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Research Article

Convergence and Divergence of Higher-Order Hermite or Hermite-Fejér Interpolation Polynomials with Exponential-Type Weights

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Let $\mathbb{R} = (-\infty, \infty)$, and let $w_\rho(x) = |x|^\rho e^{-Q(x)}$, where $\rho > -1/2$ and $Q \in C^1(\mathbb{R}) : \mathbb{R} \to \mathbb{R}^+ = [0, \infty)$ is an even function. Then we can construct the orthonormal polynomials $p_n(w_\rho^2; x)$ of degree n for $w_\rho^2(x)$. In this paper for an even integer $\nu \geq 2$ we investigate the convergence theorems with respect to the higher-order Hermite and Hermite-Fejér interpolation polynomials and related approximation process based at the zeros $\{x_{k,n,\rho}\}_{k=1}^n$ of $p_n(w_\rho^2; x)$. Moreover, for an odd integer $\nu \geq 1$, we give a certain divergence theorem with respect to the higher-order Hermite-Fejér interpolation polynomials based at the zeros $\{x_{k,n,\rho}\}_{k=1}^n$ of $p_n(w_\rho^2; x)$.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^1(\mathbb{R}) : \mathbb{R} \to \mathbb{R}^+ = [0, \infty)$ be an even function. Consider the weight $w(x) = \exp(-Q(x))$, and define, for $\rho > -1/2$,

$$w_\rho(x) := |x|^\rho w(x), \quad x \in \mathbb{R}. \tag{1.1}$$

Suppose that $\int_0^\infty x^n w_\rho(x) dx < \infty$, for all $n = 0, 1, 2, \ldots$. Then we can construct the orthonormal

polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ of degree n for $w_\rho^2(x)$; that is,

$$\int_{-\infty}^{\infty} p_{n,\rho}(x) p_{m,\rho}(x) w_\rho^2(x) dx = \delta_{m,n} \quad \text{(Kronecker delta)}. \tag{1.2}$$

We write $p_{n,\rho}(x)$ by

$$p_{n,\rho}(x) = \gamma_n x^n + \cdots, \quad \gamma_n = \gamma_{n,\rho} > 0, \tag{1.3}$$

and denote the zeros of $p_{n,\rho}(x)$ by

$$-\infty < x_{n,n,\rho} < x_{n-1,n,\rho} < \cdots < x_{2,n,\rho} < x_{1,n,\rho} < \infty. \tag{1.4}$$

Let $\mathcal{P}_n$ denote the class of polynomials with degree at most n. For $f \in C(\mathbb{R})$ we define the higher-order Hermite-Fejér interpolation polynomial $L_n(\nu, f; x) \in \mathcal{P}_{\nu n-1}$ based at the zeros $\{x_{k,n,\rho}\}_{k=1}^n$ as follows:

$$L_n^{(i)}(\nu, f; x_{k,n,\rho}) = \delta_{0,i} f(x_{k,n,\rho}) \quad \text{for } k = 1, 2, \ldots, n, \ i = 0, 1, \ldots, \nu - 1. \tag{1.5}$$

We note that $L_n(1, f; x)$ is the Lagrange interpolation polynomial, $L_n(2, f; x)$ is the ordinary Hermite-Fejér interpolation polynomial, and $L_n(4, f; x)$ is the Krylov-Stayermann polynomial. For the general cases Kanjin and Sakai [1, 2] started to investigate the so-called Freud-type weights. The fundamental polynomials $h_{k,n,\rho}(\nu; x) \in \mathcal{P}_{\nu n-1}$ for the higher-order Hermite-Fejér interpolation polynomial $L_n(\nu, f; x)$ are defined as follows:

$$\begin{gathered}
h_{k,n,\rho}(\nu; x) = l_{k,n,\rho}^{\nu}(x) \sum_{i=0}^{\nu-1} e_i(\nu, k, n)(x - x_{k,n,\rho})^i, \\
l_{k,n,\rho}(x) = \frac{p_n\left(w_\rho^2; x\right)}{(x - x_{k,n,\rho}) p_n'(w_\rho^2; x_{k,n,\rho})}, \\
h_{k,n,\rho}(\nu; x_{p,n,\rho}) = \delta_{k,p}, \qquad h_{k,n,\rho}^{(i)}(\nu; x_{p,n,\rho}) = 0, \\
k, p = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, \nu - 1.
\end{gathered} \tag{1.6}$$

Using them, we can write

$$L_n(\nu, f; x) = \sum_{k=1}^{n} f(x_{k,n,\rho}) h_{k,n,\rho}(\nu; x). \tag{1.7}$$

