Research Article

Convergence and Divergence of Higher-Order Hermite or Hermite-Fejér Interpolation Polynomials with Exponential-Type Weights

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Let \( \mathbb{R} = (-\infty, \infty) \), and let \( Q \in C^1(\mathbb{R}) : \mathbb{R} \rightarrow \mathbb{R}^+ = [0, \infty) \) be an even function. Consider the weight \( w(x) = \exp(-Q(x)) \), where \( \rho > -1/2 \). Let \( w_\rho(x) := |x|^\rho w(x) \), \( x \in \mathbb{R} \).

1. Introduction

Let \( \mathbb{R} = (-\infty, \infty) \), and let \( Q \in C^1(\mathbb{R}) : \mathbb{R} \rightarrow \mathbb{R}^+ = [0, \infty) \) be an even function. Consider the weight \( w(x) = \exp(-Q(x)) \), and define, for \( \rho > -1/2 \),

\[
    w_\rho(x) := |x|^\rho w(x), \quad x \in \mathbb{R}
\]

Suppose that \( \int_{\mathbb{R}} x^n w_\rho(x) \, dx < \infty \), for all \( n = 0, 1, 2, \ldots \) Then we can construct the orthonormal
polynomials \( p_{n,\rho}(x) = p_n(w_\rho^2; x) \) of degree \( n \) for \( w_\rho^2(x) \); that is,

\[
\int_{-\infty}^{\infty} p_{n,\rho}(x)p_{m,\rho}(x)w_\rho^2(x)dx = \delta_{m,n} \quad \text{(Kronecker delta).}
\] (1.2)

We write \( p_{n,\rho}(x) \) by

\[
p_{n,\rho}(x) = \gamma_n x^n + \cdots, \quad \gamma_n = \gamma_{n,\rho} > 0,
\] (1.3)

and denote the zeros of \( p_{n,\rho}(x) \) by

\[
-\infty < x_{n,n,\rho} < x_{n-1,n,\rho} < \cdots < x_{2,n,\rho} < x_{1,n,\rho} < \infty.
\] (1.4)

Let \( \mathcal{P}_n \) denote the class of polynomials with degree at most \( n \). For \( f \in C(\mathbb{R}) \) we define the higher-order Hermite-Fejér interpolation polynomial \( L_n(\nu, f; x) \in \mathcal{P}_{\nu n-1} \) based at the zeros \( \{x_{k,n,\rho}\}_{k=1}^n \) as follows:

\[
L_n^{(i)}(\nu, f; x_{k,n,\rho}) = \delta_{0,i}f(x_{k,n,\rho}) \quad \text{for } k = 1, 2, \ldots, n, \quad i = 0, 1, \ldots, \nu - 1.
\] (1.5)

We note that \( L_n(1, f; x) \) is the Lagrange interpolation polynomial, \( L_n(2, f; x) \) is the ordinary Hermite-Fejér interpolation polynomial, and \( L_n(4, f; x) \) is the Krylov-Stayermann polynomial. For the general cases Kanjin and Sakai [1, 2] started to investigate the so-called Freud-type weights. The fundamental polynomials \( h_{k,n,\rho}(\nu; x) \in \mathcal{P}_{\nu n-1} \) for the higher-order Hermite-Fejér interpolation polynomial \( L_n(\nu, f; x) \) are defined as follows:

\[
h_{k,n,\rho}(\nu; x) = \sum_{i=0}^{\nu-1} e_i(\nu, k, n)(x - x_{k,n,\rho})^i,
\]

\[
l_{k,n,\rho}(x) = \frac{p_n(w_\rho^2; x)}{(x - x_{k,n,\rho})p_n(w_\rho^2; x_{k,n,\rho})},
\]

\[
h_{k,n,\rho}(\nu; x_{p,n,\rho}) = \delta_{k,p}, \quad h_{k,n,\rho}^{(i)}(\nu; x_{p,n,\rho}) = 0, \quad k, p = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, \nu - 1.
\]

Using them, we can write

\[
L_n(\nu, f; x) = \sum_{k=1}^{n} f(x_{k,n,\rho})h_{k,n,\rho}(\nu; x).
\] (1.7)
Furthermore, we extend the operator $L_n(v; f; x)$. Let $l$ be a nonnegative integer, and let

$$v - 1 \geq l.$$  

For $f \in C^l(\mathbb{R})$ we define the $(l, v)$-order Hermite-Fejér interpolation polynomials $L_n(l, v; f; x) \in \mathcal{D}_{vn-1}$ as follows: for each $k = 1, 2, \ldots, n$,

$$L_n(l, v; f; x_{k,n,p}) = f(x_{k,n,p}),$$  

$$L_n^{(j)}(l, v; f; x_{k,n,p}) = f^{(j)}(x_{k,n,p}), \quad j = 1, 2, \ldots, l,$$  

(1.8)  

$$L_n^{(j)}(l, v; f; x_{k,n,p}) = 0, \quad j = l + 1, l + 2, \ldots, v - 1.$$  

Especially, $L_n(0, v; f; x)$ is equal to $L_n(v; f; x)$, and for each $P \in \mathcal{D}_{vn-1}$ we see $L_n(v-1, v; P; x) = P(x)$; that is, for $f \in C^{v-1}(\mathbb{R})$, $L_n(v-1, v; f; x)$ is the Hermite interpolation polynomial. The fundamental polynomials $h_{s,k,n,p}(l; v; x) \in \mathcal{D}_{vn-1}$, $k = 1, 2, \ldots, n$, of $L_n(l, v; f; x)$ are defined by

$$h_{s,k,n,p}(l; v; x) = \frac{P_x}{P_k}(x) \sum_{i=0}^{v-1} e_{s,i}(v, k, n) (x - x_{k,n,p})^i,$$  

(1.9)  

$$h_{s,k,n,p}^{(j)}(l; v; x_{p,n,p}) = \delta_{s,j} \delta_{k,p}, \quad j, s = 0, 1, \ldots, v - 1, \quad p = 1, 2, \ldots, n.$$  

Then we have

$$L_n(l, v; f; x) = \sum_{k=1}^{n} \sum_{s=0}^{l} f^{(s)}(x_{k,n,p}) h_{s,k,n,p}(l; v; x).$$  

(1.10)  

For the ordinary Hermite and Hermite-Fejér interpolation polynomial $L_n(1, 2; f; x)$, $L_n(2, f; x)$ and the related approximation process, Lubinsky [3] gave some interesting convergence theorems.

Our purpose in this paper is to study $L_n(v; f; x)$ and $L_n(l, v; f; x)$ as certain analogies of the Lubinsky theorems in [3] and the related approximation process for the exponential-type weights. Kasuga and Sakai [4–8] investigated the convergence and divergence theorems for the Freud-type weights. Then for an even integer $v \geq 2$ we give the convergence theorems for them; moreover, for an odd integer $v \geq 1$, we obtain a certain divergence theorem with respect to $L_n(v, f; x)$. In Section 1, we give the preliminaries for these studies, and in Section 2 we write some preliminary description. In Section 3 we report our theorems with some lemmas, and in Section 4 we prove the theorems. Finally, in Section 5, for an odd integer $v \geq 1$ we obtain a certain divergence theorem with respect to $L_n(v, f; x)$.

In what follows we abbreviate several notations as $x_{k,n} := x_{k,n,p}$, $h_{k,n}(x) := h_{k,n,p}(v, x)$, $l_{k,n}(x) := l_{k,n,p}(x)$, $h_{k,n}(x) := h_{k,n,p}(v, x)$ and $p_{n}(x) := p_{n,p}(x)$ if there is no confusion. For arbitrary nonzero real-valued functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if there exist constants $C_1, C_2 > 0$ independent of $x$ such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ for all $x$. For arbitrary positive sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$, we define $c_n \sim d_n$ similarly.

Throughout this paper $C, C_1, C_2, \ldots$ denote positive constants independent of $n, x, t$, or polynomials $P_n(x)$, and the same symbol does not necessarily denote the same constant in different occurrences.
2. Preliminaries

A function \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is said to be quasi-increasing if there exists \( C > 0 \) such that \( f(x) \leq Cf(y) \) for \( 0 < x < y \).

**Definition 2.1** (see [9]). Let \( Q : \mathbb{R} \rightarrow \mathbb{R}^+ \) be a continuous even function satisfying the following properties.

1. \( Q'(x) \) is continuous in \( \mathbb{R} \) and \( Q(0) = 0 \).
2. \( Q''(x) \) exists and is positive in \( \mathbb{R} \setminus \{0\} \).
3. \( \lim_{x \to \infty} Q(x) = \infty \).
4. The function

\[
T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0,
\]

is quasi-increasing in \((0, \infty)\), with \( T(x) \geq \Lambda > 1 \) for \( x \in \mathbb{R}^+ \setminus \{0\} \).

5. There exists \( C_1 > 0 \) such that

\[
\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus \{0\}.
\]

Then we say that \( w = \exp(-Q) \) is in the class \( \mathcal{F}(C^2) \). Besides, if there exist a compact subinterval \( J(\ni 0) \) of \( \mathbb{R} \) and \( C_2 > 0 \) such that

\[
\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J,
\]

then we say that \( w = \exp(-Q) \) is in the class \( \mathcal{F}(C^2+) \).

**Example 2.2.** We present some typical examples of \( Q(x) \) satisfying \( w = \exp(-Q) \in \mathcal{F}(C^2+) \).

1. If a continuous exponential \( Q(x) \) satisfies

\[
1 < \Lambda_1 \leq \frac{(xQ'(x))'}{Q'(x)} \leq \Lambda_2
\]

with some constants \( \Lambda_i, i = 1, 2 \), then \( w(x) = \exp(-Q(x)) \) is called a Freud weight. The class \( \mathcal{F}(C^2+) \) contains the Freud weights.

2. (See [9]) For \( \alpha > 1 \) and a nonnegative integer \( r \), we put

\[
Q(x) = Q_{r,\alpha}(x) := \exp_{r,\alpha}(|x|^{\alpha}) - \exp_{r,\alpha}(0),
\]
where, for $r \geq 1$,

$$\exp_r(x) := \exp(\exp(\cdots \exp x \cdots)) \quad (r\text{-times})$$

(2.6)

and $\exp_0(x) := x$.

