tilde{f}(w(s)) ds \\
 &= \frac{\lambda}{r^{N-1}} \int_0^\epsilon s^{N-1} \tilde{f}(b) ds \quad (\text{since } \rho(s) = 1, s < \epsilon) \\
 &\geq \frac{\lambda \tilde{f}(b)}{R^{N-1}} \int_0^\epsilon s^{N-1} ds \\
 &= \frac{\lambda f(b)}{R^{N-1}} \frac{\epsilon^N}{N} \quad \left(\text{since } b \geq m, \tilde{f}(s) = f(s) \text{ for } s \geq m \right).
 \end{aligned} \tag{2.8}$$

We also know that $|w'(r)| \leq b\alpha\beta/(R - \epsilon)$. Hence, $|\tilde{\psi}'_2(r)| > |w'(r)|$ if $\lambda > \alpha\beta(b/f(b))(NR^{N-1}/(R - \epsilon)\epsilon^N)$. But $\min_{0 < \epsilon < R} (1/(R - \epsilon)\epsilon^N) = (N + 1)^{N+1}/N^N R^{N+1}$, and this minimum is achieved at $\epsilon_0 = NR/(N + 1)$. Since $\lambda > (b/f(b))(N^2/R^2)((N + 1)/N)^{N+1} = (b/f(b))(NR^{N-1}/(R - \epsilon_0)\epsilon_0^N)$, we can choose $\epsilon = \epsilon_0$ and $\alpha, \beta > 1$ such that $\lambda > \alpha\beta(b/f(b))(NR^{N-1}/(R - \epsilon_0)\epsilon_0^N)$. Hence, $|\tilde{\psi}'_2(r)| > |w'(r)|$ on $(0, R)$. This implies $w < \tilde{\psi}_2$. Thus, ψ_2 is a strict subsolution of (1.1) if $(b/f(b))(N^2/R^2)((N + 1)/N)^{N+1} < \lambda < 2NM/f(b)R^2$. Furthermore, $\tilde{\psi}_2(0) > w(0) = b > a = \|Z_1\|_\infty$, that is, $\psi_2 \not\leq Z_1$. Moreover, $M(\lambda)$ can be chosen large enough so that $\psi_2 < Z_2$ and $Z_1 < Z_2$. Hence, by Lemma 1.3, Theorem 1.1 holds.

3. Results for the Example $f(u) = K - u + c(u^4/(1 + u^4))$

First, we will analyze some properties of this nonlinearity. We will show that for large c we can find values of K for which the function $f(u) = K - u + c(u^4/(1 + u^4))$ satisfies (H1), and we will also identify m and M such that f is increasing in (m, M) . Clearly, $f \in C^2([0, \infty))$, $f(0) = K$, and $f'(0) = -1$.

Proposition 3.1. *If $c > 16/5\sqrt[4]{135}$, then there exists two points m_1 and m_2 such that $0 < m_1 < m_2$ and $f'(m_i) = 0$ for $i = 1, 2$.*

Proof. We have $f'(u) = -1 + (4cu^3/(1 + u^4)^2)$. So $f'(u) = 0$, when $1 = 4cu^3/(1 + u^4)^2$. Let $g(u) := 4cu^3/(1 + u^4)^2$, and let $h(u) := 1$. We have $g(u) \geq 0$, $g(0) = 0$, and $\lim_{u \rightarrow \infty} g(u) = 0$. Since $g'(u) = 4cu^2(3 - 5u^4)/(1 + u^4)^3$, we can see that $g(u)$ achieves a maximum of $(5c/16)\sqrt[4]{135}$ at $u = \sqrt[4]{3/5}$. If $\max_{u \in (0, \infty)} g(u) = (5c/16)\sqrt[4]{135} > 1$, then the line $h(u)$ will cut $g(u)$ at exactly two different points. Hence, if $c > 16/5\sqrt[4]{135}$, then there are exactly two positive points $m_1 < m_2$ such that $f'(m_i) = 0$ for $i = 1, 2$. $\square$