Furthermore, we extend the operator $L_n(\nu, f; x)$. Let l be a nonnegative integer, and let $\nu - 1 \geqslant l$. For $f \in C^l(\mathbb{R})$ we define the (l, ν)-order Hermite-Fejér interpolation polynomials $L_n(l, \nu, f; x) \in \mathcal{P}_{\nu n-1}$ as follows: for each $k = 1, 2, \ldots, n$,

$$\begin{aligned} L_n(l, \nu, f; x_{k,n,\rho}) &= f(x_{k,n,\rho}), \\ L_n^{(j)}(l, \nu, f; x_{k,n,\rho}) &= f^{(j)}(x_{k,n,\rho}), \quad j = 1, 2, \ldots, l, \\ L_n^{(j)}(l, \nu, f; x_{k,n,\rho}) &= 0, \quad j = l+1, l+2, \ldots, \nu - 1. \end{aligned} \tag{1.8}$$

Especially, $L_n(0, \nu, f; x)$ is equal to $L_n(\nu, f; x)$, and for each $P \in \mathcal{P}_{\nu n-1}$ we see $L_n(\nu-1, \nu, P; x) = P(x)$; that is, for $f \in C^{(\nu-1)}(\mathbb{R})$, $L_n(\nu - 1, \nu, f; x)$ is the Hermite interpolation polynomial. The fundamental polynomials $h_{s,k,n,\rho}(l, \nu; x) \in \mathcal{P}_{\nu n-1}$, $k = 1, 2, \ldots, n$, of $L_n(l, \nu, f; x)$ are defined by

$$\begin{aligned} h_{s,k,n,\rho}(l, \nu; x) &= l_{k,n,\rho}^{\nu}(x) \sum_{i=s}^{\nu-1} e_{s,i}(\nu, k, n)(x - x_{k,n,\rho})^i, \\ h_{s,k,n,\rho}^{(j)}(l, \nu; x_{p,n,\rho}) &= \delta_{s,j}\delta_{k,p}, \quad j, s = 0, 1, \ldots, \nu - 1, \ p = 1, 2, \ldots, n. \end{aligned} \tag{1.9}$$

Then we have

$$L_n(l, \nu, f; x) = \sum_{k=1}^{n} \sum_{s=0}^{l} f^{(s)}(x_{k,n,\rho}) h_{s,k,n,\rho}(l, \nu; x). \tag{1.10}$$

For the ordinary Hermite and Hermite-Fejér interpolation polynomial $L_n(1, 2, f; x)$, $L_n(2, f; x)$ and the related approximation process, Lubinsky [3] gave some interesting convergence theorems.

Our purpose in this paper is to study $L_n(\nu, f; x)$ and $L_n(l, \nu, f; x)$ as certain analogies of the Lubinsky theorems in [3] and the related approximation process for the exponential-type weights. Kasuga and Sakai [4–8] investigated the convergence and divergence theorems for the Freud-type weights. Then for an even integer $\nu \geq 2$ we give the convergence theorems for them; moreover, for an odd integer $\nu \geqslant 1$, we obtain a certain divergence theorem with respect to $L_n(\nu, f; x)$. In Section 1, we give the preliminaries for these studies, and in Section 2 we write some preliminary description. In Section 3 we report our theorems with some lemmas, and in Section 4 we prove the theorems. Finally, in Section 5, for an odd integer $\nu \geqslant 1$ we obtain a certain divergence theorem with respect to $L_n(\nu, f; x)$.

In what follows we abbreviate several notations as $x_{k,n} := x_{k,n,\rho}$, $h_{kn}(x) := h_{k,n,\rho}(\nu, x)$, $l_{kn}(x) := l_{k,n,\rho}(x)$, $h_{skn}(x) := h_{s,k,n,\rho}(\nu, x)$ and $p_n(x) := p_{n,\rho}(x)$ if there is no confusion. For arbitrary nonzero real-valued functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if there exist constants $C_1, C_2 > 0$ independent of x such that $C_1 g(x) \leqslant f(x) \leqslant C_2 g(x)$ for all x. For arbitrary positive sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{=1}^{\infty}$, we define $c_n \sim d_n$ similarly.

Throughout this paper $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t, or polynomials $P_n(x)$, and the same symbol does not necessarily denote the same constant in different occurrences.

2. Preliminaries

A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be quasi-increasing if there exists $C > 0$ such that $f(x) \leqslant Cf(y)$ for $0 < x < y$.

Definition 2.1 (see [9]). Let $Q : \mathbb{R} \to \mathbb{R}^+$ be a continuous even function satisfying the following properties.

(a) $Q'(x)$ is continuous in $\mathbb{R}$ and $Q(0) = 0$.

(b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.

(c) $\lim_{x\to\infty} Q(x) = \infty$.

(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0, \tag{2.1}$$

is quasi-increasing in $(0, \infty)$, with $T(x) \geqslant \Lambda > 1$ for $x \in \mathbb{R}^+ \setminus \{0\}$.