(3) (See [10]) For $m \geq 0$, $\alpha \geq 0$ with $\alpha + m > 1$, we put

$$Q(x) = Q_{r,a,m}(x) := |x|^m \{ \exp_r(|x|^\alpha) - \alpha^* \exp_r(0) \},$$

(2.7)

where for $r > 0$ we suppose $\alpha^* = 0$ if $\alpha = 0$; $\alpha^* = 1$ otherwise. For $r = 0$ we suppose $m > 1$ and $\alpha = 0$.

(4) (See [10]) For $\alpha > 1$, we put

$$Q(x) = Q_{\alpha}(x) := (1 + |x|)^{\alpha} - 1.$$

(2.8)

For $x > 0$, we define the Mhaskar-Rakhmanov-Saff number $a_x$ by the equation

$$x = \frac{2}{\pi} \int_0^1 a_x u Q'(a_x u) \frac{du}{(1 - u^2)^{1/2}}.$$

(2.9)

We have the following estimates for the coefficients $e_{s,i}(\nu,k,n)(e_i(\nu,k,n) := e_{0,i}(\nu,k,n))$ in (1.6) or (1.9).

**Lemma 2.3** (see [11, Theorem 2.6]). Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$. For each $k = 1,2,\ldots,n$ and $s = 0,1,\ldots,\nu - 1$, we have $e_{0,0}(\nu,k,n) = e_0(\nu,k,n) = 1$,

$$|e_{s,i}(\nu,k,n)| \leq C \left( \frac{n}{\sqrt{a_{2n}^2 - \lambda_{k,n}^2}} \right)^{i-s} \quad s = 1,2,\ldots,\nu - 1, \ i = s, s + 1,\ldots,\nu - 1.$$

(2.10)

If we consider the higher-order Hermite-Fejér interpolation polynomial $L_n(\nu,f;x)$ on a certain finite interval, then we can see a remarkable difference between the parity of $\nu$, for example, the Lagrange interpolation polynomial $L_n(1,f;x)$ and the ordinary Hermite-Fejér interpolation polynomial $L_n(2,f;x)$ ([12–17]). Also, we can see a similar phenomenon in the case of the infinite intervals ([1, 2, 4–8]). To describe these aspects, however, we need a further strengthened definition for $\nu \geq 2$ than Definition 2.1.

**Definition 2.4.** Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$, and let $\nu \geq 1$ be an integer. Assume that $Q(x)$ is a $\nu$-times continuously differentiable function on $\mathbb{R}$ and satisfies the following.

(a) $Q^{(\nu+1)}(x)$ exists and $Q^{(i)}(x), 0 \leq i \leq \nu + 1$, are positive for $x > 0$.  


(b) There exist constants $C_i > 0$ such that

$$
\left|Q^{(i+1)}(x)\right| \leq C_i \left|Q^{(i)}(x)\right| \frac{|Q(x)|}{Q(x)}, \quad x \in \mathbb{R} \setminus \{0\}, \quad i = 1, 2, \ldots, v. \tag{2.11}
$$

(c) There exist $0 \leq \delta < 1$ and $c_1 > 0$ such that

$$
Q^{(v+1)}(x) \leq C \left(\frac{1}{x}\right)^{\delta}, \quad x \in (0, c_1]. \tag{2.12}
$$

Then we say that $w(x) = \exp(-Q(x))$ is in the class $F_v(C^2+)$. 

(d) Suppose one of the following.

\begin{itemize}
  \item[(d-1)] $Q'(x)/Q(x)$ is quasi-increasing on a certain positive interval $[c_2, \infty)$.
  \item[(d-2)] $Q^{(v+1)}(x)$ is nondecreasing on a certain positive interval $[c_2, \infty)$.
  \item[(d-3)] There exist constants $C > 0$ and $0 \leq \delta < 1$ such that $Q^{(v+1)}(x) \leq C(1/x)^{\delta}$ on $(0, \infty)$.
\end{itemize}

Then one says that $w(x) = \exp(-Q(x))$ is in the class $\tilde{F}_v(C^2+)$. 

**Example 2.5** (cf. [10, Theorem 3.1]). Let $v$ be a positive integer, and let $Q_{r,a,m}$ be defined in (2.7).

(1) Let $m$ and $a$ be nonnegative even integers with $m + a > 1$. Then $w(x) = \exp(-Q_{r,a,m}) \in \tilde{F}_v(C^2+)$. 

\begin{itemize}
  \item[(a)] If $r > 0$, then we see that $Q_{r,a,m}(x)/Q_{r,a,m}(x)$ is quasi-increasing on a certain positive interval $(c_1, \infty)$ and $Q_{r,a,m}(x)$ is nondecreasing on $(0, \infty)$.
  \item[(b)] If $r = 0$, then we see that $Q_{0,0,m}(x), m \geq 2$, is nondecreasing on $(0, \infty)$.
\end{itemize}

Hence, $w(x) = \exp(-Q_{r,a,m}) \in \tilde{F}_v(C^2+)$. 

(2) Let $m + a - v > 0$. Then $w(x) = \exp(-Q_{r,a,m}) \in \tilde{F}_v(C^2+)$, and one has the following.

\begin{itemize}
  \item[(c)] If $r \geq 2$ and $a > 0$, then there exists a constant $c_1 > 0$ such that $Q_{r,a,m}(x)/Q_{r,a,m}(x)$ is quasi-increasing on $(c_1, \infty)$.
  \item[(d)] Let $r = 1$. If $a \geq 1$, then there exists a constant $c_2 > 0$ such that $Q_{1,a,m}(x)/Q_{1,a,m}(x)$ is quasi-increasing on $(c_2, \infty)$ and, if $0 < a < 1$, then $Q_{1,a,m}(x)/Q_{1,a,m}(x)$ is quasi-decreasing on $(c_2, \infty)$.
  \item[(e)] Let $r = 1$ and $0 < a < 1$, then $Q_{1,a,m}(x)$ is nondecreasing on a certain positive interval on $(c_2, \infty)$.
\end{itemize}

Hence, $w(x) = \exp(-Q_{r,a,m}) \in \tilde{F}_v(C^2+)$. 

**Definition 2.6.** One uses the following notation:

$$
\varphi_u(x) = \begin{cases} \frac{a_u}{u} \left(1 - \frac{|x|}{a_u} \right), & |x| \leq a_u; \\
\frac{a_u}{u} \sqrt{1 - |x|/a_u} + \delta_u, & a_u < |x|; \\
\varphi_u(a_u), & \delta_u = (uT(a_u))^{-2/3}, \quad u > 0. \tag{2.13} \end{cases}
$$
Lemma 2.7 (see [18, Corollary 4.5]). Let \( \omega_{r}(x) = |x|^r \exp(-Q(x)), \exp(-Q) \in \mathcal{F}_{r}(C^2+) \). If \( x_{k,n} \neq 0 \) and \( |x_{k,n}| \leq a_n(1 + \delta_n) \), then \( e_0(v, k, n) = 1 \) and, for \( i = 1, 2, \ldots, v - 1, \)

\[
|e_i(v, k, n)| \leq C \left\{ \frac{T(a_n)}{a_n} + \left| Q'(x_{k,n}) \right| + \frac{1}{|x_{k,n}|} \right\}^{(i)} \left( \frac{n}{a_{2n} - |x_{k,n}|} + \frac{T(a_n)}{a_n} \right)^{-i}, \tag{2.14}
\]

where

\[
(i) = \begin{cases} 
1, & i: \text{odd}, \\
0, & i: \text{even}. 
\end{cases} \tag{2.15}
\]

For \( x_{k,n} = 0 \) one has

\[
e_0(v, k, n) = 1, \quad |e_i(v, k, n)| \leq C \left( \frac{n}{a_n} \right)^i, \quad i = 1, 2, \ldots, v - 1. \tag{2.16}
\]

Remark 2.8. In [19, Theorem 2.2] we see that \( x_{1,n} < a_n \) if \( n \) is large enough. Therefore Lemma 2.7 holds for all \( x_{k,n}, k = 1, 2, \ldots, n \).

Levin and Lubinsky (see [9, Lemma 3.7]) showed that there exists \( C > 0 \) such that for some \( \varepsilon > 0 \) and for large enough \( t \),

\[
T(a_t) \leq Ct^{2-\varepsilon}. \tag{2.17}
\]

In [20] we have the following estimations.

Lemma 2.9 (see [20, Theorem 1.6]). Let \( \omega = \exp(-Q) \in \mathcal{F}(C^2+) \).

(1) Let \( T(x) \) be unbounded. Then, for any \( \eta > 0 \), there exists \( C(\eta) > 0 \) such that, for \( t \geq 1 \),

\[
a_t \leq C(\eta)t^{\eta}. \tag{2.18}
\]

(2) Let \( \lambda := C_1 \) be the constant in Definition 2.1(e), that is,

\[
\frac{Q''(x)}{Q'(x)} \leq \lambda \frac{Q'(x)}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus \{0\}. \tag{2.19}
\]

If \( \lambda > 1 \), then there exists \( C(\lambda, \eta) \) such that

\[
T(a_t) \leq C(\lambda, \eta)t^{2(\eta+\lambda-1)/(\lambda+1)}, \quad t \geq 1, \tag{2.20}
\]

and, if \( 0 < \lambda \leq 1 \), then for any \( \eta > 0 \) there exists \( C(\lambda, \eta) \) such that

\[
T(a_t) \leq C(\lambda, \eta)t^{\eta}, \quad t \geq 1. \tag{2.21}
\]
Remark 2.10. (1) If $T(x)$ is bounded, then $w$ is called a Freud-type weight, and, if $T(x)$ is unbounded, then $w$ is called an Erdős-type weight.

(2) In (2.20) and (2.21), we set $0 < \eta < 2$ and

$$\varepsilon = \begin{cases} 2 - \eta, & 0 < \lambda \leq 1, \\ 2(2 - \eta) \lambda + 1, & \lambda > 1. \end{cases}$$ (2.22)

Then (2.17) holds.

(3) If

$$\limsup_{x \to \infty} \frac{Q(x)Q''(x)}{(Q'(x))^2} \leq 1,$$ (2.23)

then we have (2.21). Note that all the examples in Example 2.5 satisfy this inequality.

(4) For the Freud-type exponent $Q(x) = |x|^m$, $m > 1$, we have

$$T(a_n) = m, \quad a_n \sim t^{1/m}. \quad (2.24)$$

(5) The inequality (2.18) implies

$$0 < C \leq \frac{n}{a_n} \left\{ \frac{T(a_n)}{a_n} \right\}^{\nu-1}.$$ (2.25)

### 3. Theorems

In the rest of this paper we assume the following for the weight $w$.