Proposition 3.2. *If $K > (3/4)\sqrt[4]{3/5} - (1/4)(\sqrt[4]{3/5})^5 =: K_0$, then there exists a unique $r_0 > 0$ such that $f(r_0) = 0$.*

Proof. From Figure 5, we can see that if $f(m_1) > 0$, then $f(u)$ has a unique positive zero. Since $f'(m_1) = 0$, we obtain $cm_1^3/(1 + m_1^4) = (1 + m_1^4)/4$. So, $f(m_1) = K - m_1 + cm_1^4/(1 + m_1^4) =$

Figure 4: $g(u)$ and $h(u)$.

$K - m_1 + (m_1(1 + m_1^4)/4) = K - (3/4)m_1 + m_1^5/4$. Hence, $f(m_1) > 0$ if $K > (3/4)m_1 + m_1^5/4$. On analyzing $f''(u) = 4cu^2(3 - 5u^4)/(1 + u^4)^3$, we see that the positive inflection of $f(u)$ occurs at $u = \sqrt[4]{3/5}$. Thus, $m_1 < \sqrt[4]{3/5}$, and hence $K > K_0$ ensures that there exists a unique r_0 such that $f(r_0) = 0$. $\square$

Choose $m = m_1$ and $M = m_2$. Thus, given $K > K_0$, we can find c large so that $f(u)$ is increasing on (m, M) , and there exists a unique $r_0 > 0$ such that $f(r_0) = 0$, that is, $f(u)$ satisfies (H1). Now, we will prove that the other assumptions in Theorem 1.1 hold in the given example.

We will select $b \in [m, M]$ and $a \in (0, b)$ such that $Q(a, b, \Omega) < 1$. The point at which the function $u/f(u)$ has a minimum would be an ideal choice for b .

Proposition 3.3. *If $K < 9c/16$, then $u/f(u)$ has the shape given in Figure 6.*

Proof. We have $(u/f(u))' = (f(u) - uf'(u))/(f(u))^2$. Hence, the critical points of $u/f(u)$ are given by $f(u) - uf'(u) = 0$, and in particular, the nonzero critical points are given by $K + c(u^8 - 3u^4)/(1 + u^4)^2 = 0$. Solving for u , we get the positive critical points as $\alpha = \sqrt[4]{3c - 2K - \sqrt{c(9c - 16K)}/2(c + K)}$ and $\beta = \sqrt[4]{3c - 2K + \sqrt{c(9c - 16K)}/2(c + K)}$. Note that if $K < 9c/16$, then α and β are positive real roots of $(u/f(u))'$ with $\alpha < \beta$. Hence, $u/f(u)$ has a relative maximum at α and a relative minimum at β . $\square$

Since $\beta \rightarrow \sqrt[4]{3}$ as $c \rightarrow \infty$ and $m < \sqrt[4]{3/5}$, we have $m < \beta$. Furthermore, it is clear from Figure 4 that $M \rightarrow \infty$ as $c \rightarrow \infty$, so $\beta < M$ for $c \gg 1$. Thus, we have $\beta \in [m, M]$ for large c and we choose $b = \beta$. We also choose $a \in (0, M)$ such that $f(a) = f^*(a) = f(0)$ (see Figure 7).

The following estimates hold for a and M for $c \gg 1$.

Proposition 3.4. *For $c \gg 1$,*

$$(i) \sqrt[5]{c} < M < \sqrt[4]{c},$$

$$(ii) 1/\sqrt[3]{c} < a < 1/\sqrt[4]{c}.$$

Proof. (i) By the shape of the graph of $f(u)$ established in Propositions 3.1 and 3.3 (see Figure 5), it is enough if we prove that $f'(\sqrt[5]{c}) > 0$ and $f'(\sqrt[4]{c}) < 0$. We have $f'(\sqrt[5]{c}) = -1 + 4c^{8/5}/(1 + c^{4/5})^2 = -1 + 4c^{8/5}/(1 + 2c^{4/5} + c^{8/5}) = -1 + 4/(1/c^{8/5} + 2/c^{4/5} + 1) > 0$,