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leqslant C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus \{0\}. \tag{2.2}$$

Then we say that $w = \exp(-Q)$ is in the class $\mathcal{F}(C^2)$. Besides, if there exist a compact subinterval $J(\ni 0)$ of $\mathbb{R}$ and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geqslant C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J, \tag{2.3}$$

then we say that $w = \exp(-Q)$ is in the class $\mathcal{F}(C^2+)$.

Example 2.2. We present some typical examples of $Q(x)$ satisfying $w = \exp(-Q) \in \mathcal{F}(C^2+)$.

(1) If a continuous exponential $Q(x)$ satisfies

$$1 < \Lambda_1 \leqslant \frac{(xQ'(x))'}{Q'(x)} \leqslant \Lambda_2 \tag{2.4}$$

with some constants Λ_i, $i = 1, 2$, then $w(x) = \exp(-Q(x))$ is called a Freud weight. The class $\mathcal{F}(C^2+)$ contains the Freud weights.

(2) (See [9]) For $\alpha > 1$ and a nonnegative integer r, we put

$$Q(x) = Q_{r,\alpha}(x) := \exp_r(|x|^\alpha) - \exp_r(0), \tag{2.5}$$

where, for $r \geqslant 1$,

$$\exp_r(x) := \exp(\exp(\exp(\cdots \exp x)\cdots)) \quad (r\text{-times}) \tag{2.6}$$

and $\exp_0(x) := x$.

(3) (See [10]) For $m \geqslant 0$, $\alpha \geqslant 0$ with $\alpha + m > 1$, we put

$$Q(x) = Q_{r,\alpha,m}(x) := |x|^m \{\exp_r(|x|^\alpha) - \alpha^* \exp_r(0)\}, \tag{2.7}$$

where for $r > 0$ we suppose $\alpha^* = 0$ if $\alpha = 0$; $\alpha^* = 1$ otherwise. For $r = 0$ we suppose $m > 1$ and $\alpha = 0$.

(4) (See [10]) For $\alpha > 1$, we put

$$Q(x) = Q_\alpha(x) := (1 + |x|)^{|x|^\alpha} - 1. \tag{2.8}$$

For $x > 0$, we define the Mhaskar-Rakhmanov-Saff number a_x by the equation

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{1/2}} du. \tag{2.9}$$

We have the following estimates for the coefficients $e_{s,i}(\nu,k,n)(e_i(\nu,k,n) := e_{0,i}(\nu,k,n))$ in (1.6) or (1.9).

Lemma 2.3 (see [11, Theorem 2.6]). *Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$. For each $k = 1,2,\ldots,n$ and $s = 0,1,\ldots,\nu-1$, we have $e_{0,0}(\nu,k,n) = e_0(\nu,k,n) = 1$,*

$$|e_{s,i}(\nu,k,n)| \leqslant C\left(\frac{n}{\sqrt{a_{2n}^2 - x_{k,n}^2}}\right)^{i-s}, \quad s = 1,2,\ldots,\nu-1,\ i = s, s+1,\ldots,\nu-1. \tag{2.10}$$

If we consider the higher-order Hermite-Fejér interpolation polynomial $L_n(\nu, f; x)$ on a certain finite interval, then we can see a remarkable difference between the parity of ν, for example, the Lagrange interpolation polynomial $L_n(1, f; x)$ and the ordinary Hermite-Fejér interpolation polynomial $L_n(2, f; x)$ ([12–17]). Also, we can see a similar phenomenon in the case of the infinite intervals ([1, 2, 4–8]). To describe these aspects, however, we need a further strengthened definition for $\nu \geqslant 2$ than Definition 2.1.

Definition 2.4. Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$, and let $\nu \geqslant 1$ be an integer. Assume that $Q(x)$ is a ν-times continuously differentiable function on $\mathbb{R}$ and satisfies the following.

(a) $Q^{(\nu+1)}(x)$ exists and $Q^{(i)}(x)$, $0 \leqslant i \leqslant \nu+1$, are positive for $x > 0$.

(b) There exist constants $C_i > 0$ such that

$$\left|Q^{(i+1)}(x)\right| \leqslant C_i \left|Q^{(i)}(x)\right| \frac{|Q'(x)|}{Q(x)}, \quad x \in \mathbb{R} \setminus \{0\}, \ i = 1, 2, \ldots, \nu. \tag{2.11}$$

(c) There exist $0 \leqslant \delta < 1$ and $c_1 > 0$ such that

$$Q^{(\nu+1)}(x) \leqslant C\left(\frac{1}{x}\right)^{\delta}, \quad x \in (0, c_1]. \tag{2.12}$$

Then we say that $w(x) = \exp(-Q(x))$ is in the class $\mathcal{F}_\nu(C^2+)$.