**Assumption 3.1.** Consider the weight $w_\rho(x) = |x|^\rho \exp(-Q(x)), \exp(-Q(x)) \in \tilde{\mathcal{F}}(\mathbb{C}^2), \rho \geq 0$.

(a) (cf. [20, Theorem 1.4]) If $T(x)$ is bounded, then we suppose, for $\delta$ in (2.12),

$$a_n \leq C n^{1/(1+\nu-\delta)}. \quad (3.1)$$

(b) There exist $0 \leq \gamma < 1$ and $C(\gamma) > 0$ such that

$$T(a_n) \leq C(\gamma) n^\gamma; \quad (3.2)$$

here, if $T(x)$ is bounded, that is, a Freud-type weight, then we set $\gamma = 0$, and if $T(x)$ is unbounded, that is, an Erdős-type weight, then we set $0 < \gamma < 1$. Define

$$\varepsilon_n := \begin{cases} \frac{a_n}{n}, & \frac{T(a_n)}{a_n} < 1, \\ \frac{1}{n^{1-\gamma}}, & \frac{T(a_n)}{a_n} \geq 1. \end{cases} \quad (3.3)$$
Remark 3.2. (1) If $T(x)$ is unbounded, then (3.1) holds because of Lemma 2.9 (2.21). (2) (3.2) holds for

$$
\gamma = \frac{2(\eta + \lambda - 1)}{\lambda + 1}, \quad 0 < \lambda < 3. \quad (3.4)
$$

(3) In (3.3) we note that $\varepsilon_n \log n \to 0$ as $n \to \infty$.

We shall state our theorems. Put

$$\begin{align*}
X_n(v, f; x) &:= \sum_{j=1}^{n} f(x_{j,n,p}) \Gamma_{j,n,p}^{(v)}(x) \sum_{i=0}^{v-2} e_i(v, j, n) (x - x_{j,n,p})^i, \\
Y_n(v, f; x) &:= \sum_{j=1}^{n} f(x_{j,n,p}) \Gamma_{j,n,p}^{(v-1)}(x) e_{v-1}(v, j, n) (x - x_{j,n,p})^{v-1}, \\
Z_n(l, v, f; x) &:= \sum_{j=1}^{n} \sum_{s=l+1}^{v} f^{(s)}(x_{j,n,p}) \Gamma_{j,n,p}^{(v)}(x) \sum_{i=s}^{v-1} e_{s}(v, j, n) (x - x_{j,n,p})^i.
\end{align*}$$

Furthermore, we consider the class $G = \{g_s \in C(\mathbb{R}), s = l+1,l+2,\ldots,v-1\}$ and construct the following interpolation polynomial:

$$W_n(l, v, G; x) := \sum_{j=1}^{n} \sum_{s=l+1}^{v} g_s(x_{j,n,p}) \Gamma_{j,n,p}^{(v)}(x) \sum_{i=s}^{v-1} e_{s}(v, j, n) (x - x_{j,n,p})^i. \quad (3.6)$$

Then we have

$$\begin{align*}
L_n(v, f; x) &= X_n(v, f; x) + Y_n(v, f; x), \\
L_n(l, v, f; x) &= L_n(v, f; x) + Z_n(l, v, f; x), \\
\tilde{L}_n(l, v, f \oplus G; x) &= L_n(v, f; x) + Z_n(l, v, f; x) + W_n(l, v, G; x).
\end{align*}$$

Define

$$\Phi(x) := \frac{1}{(1 + Q(x))^{2/3} T(x)}. \quad (3.8)$$

Here we note that, for some $d > 0$,

$$\Phi(x) \sim \frac{Q(x)^{1/3}}{x Q(x)}, \quad |x| \geq d > 0. \quad (3.9)$$

Moreover, we define

$$\Phi_n(x) := \max\left\{ \delta_n, 1 - \frac{|x|}{a_n} \right\}, \quad n = 1, 2, 3, \ldots. \quad (3.10)$$
Lemma 3.3. Let \( \omega = \exp(-Q) \in \mathcal{F}(C^{2+}) \). Let \( L > 0 \) be fixed. Then one has the following:

(a) (See [9, Lemma 3.5(a)]) Uniformly for \( t > 0 \),

\[
a_{Lt} \sim a_t. \quad (3.11)
\]

(b) (See [9, Lemma 3.5(b)]) Uniformly for \( t > 0 \),

\[
Q^{(j)}(a_{Lt}) \sim Q^{(j)}(a_t), \quad j = 0, 1. \quad (3.12)
\]

Moreover,

\[
T(a_{Lt}) \sim T(a_t). \quad (3.13)
\]

(c) (See [9, Lemma 3.11 (3.52)]) Uniformly for \( t > 0 \),

\[
\left| 1 - \frac{a_{Lt}}{a_t} \right| \sim \frac{1}{T(a_t)}. \quad (3.14)
\]

(d) (See [9, Lemma 3.4 (3.17), (3.18)]) Uniformly for \( t > 0 \), one has

\[
Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}}, \quad Q'(a_t) \sim \frac{t \sqrt{T(a_t)}}{a_t}. \quad (3.15)
\]

Lemma 3.4. For \( x \in \mathbb{R} \), we have

\[
\Phi(x) \leq C\Phi_n(x), \quad n \geq 1. \quad (3.16)
\]

Proof. Let \( x = a_u, u \geq 1 \). By Lemma 3.3(d) we have

\[
u \sim Q(a_u) \sqrt{T(a_u)}. \quad (3.17)
\]

So, we have

\[
\delta_{a_u}^{-1} \sim (a_u)^{2/3} T(a_u) = \frac{a_u Q'(a_u)}{Q(a_u)^{1/3}} = \frac{xQ'(x)}{Q^{1/3}(x)} \sim Q^{-1}(a_u). \quad (3.18)
\]
Now, if \( u \leq n/2 \), then

\[
1 - \frac{a_u}{a_n} \geq 1 - \frac{a_{n/2}}{a_n} - \frac{1}{T(a_n)} \quad \text{(by Lemma 3.2(c))}
\]

\[
\geq \frac{1}{(nT(a_n))^{2/3}} = \delta_n \quad \text{(by (2.17)).}
\]

So, we have

\[
\Phi_n(x) = 1 - \frac{a_u}{a_n} \geq 1 - \frac{a_u}{a_{2u}} - \frac{1}{T(a_u)} \quad \text{(by Lemma 3.2(c))}
\]

\[
\geq \frac{1}{(uT(a_u))^{2/3}} = \delta_u - \Phi(x) \quad \text{(by (2.17) and (3.18)).}
\]

Let \( n/2 < u < n \). Then we have

\[
\Phi_n(x) \geq \delta_n - \delta_u - \Phi(x) \quad \text{(by Lemmas 3.2(a), (2.17), and (3.18)).}
\]  

Let \( C(f) \) denote a positive constant depending only on \( f \).

**Assumption 3.5.** Let \( w_\nu(x) = |x|^\nu w(x) \) and \( w(x) = \exp(-Q(x)) \in C^\nu(C^2+) \), \( \rho \geq 0 \). Suppose the following.

(A-1) Let \( f \in C(\mathbb{R}) \) satisfy that, for a given \( 0 < \delta < 1 \),

\[
|f(x)|w^{\nu-\delta}(x)\left\{\left|Q'(x)\right| + \frac{1}{|x|}\right\} \leq C(f), \quad x \in \mathbb{R},
\]

where we suppose \( \limsup_{|x| \to 0} |f(x)|/x | \leq C(f) \).

(A-2) Let \( f \in C^l(\mathbb{R}) \) for a certain \( 0 < l \leq \nu - 1 \). Then we suppose that \( f \) satisfies

\[
|f^{(s)}(x)|\left\{\Phi^{-3/4}(x)w_\rho(x)\right\}^\nu \leq C(f), \quad x \in \mathbb{R}, \ s = 1, 2, \ldots, l.
\]

(A-3) Let \( G = \{g_s \in C(\mathbb{R}), s = l + 1, l + 2, \ldots, \nu - 1\} \). Then we suppose that there exists a constant \( C > 0 \) such that

\[
|g_s(x)|\left\{\Phi^{-3/4}(x)w_\rho(x)\right\}^\nu \leq C(G), \quad x \in \mathbb{R}, \ s = l + 1, l + 2, \ldots, \nu - 1.
\]
Remark 3.6. In (3.22), we have the following.

1. \( f(0) = 0 \) and
\[
|f(x)| \left\{ \Phi^{-3/4}(x)w_p(x)^2 \right\} (|Q'(x)| + \frac{1}{|x|}) \leq C(f), \quad x \in \mathbb{R} \setminus \{0\}. \tag{3.25}
\]

2. For some positive constant C, we have \(|Q'(x)| + 1/|x| \geq C\). Hence, from (3.22), it follows that
\[
|f(x)| \left\{ \Phi^{-3/4}(x)w_p(x)^2 \right\} (|Q'(x)| + 1/|x|) \leq C(f), \quad x \in \mathbb{R}. \tag{3.26}
\]

We have a chain of results under Assumption 3.1.

Proposition 3.7. Let \( \nu = 1, 2, 3, \ldots \). For \( f(x) \) satisfying (3.26), one has
\[
\left\| \left\{ \Phi^{3/4}(x)w(x)^2 \right\} (|x| + \frac{a_n}{n})^\rho \right\|_{L^\infty(\mathbb{R})} \leq C(f). \tag{3.27}
\]

Proposition 3.8. Let \( \nu = 2, 4, 6, \ldots \). For \( f(x) \) satisfying (3.25), one has
\[
\left\| \left\{ \Phi^{3/4}(x)w(x)^2 \right\} (|x| + \frac{a_n}{n})^\rho \right\|_{L^\infty(\mathbb{R})} \leq C(f) \epsilon_n \log n, \tag{3.28}
\]
where \( \epsilon_n \) is defined by (3.3).