Figure 5: Graph of $f(u)$ with m and M . Figure 6: Graph of $u/f(u)$.

and $f'(\sqrt[4]{c}) = -1 + 4c^{7/4}/(1+c)^2 = -1 + (4/(1/c^{7/4} + 2/c^{3/4} + c^{1/4})) < 0$ for $c \gg 1$. Thus, $\sqrt[5]{c} < M < \sqrt[4]{c}$ for large c .

(ii) We have $f(a) = K$, simplifying which we get $a^4 - ca^3 + 1 = 0$. Define $j(u) := u^4 - cu^3 + 1$. If $u < a$, then $j(u) > 0$, and if $u \in (a, M)$, then $j(u) < 0$. We have $j(1/\sqrt[3]{c}) = (1/\sqrt[3]{c})^4 - c(1/\sqrt[3]{c})^3 + 1 = 1/c^{4/3} > 0$, and $j(1/\sqrt[4]{c}) = (1/\sqrt[4]{c})^4 - c(1/\sqrt[4]{c})^3 + 1 = 1/c - \sqrt[4]{c} + 1 < 0$ for $c \gg 1$. Hence, $1/\sqrt[3]{c} < a < 1/\sqrt[4]{c}$ for large c . Note that since $b = \beta \rightarrow \sqrt[4]{3}$ and $a < 1/\sqrt[4]{c} \rightarrow 0$ as $c \rightarrow \infty$, $a \in (0, b)$ for $c \gg 1$. $\square$

Now we will discuss the existence of at least three positive solutions for a certain range of λ (see Theorem 1.1). Our aim is to prove that for $c \gg 1$ and $K_0 < K < 9c/16$, $Q(a, b, \Omega) < 1$. It is enough if we prove that

$$\frac{b}{f(b)} \left(\frac{N+1}{N} \right)^{N+1} \frac{N^2}{R^2} < \min \left\{ \frac{a}{\|e_\Omega\|_\infty f^*(a)}, \frac{2NM}{f(b)R^2} \right\}. \quad (3.1)$$

Figure 7: Graph of $f^*(u)$.

Note that for $c \gg 1$ $(b/f(b)) = \sqrt[4]{3}/(K - \sqrt[4]{3} + 3c/4)$ and $M/f(b) = M/(K - \sqrt[4]{3} + 3c/4) > \sqrt[5]{c}/(K - \sqrt[4]{3} + 3c/4)$. Also, $a/f^*(a) = a/k > 1/K\sqrt[3]{c}$. Applying the estimates we obtained for $a/f^*(a)$, $b/f(b)$, and $M/f(b)$ to the above inequality, we get the following:

$$\left(\frac{\sqrt[4]{3}}{k - \sqrt[4]{3} + 3c/4}\right) \left(\frac{N+1}{N}\right)^{N+1} \frac{N^2}{R^2} < \min \left\{ \frac{1}{\|e_\Omega\|_\infty k \sqrt[3]{c}}, \frac{2N}{R^2} \left(\frac{\sqrt[5]{c}}{k - \sqrt[4]{3} + 3c/4}\right) \right\}. \quad (3.2)$$

Simplifying the above, we can see that $Q(a, b, \Omega) < 1$ if

$$\left(\frac{\sqrt[4]{3}}{k/c - \sqrt[4]{3}/c + 3/4}\right) \left(\frac{N+1}{N}\right)^{N+1} \frac{N^2}{R^2} < \min \left\{ \frac{c^{2/3}}{\|e_\Omega\|_\infty k}, \frac{2N}{R^2} \left(\frac{c^{1/5}}{k/c - \sqrt[4]{3}/c + 3/4}\right) \right\}. \quad (3.3)$$

Clearly, this inequality is true for $c \gg 1$; hence, Theorem 1.1 holds.