(d) Suppose one of the following.

(d-1) $Q'(x)/Q(x)$ is quasi-increasing on a certain positive interval $[c_2, \infty)$.

(d-2) $Q^{(\nu+1)}(x)$ is nondecreasing on a certain positive interval $[c_2, \infty)$.

(d-3) There exist constants $C > 0$ and $0 \leqslant \delta < 1$ such that $Q^{(\nu+1)}(x) \leqslant C(1/x)^{\delta}$ on $(0, \infty)$.

Then one says that $w(x) = \exp(-Q(x))$ is in the class $\widetilde{\mathcal{F}}_\nu(C^2+)$.

Example 2.5 (cf. [10, Theorem 3.1]). Let ν be a positive integer, and let $Q_{r,\alpha,m}$ be defined in (2.7).

(1) Let m and α be nonnegative even integers with $m + \alpha > 1$. Then $w(x) = \exp(-Q_{r,\alpha,m}) \in \mathcal{F}_\nu(C^2+)$.

(a) If $r > 0$, then we see that $Q'_{r,\alpha,m}(x)/Q_{r,\alpha,m}(x)$ is quasi-increasing on a certain positive interval (c_1, ∞) and $Q_{r,0,m}(x)$ is nondecreasing on $(0, \infty)$.

(b) If $r = 0$, then we see that $Q_{0,0,m}(x), m \geqslant 2$, is nondecreasing on $(0, \infty)$.
Hence, $w(x) = \exp(-Q_{r,\alpha,m}) \in \widetilde{\mathcal{F}}_\nu(C^2+)$.

(2) Let $m + \alpha - \nu > 0$. Then $w(x) = \exp(-Q_{r,\alpha,m}) \in \mathcal{F}_\nu(C^2+)$, and one has the following.

(c) If $r \geqslant 2$ and $\alpha > 0$, then there exists a constant $c_1 > 0$ such that $Q'_{r,\alpha,m}(x)/Q_{r,\alpha,m}(x)$ is quasi-increasing on (c_1, ∞).

(d) Let $r = 1$. If $\alpha \geqslant 1$, then there exists a constant $c_2 > 0$ such that $Q'_{1,\alpha,m}(x)/Q_{1,\alpha,m}(x)$ is quasi-increasing on (c_2, ∞), and, if $0 < \alpha < 1$, then $Q'_{1,\alpha,m}(x)/Q_{1,\alpha,m}(x)$ is quasi-decreasing on (c_2, ∞).

(e) Let $r = 1$ and $0 < \alpha < 1$, then $Q^{(\nu+1)}_{1,\alpha,m}(x)$ is nondecreasing on a certain positive interval on (c_2, ∞).
Hence, $w(x) = \exp(-Q_{r,\alpha,m}) \in \widetilde{\mathcal{F}}_\nu(C^2+)$.

Definition 2.6. One uses the following notation:

$$\varphi_u(x) = \begin{cases} \dfrac{a_u}{u} \dfrac{1 - |x|/a_{2u}}{\sqrt{1 - |x|/a_u + \delta_u}}, & |x| \leqslant a_u; \\ \varphi_u(a_u), & a_u < |x|, \end{cases} \qquad \delta_u = (uT(a_u))^{-2/3}, \ u > 0. \tag{2.13}$$

Lemma 2.7 (see [18, Corollary 4.5]). *Let* $w_\rho(x) = |x|^\rho \exp(-Q(x))$, $\exp(-Q) \in \widetilde{\mathcal{F}}_\nu(C^2+)$. *If* $x_{k,n} \neq 0$ *and* $|x_{k,n}| \leqslant a_n(1+\delta_n)$, *then* $e_0(\nu,k,n) = 1$ *and, for* $i = 1,2,\ldots,\nu-1$,

$$|e_i(\nu,k,n)| \leqslant C\left\{\frac{T(a_n)}{a_n} + \left|Q'(x_{k,n})\right| + \frac{1}{|x_{k,n}|}\right\}^{\langle i\rangle}\left(\frac{n}{a_{2n}-|x_{k,n}|} + \frac{T(a_n)}{a_n}\right)^{i-\langle i\rangle}, \tag{2.14}$$

where

$$\langle i\rangle = \begin{cases} 1, & i\text{: odd}, \\ 0, & i\text{: even}. \end{cases} \tag{2.15}$$

For $x_{k,n} = 0$ *one has*

$$e_0(\nu,k,n) = 1, \qquad |e_i(\nu,k,n)| \leqslant C\left(\frac{n}{a_n}\right)^i, \quad i = 1,2,\ldots,\nu-1. \tag{2.16}$$

Remark 2.8. In [19, Theorem 2.2] we see that $x_{1,n} < a_n$ if n is large enough. Therefore Lemma 2.7 holds for all $x_{k,n}$, $k = 1,2,\ldots,n$.