Proposition 3.9. Let \( \nu = 1, 2, 3, \ldots \). For \( f(x) \) satisfying (3.23), one has
\[
\left\| \left\{ \Phi^{3/4}(x)w(x)^2 \right\} (|x| + \frac{a_n}{n})^\rho \right\|_{L^\infty(\mathbb{R})} \leq C(f) \frac{a_n \log n}{n}, \tag{3.29}
\]
and for \( f(x) \) satisfying (3.24) one has
\[
\left\| \left\{ \Phi^{3/4}(x)w(x)^2 \right\} (|x| + \frac{a_n}{n})^\rho \right\|_{L^\infty(\mathbb{R})} \leq C(G) \frac{a_n \log n}{n}. \tag{3.30}
\]

Proposition 3.10. Let \( \nu = 1, 2, 3, \ldots \). Let \( P \in \mathcal{P}_m \) be fixed. Then one has
\[
\left\| \left\{ \Phi^{3/4}(x)w(x)^2 \right\} (|x| + \frac{a_n}{n})^\rho \right\|_{L^\infty(\mathbb{R})} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.31}
\]

Proposition 3.11. Let \( \nu = 2, 4, 6, \ldots \). Let \( P \in \mathcal{P}_m \) with \( P(0) = 0 \) be fixed. Then one has
\[
\left\| \left\{ \Phi^{3/4}(x)w(x)^2 \right\} (|x| + \frac{a_n}{n})^\rho \right\|_{L^\infty(\mathbb{R})} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.32}
\]
Theorem 3.12. Let \( \nu = 2, 4, 6, \ldots \) For \( f(x) \) satisfying (3.22), one has

\[
\left\| \left\{ \Phi^{3/4}(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\}^\nu \left( L_n(\nu, f; x) - f(x) \right) \right\|_{L_\infty \mathbb{R}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.33}
\]

Theorem 3.13. Let \( \nu = 2, 4, 6, \ldots \) For \( f(x) \) satisfying (3.22) and (3.23), one has

\[
\left\| \left\{ \Phi^{3/4}(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\}^\nu \left( L_n(l, \nu, f; x) - f(x) \right) \right\|_{L_\infty \mathbb{R}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.34}
\]

Corollary 3.14. Let \( \nu = 2, 4, 6, \ldots \) For \( f(x) \) satisfying (3.22), one has

\[
\left\| \left\{ \Phi^{3/4}(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\}^\nu \left( X_n(\nu, f; x) - f(x) \right) \right\|_{L_\infty \mathbb{R}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.35}
\]

Theorem 3.15. Let \( \nu = 2, 4, 6, \ldots \) For \( f(x) \) satisfying (3.22), (3.23) and (3.24), one has

\[
\left\| \left\{ \Phi^{3/4}(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\}^\nu \left( \tilde{L}_n(l, \nu, f; x) - f(x) \right) \right\|_{L_\infty \mathbb{R}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.36}
\]

Define

\[
I_n[k, H_n(f)] := \int_{-\infty}^{\infty} k(x) H_n(f; x) \, dx, \tag{3.37}
\]

where \( H_n(f) \) equals to one of the following:

\[
L_n(\nu, f), L_n(l, \nu, f), \tilde{L}_n(l, \nu, f \circ G), X_n(\nu, f). \tag{3.38}
\]

One also defines

\[
I[k, f] := \int_{-\infty}^{\infty} k(x) f(x) \, dx. \tag{3.39}
\]

Theorem 3.16. Let \( \nu = 2, 4, 6, \ldots \) Let \( f(x) \) satisfy (3.22), (3.23) and (3.24). If

\[
\int_{-\infty}^{\infty} k(x) \left\{ \Phi^{3/4}(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\}^\nu \, dx < \infty, \quad n = 1, 2, 3, \ldots, \tag{3.40}
\]

then we have

\[
\lim_{n \to \infty} I_n[k, H_n(f)] = I[k, f]. \tag{3.41}
\]
For example, we can take $k(x)$ in the following way. If $T(x)$ is unbounded, we have, for $\Delta \in \mathbb{R}$,

$$
\int_1^{\infty} (1 + |x|)^{-\Delta} \Phi^{3/4}(x)\,dx \leq \int_1^{\infty} (1 + |x|)^{-\Delta} \left( \frac{1}{T(x)Q(x)^{2/3}} \right)^{3\nu/4} \,dx < \infty \quad (3.42)
$$

(see [10, Lemma 2.1(b)]). Then we set

$$
k(x) := (1 + |x|)^{\nu-\Delta} \left[ \Phi^{3/2}(x)w(x) \right]^\nu. \quad (3.43)
$$

If $T(x)$ is bounded and $w(x) = \exp(-|x|^{\beta})$, $\beta > 1$, then we take $\Delta$ as $\nu\beta/2 + \Delta > 1$ so that (3.43) can hold. Then $k(x)$ defined by (3.43) satisfies (3.40).

### 4. Proof of Theorems

For constants $C, C_1 > 0$, the same symbol does not necessarily denote the same constant in different occurrences.

**Lemma 4.1.** One has the following. (1) (See [19, Theorem 2.3]) Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2)$ and $\rho > -1/2$. Then, uniformly for $n \geq 1$ one has

$$
\sup_{x \in \mathbb{R}} |p_{n,\rho}(x)w(x)| \left( |x| + \frac{a_n}{n} \right)^{\rho} \left| x^2 - a_n^2 \right|^{1/4} \sim 1, \quad (4.1)
$$

and for $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$,

$$
\sup_{x \in \mathbb{R}} |p_{n,\rho}(x)w(x)| \left( |x| + \frac{a_n}{n} \right)^{\rho} \sim a_n^{-1/2}(nT(a_n))^{1/6}. \quad (4.2)
$$

(2) (See [19, Theorem 2.5(d)]) Let $w \in \mathcal{F}(C^2+)$ and $\rho > -1/2$. Let $1 \leq j \leq n - 1$ and $x \in [x_{j+1,n}, x_{j,n}]$. Then

$$
|p_n(x)w(x)| \left( |x| + \frac{a_n}{n} \right)^{\rho} \sim \min \left\{ |x - x_{j,n}|, |x - x_{j+1,n}| \right\} \phi_n(x_{j,n})^{-1} \left| a_n^2 - x_{j,n}^2 \right|^{-1/4}. \quad (4.3)
$$

(3) (See [19, Theorem 2.5(c)]) Let $w \in \mathcal{F}(C^2+)$ and $\rho > -1/2$. Then one has

$$
\max_{x \in \mathbb{R}} \left| \frac{p_n(x)w(x)(|x| + a_n/n)^{\rho}}{(x - x_{j,n})p_n'(x_{j,n})w(x_{j,n})(|x_{j,n}| + a_n/n)^{\rho}} \right| = \max_{x \in \mathbb{R}} \left| p_n(x)w(x) \left( |x| + \frac{a_n}{n} \right)^{\rho} \right| - 1. \quad (4.4)
$$
(4) (See [19, Theorem 2.5(a)]) Let \( w \in \mathcal{F}(C^2+) \) and \( \rho > -1/2 \). For \( 1 \leq j \leq n \) we have
\[
|p_n'(x)| (\left| x_{j,n} + \frac{a_n}{n} \right|)^\rho \sim (x_{j,n})^{-1/4}. \tag{4.5}
\]

**Lemma 4.2** (see [19, Theorem 2.2]). Let \( w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+) \) and \( \rho > -1/2 \). For the zeros \( x_{j,n} = x_{j,n,\rho} \), one has the following:

(1) For \( n \geq 1 \) and \( 1 \leq j \leq n - 1 \),
\[
x_{j,n} - x_{j+1,n} - \varphi_n(x_{j,n}),
\]
\[
\varphi_n(x_{j,n}) - \varphi_n(x_{j+1,n}) \quad \text{(see ([19, Lemma A.1(A.3)]))}. \tag{4.6}
\]

(2) For the minimum positive zero \( x_{[n/2],n} \) (\( \lfloor n/2 \rfloor \) is the largest integer \( \leq n/2 \)), one has
\[
x_{[n/2],n} - a_n n^{-1}. \tag{4.7}
\]

and for large enough \( n \),
\[
1 - \frac{x_{1,n}}{a_n} \sim \delta_n. \tag{4.8}
\]

(3) (See [19, Lemma 4.7]) \( b_n = y_{n-1} / y_n - a_n \sim x_{1,n} \), where \( y_n \) is defined by (1.3).

**Lemma 4.3.** Let \( w \in \mathcal{F}(C^2+) \). Then there exist \( C_1, C_2 > 0 \) such that
\[
\sup_{x \in \mathbb{R}} \left| p_n(x) w(x) \right| \left( |x| + \frac{a_n}{n} \right)^\rho \Phi_n^{1/4}(x) \leq C_1 \sup_{x \in \mathbb{R}} \left| p_n(x) w(x) \right| \left( |x| + \frac{a_n}{n} \right)^\rho \Phi_n^{1/4}(x) \leq C_2 a_n^{1/2}. \tag{4.9}
\]

**Proof.** The first inequality follows from Lemma 3.4. We show the second inequality. Noting (4.8), from Lemma 4.1 (4.1) we have
\[
C \geq \sup_{|x| \leq x_{1,n}} \left| p_n(x) w(x) \right| \left( |x| + \frac{a_n}{n} \right)^\rho \left| x^2 - a_n^2 \right|^{1/4}
\]
\[
\sim \sup_{|x| \leq x_{1,n}} \left| p_n(x) w(x) \right| \left( |x| + \frac{a_n}{n} \right)^\rho a_n^{1/2} \Phi_n^{1/4}(x). \tag{4.10}
\]
From (4.2) we see that

\[ a_n^{-1/2} \geq C \sup_{|x| > x_{l,n}} |p_{n,\rho}(x) w(x)| \left( \left| x + \frac{a_n}{n} \right| \Phi_n^{-1/4}(x) \right), \]

\[ \sim C \sup_{|x| > x_{l,n}} |p_{n,\rho}(x) w(x)| \left( \left| x + \frac{a_n}{n} \right|^\rho \Phi_n^{1/4}(x). \right. \] (4.11)

Therefore we have the result.