Appendix

Consider the two-point boundary value problem

$$\begin{aligned} -u'' &= \lambda f(u), & x \in (0, 1), \\ u(0) &= 0 = u(1), \end{aligned} \quad (A.1)$$

where f satisfies the following hypotheses:

- (G1) $f \in C^2([0, \infty))$, $f(u) > 0$ for $0 < u < r_0$ and $f(u) < 0$ for $u > r_0$ for some $r_0 > 0$,
- (G2) there exists $k \geq 0$ such that $f(u) - f(v) \geq -k(u - v)$ for all $u, v \in [0, r_0]$ with $u > v$.

Using the quadrature method (see [12]), it follows that (A.1) has a positive solution if and only if

$$\sqrt{\lambda} = \sqrt{2} \int_0^p \frac{dz}{\sqrt{[F(\rho) - F(z)]}} := G(\rho), \quad (A.2)$$

Figure 8: Graph of $H(u)$. Figure 9: Bifurcation diagram with $K = 1$ and $c = 6$. Here, $r_0 = 6.9975$.

where $F(s) := \int_0^s f(t)dt$ and $\rho = u(1/2) = \|u\|_\infty$. Further, $u(x)$ is symmetric about $x = 1/2$ and is given by

$$\int_0^{u(x)} \frac{dz}{\sqrt{[F(\rho) - F(z)]}} = \sqrt{2\lambda}x, \quad 0 < x < \frac{1}{2}. \quad (\text{A.3})$$

Equation (A.2) describes the bifurcation curve of positive solutions of (A.1), and it follows by results in [12] that $\lim_{\rho \rightarrow 0^+} G(\rho) = 0$ and $\lim_{\rho \rightarrow r_0^-} G(\rho) = \infty$. Furthermore, from [4], we have

$$G'(\rho) = \sqrt{2} \int_0^1 \frac{H(\rho) - H(\rho s)}{[F(\rho) - F(\rho s)]^{3/2}} ds, \quad (\text{A.4})$$

where $H(u) = F(u) - u/2f(u)$. Note that $H(0) = 0$ and $H'(0) = 1/2f(0) > 0$. Hence, if there exists a point $\rho_0 \in (0, r_0)$ such that $H(\rho_0) < 0$, (see Figure 8) then $G'(\rho) < 0$ for a certain range of ρ ; thus, the bifurcation diagram must be at least S-shaped. We will now prove that such a ρ_0 exists when $K > K_0$ and $c > 5.20626K$.

Consider the case $f(u) = K - u + c(u^4/(1+u^4))$. Clearly, given $K > K_0$, then for $c \gg 1$ f satisfies (G1)-(G2) (see Proposition 3.3). Hence, $G(\rho)$ is defined for all $\rho \in S = (0, r_0)$.

Figure 10: Bifurcation diagram with $K = 1$ and $c = 15$. Here, $r_0 = 15.9998$.

We have $H(u) = F(u) - u/2f(u) = Ku/2 + cu - c(u^5/2(1 + u^4)) - c/4\sqrt{2}\{-2\tan^{-1}(1 - \sqrt{2}u) + 2\tan^{-1}(1 + \sqrt{2}u) - \ln(1 + \sqrt{2}u + u^2)/(1 - \sqrt{2}u + u^2)\}$. Clearly, $\sqrt[4]{3/5} < r_0$ (see Proposition 3.3); choose $\rho_0 = \sqrt[4]{3/5}$. Thus, $H(\rho_0) = .440056K - .0845244c$, and hence, $H(\rho_0) < 0$ if $c > 5.20626K$.

We finally used Mathematica to compute $\sqrt{\lambda} = G(\rho)$ in the case when $f(u) = K - u + c(u^4/(1 + u^4))$ and plotted the bifurcation diagrams. We found that the bifurcation diagrams are, in fact, exactly S-shaped when multiplicity occurred. Figures 9 and 10 describe the bifurcation diagrams for a certain value of c and K .

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