Levin and Lubinsky (see [9, Lemma 3.7]) showed that there exists $C > 0$ such that for some $\varepsilon > 0$ and for large enough t,

$$T(a_t) \leqslant Ct^{2-\varepsilon}. \tag{2.17}$$

In [20] we have the following estimations.

Lemma 2.9 (see [20, Theorem 1.6]). *Let* $w = \exp(-Q) \in \mathcal{F}(C^2+)$.

(1) *Let* $T(x)$ *be unbounded. Then, for any* $\eta > 0$, *there exists* $C(\eta) > 0$ *such that, for* $t \geqslant 1$,

$$a_t \leqslant C(\eta)t^\eta. \tag{2.18}$$

(2) *Let* $\lambda := C_1$ *be the constant in Definition 2.1(e), that is,*

$$\frac{Q''(x)}{Q'(x)} \leqslant \lambda\frac{Q'(x)}{Q(x)}, \quad a.e.\ x \in \mathbb{R}\setminus\{0\}. \tag{2.19}$$

If $\lambda > 1$, *then there exists* $C(\lambda,\eta)$ *such that*

$$T(a_t) \leqslant C(\lambda,\eta)t^{2(\eta+\lambda-1)/(\lambda+1)}, \quad t \geqslant 1, \tag{2.20}$$

and, if $0 < \lambda \leqslant 1$, *then for any* $\eta > 0$ *there exists* $C(\lambda,\eta)$ *such that*

$$T(a_t) \leqslant C(\lambda,\eta)t^\eta, \quad t \geqslant 1. \tag{2.21}$$

Remark 2.10. (1) If $T(x)$ is bounded, then w is called a Freud-type weight, and, if $T(x)$ is unbounded, then w is called an Erdős-type weight.

(2) In (2.20) and (2.21), we set $0 < \eta < 2$ and

$$\varepsilon = \begin{cases} 2-\eta, & 0 < \lambda \leqslant 1, \\ \dfrac{2(2-\eta)}{(\lambda+1)}, & \lambda > 1. \end{cases} \tag{2.22}$$

Then (2.17) holds.

(3) If

$$\limsup_{x\to\infty} \frac{Q(x)Q''(x)}{(Q'(x))^2} \leqslant 1, \tag{2.23}$$

then we have (2.21). Note that all the examples in Example 2.5 satisfy this inequality.

(4) For the Freud-type exponent $Q(x) = |x|^m, m > 1$, we have

$$T(a_t) = m, \quad a_t \sim t^{1/m}. \tag{2.24}$$

(5) The inequality (2.18) implies

$$0 < C \leqslant \frac{n}{a_n}\left\{\frac{T(a_n)}{a_n}\right\}^{\nu-1}. \tag{2.25}$$

3. Theorems

In the rest of this paper we assume the following for the weight w.

Assumption 3.1. Consider the weight $w_\rho(x) = |x|^\rho \exp(-Q(x)), \exp(-Q(x)) \in \widetilde{\mathcal{F}}_\nu(C^2+), \rho \geqslant 0$.

(a) (cf. [20, Theorem 1.4]) If $T(x)$ is bounded, then we suppose, for δ in (2.12),

$$a_n \leqslant Cn^{1/(1+\nu-\delta)}. \tag{3.1}$$

(b) There exist $0 \leqslant \gamma < 1$ and $C(\gamma) > 0$ such that

$$T(a_n) \leqslant C(\gamma)n^\gamma; \tag{3.2}$$

here, if $T(x)$ is bounded, that is, a Freud-type weight, then we set $\gamma = 0$, and if $T(x)$ is unbounded, that is, an Erdős-type weight, then we set $0 < \gamma < 1$. Define

$$\varepsilon_n := \begin{cases} \dfrac{a_n}{n}, & \dfrac{T(a_n)}{a_n} < 1, \\ \dfrac{1}{n^{1-\gamma}}, & \dfrac{T(a_n)}{a_n} \geqslant 1. \end{cases} \tag{3.3}$$

Remark 3.2. (1) If $T(x)$ is unbounded, then (3.1) holds because of Lemma 2.9 (2.21).

(2) (3.2) holds for

$$\gamma = \frac{2(\eta + \lambda - 1)}{\lambda + 1}, \quad 0 < \lambda < 3. \tag{3.4}$$

(3) In (3.3) we note that $\varepsilon_n \log n \to 0$ as $n \to \infty$.