Proof of Proposition 3.7. We recall the definition of \( X_n(\nu, f; x) \):

\[ X_n(\nu, f; x) = \sum_{j=1}^{n} f(x_{j,n}) l_{j,n}^\nu(x) \sum_{i=0}^{\nu-2} c_i(n, \nu, j, n) (x - x_{j,n})^i \]

\[ = \sum_{i=0}^{\nu-2} \sum_{j=1}^{n} f(x_{j,n}) l_{j,n}^\nu(x) (x - x_{j,n})^i c_i(n, \nu, j, n). \] (4.12)

Using Lemma 2.3, we may estimate, for \( i = 0, 1, \ldots, \nu - 2 \) and \( 1 \leq j \leq n \),

\[ A_{i,j}(x) := \left( \Phi_3^{3/4}(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\nu \right) \left( |f(x_{j,n})| \right)^\nu \left| l_{j,n}(x) \right| |x - x_{j,n}|^i \left( \frac{n}{\sqrt{a_n^2 - x_{j,n}^2}} \right)^i. \] (4.13)

Let \((x_{m+1,n} + x_{m,n})/2 < x \leq x_{m,n}\) or \(x_{m,n} \leq x < (x_{m-1,n} + x_{m,n})/2\). For simplicity, we assume \( x > 0 \), and let \( x_{0,n} := x_{1,n} + \delta \varphi_n(x_{1,n}) \) for a fixed \( \delta > 0 \) small enough. Then we can assume that there exists a constant \( c > 0 \) such that

\[ x_{0,n} := x_{1,n} + \delta \varphi_n(x_{1,n}) < a_n - c\delta_n. \] (4.14)

Assume that \( x < x_{0,n} \). Then we first estimate \( A_{i,j}(x) \). By Lemma 4.1(3) and the definition of \( \varphi_n(x), \) we have

\[ \left( \Phi_3^{3/4}(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\nu \right)^\nu \left| l_{m,n}(x) \right| |x - x_{m,n}|^i \leq C \Phi_3^{3/4}(x) \left( w(x_{m,n}) \left( |x_{m,n}| + \frac{a_n}{n} \right)^\rho \right)^\nu, \] (4.15)

\[ \left| x - x_{m,n} \right|^i \left( \frac{n}{\sqrt{a_n^2 - x_{m,n}^2}} \right)^i \leq C \left( \frac{n \varphi_n(x_{m,n})}{\sqrt{1 - |x_{m,n}|/a_n}} \right)^i \leq C \left( \frac{1 - |x_{m,n}|/a_n}{\sqrt{1 - |x_{m,n}|/a_n}} \right)^i, \] (4.16)

and by Remark 3.6

\[ |f(x_{m,n})| \leq C(f) \Phi_3^{3/4}(x_{m,n}) \left( w(x_{m,n}) \left( |x_{m,n}| + \frac{a_n}{n} \right)^\rho \right)^{-\nu}. \] (4.17)
Therefore, we have

\[
A_{i,m}(x) \leq C(f) \Phi^{3v/4}(x) \Phi^{3v/4}(x_m) \left( \frac{\sqrt{1 - |x_m|/a_{2n}}}{\sqrt{1 - |x_m|/a_n}} \right)^i
\]

\[
= C(f) \Phi^{3v/4}(x) \left( \max \left\{ 1 - \frac{|x_m|}{a_n} \delta_n \right\} \right)^{3v/4} \left( \frac{\sqrt{1 - |x_m|/a_{2n}}}{\sqrt{1 - |x_m|/a_n}} \right)^i
\]

\[
= C(f) \Phi^{3v/4}(x) \left( 1 - \frac{|x_m|}{a_n} \right)^{3v/4 - 1/2} \left( \frac{\sqrt{1 - |x_m|/a_{2n}}}{\sqrt{1 - |x_m|/a_n}} \right)^i
\]

\[
\leq C(f).
\]

Next, we estimate \( \sum_{1 \leq j \leq n, j \neq m} A_{i,j}(x) \). For \( 1 \leq j \leq n, j \neq m \), we have, by Lemma 4.3 and Lemma 4.4,

\[
\left( \Phi^{3v/4}(x) w(x) \left| |x| + a_n/n \right|^\rho \right)^v \left| |x - x_{i,n}| \right|^i
\]

\[
= \Phi^{v/2}(x) \left[ \frac{\rho w(x) \Phi^{1/4}(x) w(x) \left| |x| + a_n/n \right|^\rho}{\rho w(x_{i,n}) \left| |x_{i,n}| + a_n/n \right|^\rho} \right]^v \left| \frac{w(x_{i,n}) \left( |x_{i,n}| + a_n/n \right)^\rho}{|x - x_{i,n}|^{v-i}} \right.
\]

\[
= \Phi^{v/2}(x) a_n^{-v/2} q_n^v(x_{i,n}) \left( \frac{w(x_{i,n}) \left( |x_{i,n}| + a_n/n \right)^\rho}{|x - x_{i,n}|^{v-i}} \right.
\]

Then, since we know from Remark 3.6 that

\[
\left| f(x_{i,n}) \right| \leq C(f) \Phi^{3v/4}(x_{i,n}) \left( w(x_{i,n}) \left| |x_{i,n}| + a_n/n \right|^\rho \right)^{-v},
\]

we have, by Lemma 3.4,

\[
\sum_{j \neq m} A_{i,j}(x) \leq C(f) \sum_{j \neq m} \Phi^{v/2}(x) \Phi^{3v/4}(x_{i,n}) q_n^v(x_{i,n})
\]

\[
\times \left( 1 - \frac{|x_{i,n}|}{a_n} \right)^{v/4} \left( \frac{1}{|x - x_{i,n}|} \right)^{v-i} \left( \frac{n}{\sqrt{a_{2n}^2 - x_{i,n}^2}} \right)^i.
\]

By the definition of \( q_n(x) \) and by Lemma 3.3(a) \( a_{2n} \sim a_n \), we have

\[
q_n^v(x_{i,n}) \left( \frac{n}{\sqrt{a_{2n}^2 - x_{i,n}^2}} \right)^i \leq C \left( \frac{a_n}{n} \right)^{v-i} \left( 1 - \frac{|x_{i,n}|}{a_{2n}} \right)^{v-i/2} \left( 1 - \frac{|x_{i,n}|}{a_n} \right)^{-v/2}.
\]
By Lemma 4.2(1), we have

\[
\frac{1}{|x-x_{j,n}|} \leq C \frac{1}{\sum_{j \leq s \leq m+1 \text{ or } m-1 \leq s \leq j} \varphi_n(x_{s,n})} \\
\leq C \frac{n/a_n}{\sum_{j \leq s \leq m+1 \text{ or } m-1 \leq s \leq j} (1 - |x_{s,n}|/a_{2n})/(1 - |x_{s,n}|/a_n + \delta_n)^{1/2}} \\
\leq C \frac{n/a_n}{\sum_{j \leq s \leq m+1 \text{ or } m-1 \leq s \leq j} (1 - |x_{s,n}|/a_n)/(1 - |x_{s,n}|/a_n + \delta_n)^{1/2}} \\
\leq C \frac{n/a_n}{|m-j|\left(1 - \max\{|x|, |x_{j,n}|\}/a_n\right)}^{1/2}.
\] (4.23)

Here, we note from (4.14) and (4.8) that

\[
\Phi_n(x) \sim \left(1 - \frac{|x|}{a_n}\right), \quad \Phi_n(x_{j,n}) \sim \left(1 - \frac{|x_{j,n}|}{a_n}\right).
\] (4.24)

Thus, we have, for \(j \neq m\),

\[
\Phi_n^{v/2}(x)\Phi_n^{3v/4}(x_{j,n})\varphi_n^{v}(x_{j,n}) \left(1 - \frac{|x_{j,n}|}{a_n}\right)^{v/4} \left(1 - \frac{1}{|x-x_{j,n}|}\right)^{v-i} \left(\frac{n}{\sqrt{a_{2n}^2 - x_{j,n}^2}}\right)^i \\
\leq C \left(1 - \frac{|x|}{a_n}\right)^{v/2} \left(1 - \frac{|x_{j,n}|}{a_n}\right)^{v/2} \left(1 - \frac{|x_{j,n}|}{a_{2n}}\right)^{v-i} \\
\times \left(\frac{1}{|m-j|\left(1 - \max\{|x|, |x_{j,n}|\}/a_n\right)}\right)^{v-i} \\
\leq C \left(\frac{1}{|m-j|}\right)^{v-i}.
\] (4.25)

Therefore, we have

\[
\sum_{1 \leq j \leq n, \ j \neq m} A_{i,j}(x) \leq C(f) \sum_{j \neq m} \left(\frac{1}{|m-j|}\right)^{v-i} \leq C(f),
\] (4.26)

because of \(v - i \geq 2\).

Remark 4.4. If we consider the estimate of \(\sum_{i=1}^{n} A_{i,v-1}(x)\) with (4.13), then we obtain

\[
\sum_{i=1}^{n} A_{i,v-1}(x) \leq C(f) \log n.
\] (4.27)
We continue the proof Proposition 3.7. We need to estimate \( \sum_{j=1}^{n} A_{ij}(x) \) for \( |x| > x_{0,n} \). Now, suppose \( x > x_{0,n} \). Then similarly to (4.23), we have

\[
\frac{1}{|x_{0,n} - x_{j,n}|} \leq C \frac{1}{\sum_{1 \leq s \leq j} \Phi_n(x_{s,n})} \leq C \frac{n/a_n}{j(1 - x_{0,n}/a_n)^{1/2}} \leq C \frac{1}{j^{\frac{1}{2}} a_{n}}. \quad (4.28)
\]

Similarly to (4.21), we have using (4.22)

\[
\sum A_{ij}(x) \leq C(f) \sum_j \Phi_n^{\nu/2}(x) \frac{\Phi_n^{\nu/4}(x_{j,n}) \Phi_n^{\nu}(x_{j,n})}{x_{j,n}}
\times \left(1 - \frac{|x_{j,n}|}{a_n}\right)^{\nu/4} \left(1 - \frac{1}{|x_{0,n} - x_{j,n}|}\right)^{v-i} \left(\frac{n}{a^2_{2n} - x^2_{j,n}}\right)^{i} \quad (4.29)
\]

\[
\leq C(f) \sum_j 1_{j^{v-i}} C(f),
\]

because of \( v - i \geq 2 \) and \( \Phi_n^{\nu/2}(x) \delta_n^{-(v-\nu)/2} \leq \delta_n^{\nu/2} \delta_n^{-(v-\nu)/2} \leq C. \) Consequently we achieve the result. \( \square \)

**Remark 4.5.** The above proof implies the following: there exists a constant \( C > 0 \) such that

\[
\sum_{j=1}^{n} |h_{i,n,p}(v; x)| \leq \sum_{j=1}^{n} |l_{j,n}(x)|^\nu \sum_{i=0}^{\nu-2} |e_i(v, k, n)| \left| x - x_{j,n}\right|^i \leq C, \quad x \in \mathbb{R}. \quad (4.30)
\]

**Proof of Proposition 3.8.** We use the same method to prove of Proposition 3.7. So, we let \( (x_{m+1,n} + x_{m,n})/2 < x \leq x_{m,n} \) or \( x_{m,n} < x < (x_{m-1,n} + x_{m,n})/2. \) For simplicity, we assume \( x > 0 \), and let \( x_{0,n} := x_{1,n} + \delta \Phi_n(x_{1,n}) \) for a fixed \( \delta > 0 \) small enough and there exists a constant \( \delta_1 > 0 \) such that \( x_{0,n} := x_{1,n} + \delta \Phi_n(x_{1,n}) < a_n - \delta \delta_n. \) Assume that \( x < x_{0,n} \). Since \( f(0) = 0 \), we may leave out the term with \( x_{j,n} = 0 \). Hence we consider only the term of \( x_{j,n} \neq 0. \) Noting Assumption 3.1, (3.2), and Lemma 2.7, we estimate \( \sum_{j=1}^{n} B_{v-1,j}(x) \), where

\[
B_{v-1,j}(x) := \left(\Phi_n^{\nu/4}(x) \nu(x) \left|x + \frac{a_n}{n}\right|^\nu \right)^{v}
\times \left|f(x_{j,n})\right||l_{j,n}(x)| \left|x - x_{j,n}\right|^{v-1} \left(\frac{T(a_n)}{a_n} + \left|Q'(x_{j,n})\right| + \frac{1}{\left|x_{j,n}\right|}\right) \left(\frac{n}{a_{2n} - \left|x_{j,n}\right|}\right)^{v-2}. \quad (4.31)
\]
First we estimate $B_{\nu-1,m}(x)$. By Lemma 4.2, (4.16), and the definition of $\varphi_n(x)$, we have

$$
|x - x_{m,n}|^{-1} \left( \frac{n}{\sqrt{a_{2n}^2 - x_{m,n}^2}} \right)^{-2} \leq C \varphi_n(x_{m,n}) \left( \frac{\sqrt{1 - |x_{m,n}|/a_{2n}}}{\sqrt{1 - |x_{m,n}|/a_n}} \right)^{-v - 1} \leq C \frac{a_n}{n} \left( 1 - \frac{|x_{m,n}|}{a_{2n}} \right)^{1/2} \left( \frac{\sqrt{1 - |x_{m,n}|/a_{2n}}}{\sqrt{1 - |x_{m,n}|/a_n}} \right)^{-v - 1}. 
$$

(4.32)

Since here $|Q'(x_{m,n})| + 1/|x_{m,n}| \geq C$ for some positive constant $C$, we know that

$$
\left( \frac{T(a_n)}{a_n} + |Q'(x_{m,n})| + \frac{1}{|x_{m,n}|} \right) \leq C \left\{ \left| T(a_n) \right| \right\} \left( |Q'(x_{m,n})| + \frac{1}{|x_{m,n}|} \right),
$$

(4.33)

and by (3.25)

$$
|f(x_{m,n})| \leq C(f) \Phi_n^{3\nu/4}(x_{m,n}) \left( w(x_{m,n}) \left( |x_{m,n}| + \frac{a_n}{n} \right)^{\rho} \right)^{-\nu} \left( |Q'(x_{m,n})| + \frac{1}{|x_{m,n}|} \right)^{-1}. \quad (4.34)
$$

Therefore, using (3.3) and (4.15), we have

$$
B_{\nu-1,m}(x) \leq C(f) \max \left\{ 1, \frac{T(a_n)}{a_n} \right\} \Phi_n^{3\nu/4}(x) \Phi_n^{3\nu/4}(x_{m,n}) \left( \frac{a_n}{n} \right)^{1/2} \left( 1 - \frac{|x_{m,n}|}{a_{2n}} \right)^{1/2} \left( \frac{\sqrt{1 - |x_{m,n}|/a_{2n}}}{\sqrt{1 - |x_{m,n}|/a_n}} \right)^{-v - 1}.
$$

(4.35)

Noting (4.24), we have

$$
B_{\nu-1,m}(x) \leq C(f) \varepsilon_n \Phi_n^{3\nu/4}(x) \left( 1 - \frac{|x_{m,n}|}{a_{2n}} \right)^{(v+2)/4} \left( 1 - \frac{|x_{m,n}|}{a_n} \right)^{v/2} \leq C(f) \varepsilon_n. \quad (4.36)
$$

Next, we estimate $\sum_{j \neq m} B_{\nu-1,j}(x)$. Noting (4.19) and (4.22), we have

$$
\left( \Phi_n^{\nu/2}(x) w(x) \left| \frac{|x| + a_n}{n} \right|^{\rho} \right)^v \left| l_{j,n}(x) \right|^v \left| x - x_{j,n} \right|^{v - 1} = \Phi_n^{\nu/2}(x) a_{2n}^{-\nu/2} \Phi_n^{\nu}(x_{j,n}) \left( a_{2n}^2 - x_{j,n}^2 \right)^{v/4} \frac{\left| w(x_{j,n}) \left( |x_{j,n}| + a_n/n \right)^{\rho} \right|}{\left| x - x_{j,n} \right|},
$$

(4.37)

$$
\Phi_n^{\nu}(x_{j,n}) \left( \frac{a_n^2}{\sqrt{a_{2n}^2 - x_{j,n}^2}} \right)^{-v - 2} \leq C \left( \frac{a_n}{n} \right) \left( 1 - \frac{|x_{j,n}|}{a_{2n}} \right)^{-v/2} \left( 1 - \frac{|x_{j,n}|}{a_n} \right)^{v/2 + 1}. \quad (4.37)
$$
Then by (4.33) and (3.25) (noting (4.34)), using the notation (3.3) and (4.24), we have

\[
B_{v-1,j}(x) \leq C(f) \max \left\{ 1, \frac{T(a_n)}{a_n} \right\} \left( \frac{a_n}{n} \right)^2 \Phi_n^{v/2}(x) \Phi_n^{3v/4}(x,j,n) a_n^{-v/2} \left[ a_n^2 - x_j,n^2 \right]^{v/4} \\
\times \frac{1}{|x - x_j,n|} \left( 1 - \frac{|x_j,n|}{a_n} \right)^{-v/2} \left( 1 - \frac{|x_j,n|}{a_{2n}} \right)^{v/2+1}
\]

\[
\leq C(f) \varepsilon_n \frac{a_n}{n} \left( 1 - \frac{|x|}{a_n} \right)^{v/2} \left( 1 - \frac{|x_j,n|}{a_n} \right)^{-v/4} \left( 1 - \frac{|x_j,n|}{a_{2n}} \right)^{v/2+1} \frac{1}{|x - x_j,n|}
\]

\[
\leq C(f) \varepsilon_n \frac{a_n}{n} \left( 1 - \frac{|x|}{a_n} \right)^{v/2} \left( 1 - \frac{|x_j,n|}{a_n} \right)^{-v/4} \left( 1 - \frac{|x_j,n|}{a_{2n}} \right)^{v/2+1} \frac{1}{|x - x_j,n|}.
\]

(4.38)

Therefore, since we know from (4.23) that

\[
\frac{1}{|x - x_j,n|} \leq C \frac{n/a_n}{|m - j| (1 - \max \{|x|, |x_j,n|\}/a_n)^{1/2}},
\]

noting (4.24), we have

\[
\sum_{j \neq m} B_{v-1,j}(x) \leq C(f) \varepsilon_n \left( 1 - \frac{|x|}{a_n} \right)^{v/2} \sum_{j \neq m} \left( 1 - \frac{|x_j,n|}{a_n} \right)^{-v/4} \left( 1 - \frac{|x_j,n|}{a_{2n}} \right)^{v/2+1} \frac{1}{|m - j| (1 - \max \{|x|, |x_j,n|\}/a_n)^{1/2}}
\]

\[
\leq C(f) \varepsilon_n \left( 1 - \frac{|x|}{a_n} \right)^{(v-1)/2} \sum_{j \neq m} \left( 1 - \frac{|x_j,n|}{a_n} \right)^{(v-1)/2} \left( 1 - \frac{|x_j,n|}{a_{2n}} \right)^{v/2+1} \frac{1}{|m - j|}
\]

\[
\leq C(f) \varepsilon_n \sum_{j \neq m} \frac{1}{|m - j|} \leq C(f) \varepsilon_n \log n.
\]

(4.40)

Finally, we estimate \( \sum_{j=1}^{n} B_{v-1,j}(x) \) for \( x > x_{0,n} \). Suppose \( x > x_{0,n} \). Then similar to the above computations, we have

\[
B_{v-1,j}(x) \leq C(f) \varepsilon_n \frac{a_n}{n} \Phi_n^{v/2}(x) \Phi_n^{3v/4}(x,j,n) \left( 1 - \frac{|x_j,n|}{a_n} \right)^{-v/4} \left( 1 - \frac{|x_j,n|}{a_{2n}} \right)^{v/2+1} \frac{1}{|x_{0,n} - x_j,n|}
\]

\[
\leq C(f) \varepsilon_n \frac{a_n}{n} \Phi_n^{v/2}(x) \left( 1 - \frac{|x_j,n|}{a_n} \right)^{v/2} \left( 1 - \frac{|x_j,n|}{a_{2n}} \right)^{v/2+1} \frac{1}{|x_{0,n} - x_j,n|}.
\]

(4.41)

Then, since we know by (4.28) that

\[
\frac{1}{|x_{0,n} - x_j,n|} \leq C \frac{1}{f0_n^{1/2} a_n},
\]

(4.42)
we have
\[ B_{n-1,j}(x) \leq C(f)\varepsilon_n \Phi^{\nu/2}(x) \left(1 - \frac{|x_{j,n}|}{a_n}\right)^{\nu/2} \left(1 - \frac{|x_{j,n}|}{a_{2n}}\right)^{\nu/2+1} \frac{1}{j^{\delta_n^{1/2}}}. \] (4.43)

Then, since \( \Phi^{\nu/2}(x)\delta_n^{1/2} < C \) for \( x > x_{0,n} \), we have
\[
\sum_{j=1}^{n} B_{n-1,j}(x) \leq C(f)\varepsilon_n \sum_{j=1}^{n} \frac{1}{j} \leq C(f)\varepsilon_n \log n.
\] (4.44)

Therefore, the result is proved.

\[\square\]