We shall state our theorems. Put

$$\begin{aligned} X_n(\nu, f; x) &:= \sum_{j=1}^{n} f(x_{j,n,\rho}) l_{j,n,\rho}^{\nu}(x) \sum_{i=0}^{\nu-2} e_i(\nu, j, n)(x - x_{j,n,\rho})^i, \\ Y_n(\nu, f; x) &:= \sum_{j=1}^{n} f(x_{j,n,\rho}) l_{j,n,\rho}^{\nu}(x) e_{\nu-1}(\nu, j, n)(x - x_{j,n,\rho})^{\nu-1}, \\ Z_n(l, \nu, f; x) &:= \sum_{j=1}^{n} \sum_{s=1}^{l} f^{(s)}(x_{j,n,\rho}) l_{j,n,\rho}^{\nu}(x) \sum_{i=s}^{\nu-1} e_{si}(\nu, j, n)(x - x_{j,n,\rho})^i. \end{aligned} \tag{3.5}$$

Furthermore, we consider the class $G = \{g_s \in C(\mathbb{R}), s = l+1, l+2, \ldots, \nu-1\}$ and construct the following interpolation polynomial:

$$W_n(l, \nu, G; x) := \sum_{j=1}^{n} \sum_{s=l+1}^{\nu-1} g_s(x_{j,n,\rho}) l_{j,n,\rho}^{\nu}(x) \sum_{i=s}^{\nu-1} e_{si}(\nu, j, n)(x - x_{j,n,\rho})^i. \tag{3.6}$$

Then we have

$$\begin{aligned} L_n(\nu, f; x) &= X_n(\nu, f; x) + Y_n(\nu, f; x), \\ L_n(l, \nu, f; x) &= L_n(\nu, f; x) + Z_n(l, \nu, f; x), \\ \widetilde{L}_n(l, \nu, f \oplus G; x) &:= L_n(\nu, f; x) + Z_n(l, \nu, f; x) + W_n(l, \nu, G; x). \end{aligned} \tag{3.7}$$

Define

$$\Phi(x) := \frac{1}{(1 + Q(x))^{2/3} T(x)}. \tag{3.8}$$

Here we note that, for some $d > 0$,

$$\Phi(x) \sim \frac{Q(x)^{1/3}}{xQ'(x)}, \quad |x| \geqslant d > 0. \tag{3.9}$$

Moreover, we define

$$\Phi_n(x) := \max\left\{\delta_n, 1 - \frac{|x|}{a_n}\right\}, \quad n = 1, 2, 3, \ldots. \tag{3.10}$$

Lemma 3.3. *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Let $L > 0$ be fixed. Then one has the following:*

(a) *(See [9, Lemma 3.5(a)]) Uniformly for $t > 0$,*

$$a_{Lt} \sim a_t. \tag{3.11}$$

(b) *(See [9, Lemma 3.5(b)]) Uniformly for $t > 0$,*

$$Q^{(j)}(a_{Lt}) \sim Q^{(j)}(a_t), \quad j = 0, 1. \tag{3.12}$$

Moreover,

$$T(a_{Lt}) \sim T(a_t). \tag{3.13}$$

(c) *(See [9, Lemma 3.11 (3.52)]) Uniformly for $t > 0$,*

$$\left|1 - \frac{a_{Lt}}{a_t}\right| \sim \frac{1}{T(a_t)}. \tag{3.14}$$

(d) *(See [9, Lemma 3.4 (3.17), (3.18)]) Uniformly for $t > 0$, one has*

$$Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}}, \qquad Q'(a_t) \sim \frac{t\sqrt{T(a_t)}}{a_t}. \tag{3.15}$$

Lemma 3.4. *For $x \in \mathbb{R}$, we have*

$$\Phi(x) \leqslant C\Phi_n(x), \quad n \geqslant 1. \tag{3.16}$$

Proof. Let $x = a_u, u \geqslant 1$. By Lemma 3.3(d) we have

$$u \sim Q(a_u)\sqrt{T(a_u)}. \tag{3.17}$$

So, we have

$$\delta_u^{-1} \sim Q(a_u)^{2/3}T(a_u) = \frac{a_u Q'(a_u)}{Q(a_u)^{1/3}} = \frac{xQ'(x)}{Q^{1/3}(x)} \sim \Phi^{-1}(a_u). \tag{3.18}$$

Now, if $u \leqslant n/2$, then

$$\begin{aligned} 1-\frac{a_u}{a_n} &\geqslant 1-\frac{a_{n/2}}{a_n} \sim \frac{1}{T(a_n)} \quad (\text{by Lemma 3.2(c)}) \\ &\geqslant \frac{1}{(nT(a_n))^{2/3}} = \delta_n \quad (\text{by (2.17)}). \end{aligned} \tag{3.19}$$

So, we have

$$\begin{aligned} \Phi_n(x) = 1-\frac{a_u}{a_n} &\geqslant 1-\frac{a_u}{a_{2u}} \sim \frac{1}{T(a_u)} \quad (\text{by Lemma 3.2(c)}) \\ &\geqslant \frac{1}{(uT(a_u))^{2/3}} = \delta_u \sim \Phi(x) \quad (\text{by (2.17) and (3.18)}). \end{aligned} \tag{3.20}$$

Let $n/2 < u < n$. Then we have

$$\Phi_n(x) \geqslant \delta_n \sim \delta_u \sim \Phi(x) \quad (\text{by Lemmas 3.2(a), (2.17), and (3.18)}). \tag{3.21}$$

□

Let $C(f)$ denote a positive constant depending only on f.