**Proof of Proposition 3.9.** By Lemma 2.3 we see that, for a constant \( C_1 > 0 \),
\[
|Z(n,l,v,f;x)| \leq C_1 \sum_{j=1}^{n} \sum_{s=1}^{l} |f^{(s)}(x_{j,n})||l_{j,n}(x)|^v \sum_{i=s}^{\nu-1} \left( \frac{n}{\sqrt{a_{2n}^2 - x_{j,n}^2}} \right)^i |x - x_{j,n}|^i.
\]

\[
\leq C_1 \frac{a_n}{n} \sum_{j=1}^{n} \sum_{s=1}^{l} |f^{(s)}(x_{j,n})||l_{j,n}(x)|^v \sum_{i=s}^{\nu-1} \left( \frac{n}{\sqrt{a_{2n}^2 - x_{j,n}^2}} \right)^i |x - x_{j,n}|^i
\]

\[
\leq C_1 \frac{a_n}{n} \sum_{j=1}^{n} \sum_{s=1}^{l} \sum_{i=0}^{\nu-1} |f^{(s)}(x_{j,n})||l_{j,n}(x)|^v |x - x_{j,n}|^i \left( \frac{n}{\sqrt{a_{2n}^2 - x_{j,n}^2}} \right)^i
\]

\[
=: \frac{a_n}{n} \sum_{s=1}^{l} \sum_{i=0}^{\nu-1} C_{l,s}(x),
\]

where,
\[
C_{l,s}(x) := \sum_{j=1}^{n} |f^{(s)}(x_{j,n})||l_{j,n}(x)|^v |x - x_{j,n}|^i \left( \frac{n}{\sqrt{a_{2n}^2 - x_{j,n}^2}} \right)^i.
\] (\ast)

We set
\[
X_{l,s,n}(v,f;x) := \sum_{j=1}^{n} f^{(s)}(x_{j,n})l_{j,n}(x)e_i(v,j,n)(x - x_{j,n})^i, \quad i = 0, 1, \ldots, v - 1,
\] (4.46)

and hereafter we wrote (\ast) as \(|X|_{l,s,n}(v,f;x)\). Now, we repeat the proof of Proposition 3.7 by exchanging \( f(x) \) with \( f^{(s)}(x) \), \( s = 1, 2, \ldots, v - 1 \), and we note (3.23) (and (3.26)). Then, for \( 0 \leq i \leq v - 2 \) we obtain
\[
\left\{ \Phi_n^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right) \right\}^v |X|_{l,s,n}(v,f;x) \leq C_1 C(f).
\] (4.47)
For $i = \nu - 1$, we use the estimate for $l_{mn}(x)$ in the proof of Proposition 3.7; furthermore we use Remark 4.4. Then we have

$$\left\{ \Phi_{\nu}^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right) \right\}^\nu |X|_{\nu-1,s,n}(\nu, f; x) \leq C_1 C(f) \log n.$$  

(4.48)

Consequently, we have

$$\left\{ \Phi_{\nu}^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right) \right\}^\nu |Z_n(l, \nu, f; x) | \leq C_1 C(f) \frac{a_n \log n}{n}.$$

(4.49)

Similarly, we have

$$\left\{ \Phi(x)^{3/4}w(x) \left( |x| + \frac{a_n}{n} \right) \right\}^\nu |W_n(l, \nu, G; x) | \leq C_1 C(G) \frac{a_n \log n}{n}.$$

(4.50)

**Proof of Proposition 3.10.** Let $P \in \mathcal{P}_m$ be fixed. From Proposition 3.9 we see

$$\left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right) \right)^\nu \{ L_n(\nu, P) - P \} \right\|_{L_\infty(\mathbb{R})}$$

$$= \left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right) \right)^\nu \{ L_n(\nu, P) - L_n(\nu - 1, \nu, P) \} \right\|_{L_\infty(\mathbb{R})}$$

$$= \left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right) \right)^\nu Z_n(\nu - 1, \nu, P) \right\|_{L_\infty(\mathbb{R})} \to 0 \quad \text{as } n \to \infty.$$  

(4.51)

**Proof of Proposition 3.11.** Let $P \in \mathcal{P}_m$ with $P(0) = 0$. Then $P$ satisfies the condition (A-1). By Proposition 3.8, we see

$$\left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right) \right)^\nu \gamma_n(\nu, P) \right\|_{L_\infty(\mathbb{R})} \to 0 \quad \text{as } n \to \infty.$$  

(4.52)

So, from Propositions 3.10 and 3.8 (noting (3.3)) we have

$$\left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right) \right)^\nu (X_n(\nu, P) - P) \right\|_{L_\infty(\mathbb{R})}$$

$$\leq C_1 \left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right) \right)^\nu \{ L_n(\nu, P) - P \} \right\|_{L_\infty(\mathbb{R})}$$

$$+ \left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right) \right)^\nu \gamma_n(\nu, P) \right\|_{L_\infty(\mathbb{R})} \to 0 \quad \text{as } n \to \infty.$$  

(4.53)
Proof of Theorem 3.12. Since \( f \) satisfies (3.22), we see

\[
\lim_{|x| \to \infty} |f(x)|w(x)^{\nu-\delta} = 0.
\]  

(4.54)

For a given \( \varepsilon > 0 \), there exists a polynomial \( P \in \mathcal{P}_m \) with \( P(0) = 0 \) such that

\[
\sup_{x \in \mathbb{R}} |f(x) - P(x)| \left\{ \Phi^{-3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^{\rho} \right\}^\nu < \varepsilon.
\]  

(4.55)

In fact, by [21, Theorem 1.4], there exists a polynomial \( R \in \mathcal{P}_m \) such that

\[
\sup_{x \in \mathbb{R}} |f(x) - R(x)|w^{\nu-\delta}(x) < \frac{\varepsilon}{2}.
\]  

(4.56)

Let \( P(x) := R(x) - R(0) \). Noting that

\[
\left\{ \Phi^{-3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^{\rho} \right\}^\nu \leq Cw^{\nu-\delta}(x), \quad x \in \mathbb{R} \text{ for some } C > 0
\]  

(4.57)

and \( f(0) = 0 \), we have

\[
\sup_{x \in \mathbb{R}} |f(x) - P(x)| \left\{ \Phi^{-3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^{\rho} \right\}^\nu 
\leq \sup_{x \in \mathbb{R}} |f(x) - P(x)|w^{\nu-\delta}(x) 
\leq \sup_{x \in \mathbb{R}} |f(x) - R(x)|w^{\nu-\delta}(x) + |f(0) - R(0)|w^{\nu-\delta}(0) < \varepsilon;
\]  

(4.58)

that is, we have (4.55). Here, we know that

\[
L_n(\nu, f)(x) = X_n(\nu, f - P) + (X_n(\nu, P) - P) + (P - f) + Y_n(\nu, f).
\]  

(4.59)
Therefore, by Propositions 3.7, 3.11, and 3.8, we have for \( n \) large enough

\[
\left\| \left( \Phi^{3/4}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^{\rho} \right)^v (L_n(v, f) - f) \right\|_{L^\infty(\mathbb{R})} \\
\leq C \left\| \left( \Phi^{3/4}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^{\rho} \right)^v X_n(v, P - f) \right\|_{L^\infty(\mathbb{R})} \\
+ \left\| \left( \Phi^{3/4}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^{\rho} \right)^v \left( X_n(v, P) - f \right) \right\|_{L^\infty(\mathbb{R})} \\
+ \left\| \left( \Phi^{3/4}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^{\rho} \right)^v Y_n(v, f) \right\|_{L^\infty(\mathbb{R})} \\
=: I_n + II_n + III_n + IV_n.
\]

Here we see that, by Proposition 3.7 with \( \varepsilon := C(f - P) \) (constant depending only on \( f - P \)) and (4.55),

\[
I_n \leq C\varepsilon, \quad III_n \leq \varepsilon,
\]

and for \( n \geq n_0 \) large enough, we have

\[
II_n \leq C\varepsilon, \quad \text{(by Proposition 3.9),} \\
IV_n \leq C\varepsilon, \quad \text{(by Proposition 3.6).}
\]

Consequently, noting (3.3), we have

\[
\lim_{n \to \infty} \{ I_n + II_n + III_n + IV_n \} = 0.
\]

Then we have Theorem 3.12.

Proof of Theorem 3.13. From Theorem 3.12 and Proposition 3.9, we have

\[
\left\| \left( \Phi^{3/4}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^{\rho} \right)^v (L_n(l, v, f) - f) \right\|_{L^\infty(\mathbb{R})} \\
\leq C \left\| \left( \Phi^{3/4}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^{\rho} \right)^v (L_n(v, f) - f) \right\|_{L^\infty(\mathbb{R})} \\
+ \left\| \left( \Phi^{3/4}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^{\rho} \right)^v Z_n(l, v, f) \right\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad n \to \infty.
\]
Proof of Corollary 3.14. From Theorem 3.12 and Proposition 3.8, we have

\[
\left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^\nu \right)^v (X_n(v,f) - f) \right\|_{L^1(\mathbb{R})} \\
\leq C_1 \left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^\nu \right)^v \left( L_n(l,v,f) - f \right) \right\|_{L^1(\mathbb{R})} \\
+ \left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^\nu \right)^v W_n(l,v,G) \right\|_{L^1(\mathbb{R})} \to 0 \text{ as } n \to \infty. \tag{4.65}
\]

Proof of Theorem 3.15. From Theorem 3.13 and Proposition 3.10 we have

\[
\left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^\nu \right)^v \left( \tilde{I}_n(l,v,f \oplus G) - f \right) \right\|_{L^1(\mathbb{R})} \\
\leq C \left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^\nu \right)^v \left( L_n(l,v,f) - f \right) \right\|_{L^1(\mathbb{R})} \\
+ \left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^\nu \right)^v W_n(l,v,G) \right\|_{L^1(\mathbb{R})} \to 0 \text{ as } n \to \infty. \tag{4.66}
\]

Proof of Theorem 3.16. We use Theorems 3.12, 3.13, and Corollary 3.14, and Theorem 3.15. Then we have

\[
\left| I_n[k,H_n(f);f] - I[k,f] \right| \\
= \int_{-\infty}^{\infty} k(x) \{ H_n(f)(x) - f(x) \} dx \\
\leq C \left\| \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^\nu \right)^v \left( H_n(f)(x) - f(x) \right) \right\|_{L^1(\mathbb{R})} \\
\times \int_{-\infty}^{\infty} \left( \Phi^{3/4}(x)w(x) \left( |x| + \frac{a_n}{n} \right)^\nu \right)^v k(x) dx \to 0 \text{ as } n \to \infty. \tag{4.67}
\]

\[\square\]

5. Divergence Theorem

If \( \nu \) is a positive odd integer, then we obtain the unboundedness of \( L(n,v,f;x) \). We define

\[
\Lambda_n(\nu,\mathbb{R}) = \max_{x \in \mathbb{R}} \sum_{k=1}^{n} |h_{kn}(x)|. \tag{5.1}
\]

Theorem 5.1 (cf. [1, Theorem 2]). Let \( \nu > 0 \) be an odd integer. Then there exists a constant \( C > 0 \) and \( n_0 > 0 \) such that for \( n \geq n_0 \)

\[
\Lambda_n(\nu,\mathbb{R}) \geq C \log n. \tag{5.2}
\]
Let $-\infty \leq a < b \leq \infty$, and let us define

$$\Lambda_n(\nu, (a, b)) = \max_{a \leq x \leq b} \sum_{k=1}^{n} |h_k(x)|.$$  \hfill (5.3)

Then we will show that for $n \geq n_0$

$$\Lambda_n(\nu, (a, b)) \geq C \log n.$$  \hfill (5.4)

**Remark 5.2.** For the interpolation polynomial $L_n(\nu, f; x)$, we can see a remarkable difference between the cases of an odd number $\nu$ and of an even number $\nu$. Let us consider any continuous function $f \in C([a, b]), 0 < a < b$. Then we can extend $f$ to a continuous function $f^* \in C(\mathbb{R})$ which satisfies (3.22) and $f(x) = f^*(x), x \in [a, b]$. Then from Theorem 3.12 for an even positive integer $\nu$, we see

$$\|L_n(\nu, f^*; x) - f(x)\|_{L_\infty([a,b])} \to 0 \text{ as } n \to \infty.$$ \hfill (5.5)

On the other hand, the standard argument (cf, [22, Theorem 4.3]) leads us to the following. Theorem 5.1 means that there exists a certain function $f^* \in C(\mathbb{R})$ such that

$$\|L_n(\nu, f^*; x) - f^*(x)\|_{L_\infty([a,b])} \to 0 \text{ as } n \to \infty;$$ \hfill (5.6)

that is, for $f(x) := f^*(x), a \leq x \leq b$ we see that the interpolation polynomials $L_n(\nu, f^*; x)$ do not converge to $f$. We also remark that the polynomial $L_n(\nu, f^*; x)$ interpolates $f \in C[a, b]$ at only $\{x_{j,n} \in [a, b], 1 \leq j \leq n\}.

To prove the theorem we use the following lemma.

**Lemma 5.3** (see [18, Theorem 11]). For $j = 0, 1, 2, \ldots$, there is a polynomial $\Psi_j(x)$ of degree $j$ such that $(-1)^j \Psi_j(-\nu) > 0$ for $\nu = 1, 3, 5, \ldots$, and the following relation holds. Let $0 < \varepsilon < 1$. Then one has an expression for $(1/\varepsilon)(a_n/n) \leq |x_{k,n}| \leq \varepsilon a_n$ and $s = 0, 1, \ldots, (\nu - 1)/2$:

$$e_2s(\nu, k, n) = (-1)^s \frac{1}{(2s)!} \Psi_s(-\nu) \beta_s(n, k) \left(\frac{n}{a_n}\right)^{2s} \left\{1 + \eta_{k,n}(\nu, s)\right\},$$ \hfill (5.7)

where for $\beta_s(n, k), n = 1, 2, 3, \ldots, k = 1, 2, \ldots, n, s = 0, 1, \ldots, (\nu - 1)/2, there exist the constants $C_1, C_2$ such that

$$0 < C_1 \leq \beta_s(n, k) \leq C_2.$$ \hfill (5.8)
and \( \eta_{kn}(\nu, s) \) satisfies

\[
|\eta_{kn}(\nu, s)| \leq \varepsilon_n^* \quad \varepsilon_n^* = \varepsilon_n^*(\varepsilon) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0,
\]

for \( k \) with \((1/\varepsilon)(a_n/n) \leq |x_{k,n}| \leq \varepsilon a_n\), and for \( s = 0, 1, 2, \ldots \).

**Proof of Theorem 5.1.** To get a lower bound of \( \Lambda_n(\nu, \mathbb{R}) \), it suffices to consider a lower bound (5.4) of \( \Lambda_n(\nu, [a, b]) \) with \( 0 < a < b < \infty \). Let \( c := a/2, d := 2b - a \) and we consider the intervals \( I := [a, b], J = [c, d] \). If we consider \( n \) large enough, then we have \((1/\varepsilon)(a_n/n) \leq c < 2b - a \leq \varepsilon a_n\). See the expression

\[
|h_{kn}(x)| \geq \left| p_{kn}(x) e_{\nu-1}(\nu, k, n)(x - x_{k,n})^{\nu-1} \right| - \left| p_{kn}(x) \sum_{i=0}^{\nu-2} e_i(\nu, k, n)(x - x_{k,n})^i \right|.
\]

Then we have

\[
\Lambda_n([a, b]) \geq \max_{x \in I} \sum_{x_{k,n} \in J} \left| p_{kn}(x) e_{\nu-1}(\nu, k, n)(x - x_{k,n})^{\nu-1} \right| - \max_{x \in I} \left| p_{kn}(x) \sum_{i=0}^{\nu-2} e_i(\nu, k, n)(x - x_{k,n})^i \right|
\]

\[
= \max_{x \in I} F_n(x) - \max_{x \in I} G_n(x).
\]

It follows from Remark 4.5 that \( \max_{a \leq x \leq b} G_n(x) \leq C \). Hence, it is enough to show that \( \max_{x \in I} F_n(x) \geq C \log n \). Let \( x_{k+1,n}, x_{k,n} \in J \); then by Lemma 4.2 and the definition of \( \varphi_n(x_{k,n}) \) there exists \( 0 < \alpha \leq \beta < \infty \) such that

\[
\alpha \frac{a_n}{n} \leq |x_{k,n} - x_{k+1,n}| \leq \beta \frac{a_n}{n}.
\]

For \( x \in I \) we consider only \( x_{k,n} \in J \) such that, for some positive integer \( j_k \),

\[
(j_k - 1)\alpha \frac{a_n}{n} \leq |x_{k,n} - x| \leq j_k \beta \frac{a_n}{n}.
\]

Then for each \( x \in I \) we define

\[
\Gamma(x) := \left\{ j_k; x_{k,n} \in J, (j_k - 1)\alpha \frac{a_n}{n} \leq |x_{k,n} - x| \leq j_k \beta \frac{a_n}{n} \right\}.
\]
Here, we will see that
\[ \{ j; 1 \leq j \leq \frac{d-cn}{2\beta a_n} - 1 \} \subset \Gamma(x). \] (5.15)

Let

\[ x_{m+1,n} < x \leq x_{m,n}, \]
\[ x_{k(c)+1,n} < c \leq x_{k(c),n} x_{k(d),n} \leq d < x_{k(d)-1,n}. \] (5.16)

Then we have

\[ x - x_{m+1,n} \leq x_{m,n} - x_{m+1,n} \leq \beta a_n, \]
\[ x - x_{m+2,n} \leq x_{m,n} - x_{m+2,n} \leq 2\beta a_n, \]
\[ \vdots \]
\[ x - x_{k(c),n} \leq x_{m,n} - x_{k(c),n} \leq j(c)\beta a_n, \]
\[ x_{m,n} - x \leq x_{m,n} - x_{m+1,n} \leq \beta a_n, \]
\[ x_{m-1,n} - x \leq x_{m-1,n} - x_{m+1,n} \leq 2\beta a_n, \]
\[ \vdots \]
\[ x_{k(d),n} - x \leq x_{k(d),n} - x_{m+1,n} \leq j(d)\beta a_n, \] (5.17)

where \( j(c) \) and \( j(d) \) are integers. On the other hand,

\[ \frac{(x_{k(d),n} - x_{k(c),n})}{(\beta a_n/n)} \geq \frac{(d-c-2\beta a_n/n)}{(\beta a_n/n)} = \frac{d-cn}{\beta a_n} - 2. \] (5.18)

Therefore, we have

\[ \max(j(c), j(d)) \geq \frac{1}{2} \left( \frac{d-cn}{\beta a_n} - 2 \right) = \frac{d-cn}{2\beta a_n} - 1. \] (5.19)

Now, we take a positive integer \( N(n) \) such that

\[ N(n) \leq \frac{d-cn}{2\beta a_n} - 1 < N(n) + 1. \] (5.20)
Consequently, we have the following. By Lemmas 4.2, Lemma 4.1, 4.3, 4.4, and Lemma 2.3, we have

\[ F_n(x) = \sum_{x_k \in \mathcal{J}} |l_{kn}(x)|^\nu |e_{\nu-1}(\nu, k, n)| |x_k - x|^{\nu-1} \]

\[ = \sum_{x_k \in \mathcal{J}} \left( \frac{|p_n(x)| |w_\rho(x)|}{|x - x_k|} \frac{w_\rho(x_k, n)}{w_\rho(x)} \right)^\nu |e_{\nu-1}(\nu, k, n)| |x - x_k|^{\nu-1} \]

\[ = \sum_{x_k \in \mathcal{J}} \left( \frac{|p_n(x)| |w_\rho(x)|}{|x - x_k|} \frac{w_\rho(x_k, n)}{w_\rho(x)} \right)^\nu |e_{\nu-1}(\nu, k, n)| \frac{1}{|x - x_k|} \]

\[ \geq C \sum_{j \in \mathcal{J}(x)} \left( \frac{a_n^{-1/2}}{n/a_n} \frac{w_\rho(a)}{w_\rho(a)} \right)^\nu |e_{\nu-1}(\nu, k, n)| \frac{n}{j_k \beta a_n} \]

\[ \geq C \sum_{j \in \mathcal{J}(x)} \frac{1}{j} \left( \frac{a_n}{n} \right)^{\nu-1} |e_{\nu-1}(\nu, k, n)|. \]

Here, using Lemma 5.3 and (5.15), we have

\[ F_n(x) \geq C \sum_{j \in \mathcal{J}(x)} \frac{1}{j} \left( \frac{a_n}{n} \right)^{\nu-1} \left( \frac{n}{a_n} \right)^{\nu-1} \geq C \sum_{1 \leq j \leq N(n)} \frac{1}{j}. \]

Here, there exists 0 < \eta < 1 such that \( n^3 \leq N(n) \). Therefore we see

\[ F_n(x) \geq C \sum_{1 \leq j \leq n^\eta} \frac{1}{j} \geq C \log n. \]

So we have (5.4), and consequently the proof of Theorem 5.1 is completed. \( \square \)

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**References**


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