Assumption 3.5. Let $w_\rho(x) = |x|^\rho w(x)$ and $w(x) = \exp(-Q(x)) \in \mathcal{F}_\nu(C^2+)$, $\rho \geqslant 0$. Suppose the following.

(A-1) Let $f \in C(\mathbb{R})$ satisfy that, for a given $0 < \delta < 1$,

$$|f(x)| w^{\nu-\delta}(x) \left\{ |Q'(x)| + \frac{1}{|x|} \right\} \leqslant C(f), \quad x \in \mathbb{R}, \tag{3.22}$$

where we suppose $\limsup_{|x|\to 0} |f(x)/x| \leqslant C(f)$.

(A-2) Let $f \in C^l(\mathbb{R})$ for a certain $0 < l \leqslant \nu - 1$. Then we suppose that f satisfies

$$\left| f^{(s)}(x) \right| \left\{ \Phi^{-3/4}(x) w_\rho(x) \right\}^\nu \leqslant C(f), \quad x \in \mathbb{R},\ s = 1, 2, \ldots, l. \tag{3.23}$$

(A-3) Let $G = \{g_s \in C(\mathbb{R}), s = l+1, l+2, \ldots, \nu-1\}$. Then we suppose that there exists a constant $C > 0$ such that

$$|g_s(x)| \left\{ \Phi^{-3/4}(x) w_\rho(x) \right\}^\nu \leqslant C(G), \quad x \in \mathbb{R},\ s = l+1, l+2, \ldots, \nu-1. \tag{3.24}$$

Remark 3.6. In (3.22), we have the following.

(1) $f(0) = 0$ and

$$|f(x)|\left\{\Phi^{-3/4}(x)w_\rho(x)\right\}^\nu\left(|Q'(x)| + \frac{1}{|x|}\right) \leqslant C(f), \quad x \in \mathbb{R} \setminus \{0\}. \tag{3.25}$$

(2) For some positive constant C, we have $|Q'(x)| + 1/|x| \geqslant C$. Hence, from (3.22), it follows that

$$|f(x)|\left\{\Phi^{-3/4}(x)w_\rho(x)\right\}^\nu \leqslant C(f), \quad x \in \mathbb{R}. \tag{3.26}$$

We have a chain of results under Assumption 3.1.

Proposition 3.7. *Let $\nu = 1, 2, 3, \ldots$. For $f(x)$ satisfying* (3.26)*, one has*

$$\left\|\left\{\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^\rho\right\}^\nu X_n(\nu, f; x)\right\|_{L_\infty(\mathbb{R})} \leqslant C(f). \tag{3.27}$$

Proposition 3.8. *Let $\nu = 2, 4, 6, \ldots$. For $f(x)$ satisfying* (3.25)*, one has*

$$\left\|\left\{\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^\rho\right\}^\nu Y_n(\nu, f; x)\right\|_{L_\infty(\mathbb{R})} \leqslant C(f)\varepsilon_n \log n, \tag{3.28}$$

where ε_n is defined by (3.3).

Proposition 3.9. *Let $\nu = 1, 2, 3, \ldots$. For $f(x)$ satisfying* (3.23)*, one has*

$$\left\|\left\{\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^\rho\right\}^\nu Z_n(l, \nu, f; x)\right\|_{L_\infty(\mathbb{R})} \leqslant C(f)\frac{a_n \log n}{n}, \tag{3.29}$$

and for $f(x)$ satisfying (3.24) *one has*

$$\left\|\left\{\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^\rho\right\}^\nu W_n(l, \nu, G; x)\right\|_{L_\infty(\mathbb{R})} \leqslant C(G)\frac{a_n \log n}{n}. \tag{3.30}$$

Proposition 3.10. *Let $\nu = 1, 2, 3, \ldots$. Let $P \in \mathcal{P}_m$ be fixed. Then one has*

$$\left\|\left\{\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^\rho\right\}^\nu (L_n(\nu, P, x) - P(x))\right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.31}$$

Proposition 3.11. *Let $\nu = 2, 4, 6, \ldots$. Let $P \in \mathcal{P}_m$ with $P(0) = 0$ be fixed. Then one has*

$$\left\|\left\{\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^\rho\right\}^\nu (X_n(\nu, P; x) - P(x))\right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.32}$$

Theorem 3.12. *Let $\nu = 2, 4, 6, \ldots$. For $f(x)$ satisfying* (3.22), *one has*

$$\left\| \left\{ \Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} \left(L_n(\nu, f; x) - f(x) \right) \right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.33}$$

Theorem 3.13. *Let $\nu = 2, 4, 6, \ldots$. For $f(x)$ satisfying* (3.22) *and* (3.23), *one has*

$$\left\| \left\{ \Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} \left(L_n(l, \nu, f; x) - f(x) \right) \right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.34}$$

Corollary 3.14. *Let $\nu = 2, 4, 6, \ldots$. For $f(x)$ satisfying* (3.22), *one has*

$$\left\| \left\{ \Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} \left(X_n(\nu, f; x) - f(x) \right) \right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.35}$$

Theorem 3.15. *Let $\nu = 2, 4, 6, \ldots$. For $f(x)$ satisfying* (3.22), (3.23) *and* (3.24), *one has*

$$\left\| \left\{ \Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} \left(\widetilde{L}_n(l, \nu, f \oplus G; x) - f(x) \right) \right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.36}$$

Define

$$I_n[k, H_n(f)] := \int_{-\infty}^{\infty} k(x) H_n(f; x) dx, \tag{3.37}$$

where $H_n(f)$ equals to one of the following:

$$L_n(\nu, f), L_n(l, \nu, f), \widetilde{L}_n(l, \nu, f \oplus G), X_n(\nu, f). \tag{3.38}$$

One also defines

$$I[k, f] := \int_{-\infty}^{\infty} k(x) f(x) dx. \tag{3.39}$$

Theorem 3.16. *Let $\nu = 2, 4, 6, \ldots$. Let $f(x)$ satisfy* (3.22), (3.23) *and* (3.24). *If*

$$\int_{-\infty}^{\infty} k(x) \left\{ \Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{-\nu} dx < \infty, \quad n = 1, 2, 3, \ldots, \tag{3.40}$$

then we have

$$\lim_{n \to \infty} I_n[k, H_n(f)] = I[k, f]. \tag{3.41}$$

For example, we can take $k(x)$ in the following way. If $T(x)$ is unbounded, we have, for $\Delta \in \mathbb{R}$,

$$\int_1^\infty (1+|x|)^{-\Delta}\Phi^{3\nu/4}(x)dx \leqslant \int_1^\infty (1+|x|)^{-\Delta}\left(\frac{1}{T(x)Q(x)^{2/3}}\right)^{3\nu/4} dx < \infty \tag{3.42}$$

(see [10, Lemma 2.1(b)]). Then we set

$$k(x) := (1+|x|)^{\rho\nu-\Delta}\left\{\Phi^{3/2}(x)w(x)\right\}^{\nu}. \tag{3.43}$$

If $T(x)$ is bounded and $w(x) = \exp(-|x|^\beta)$, $\beta > 1$, then we take Δ as $\nu\beta/2+\Delta > 1$ so that (3.43) can hold. Then $k(x)$ defined by (3.43) satisfies (3.40).

4. Proof of Theorems

For constants $C, C_1 > 0$, the same symbol does not necessarily denote the same constant in different occurrences.

Lemma 4.1. *One has the following.* (1) *(See [19, Theorem 2.3]) Let* $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2)$ *and* $\rho > -1/2$. *Then, uniformly for* $n \geqslant 1$ *one has*

$$\sup_{x\in\mathbb{R}}\left|p_{n,\rho}(x)w(x)\right|\left(|x|+\frac{a_n}{n}\right)^{\rho}\left|x^2-a_n^2\right|^{1/4} \sim 1, \tag{4.1}$$

and for $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$,

$$\sup_{x\in\mathbb{R}}\left|p_{n,\rho}(x)w(x)\right|\left(|x|+\frac{a_n}{n}\right)^{\rho} \sim a_n^{-1/2}(nT(a_n))^{1/6}. \tag{4.2}$$

(2) *(See [19, Theorem 2.5(d)]) Let* $w \in \mathcal{F}(C^2+)$ *and* $\rho > -1/2$. *Let* $1 \leqslant j \leqslant n-1$ *and* $x \in [x_{j+1,n}, x_{j,n}]$. *Then*

$$\left|p_n(x)w(x)\right|\left(|x|+\frac{a_n}{n}\right)^{\rho} \sim \min\{|x-x_{j,n}|, |x-x_{j+1,n}|\}\varphi_n(x_{j,n})^{-1}\left|a_n^2-x_{j,n}^2\right|^{-1/4}. \tag{4.3}$$

(3) *(See [19, Theorem 2.5(c)]) Let* $w \in \mathcal{F}(C^2+)$ *and* $\rho > -1/2$. *Then one has*

$$\begin{aligned}\max_{x\in\mathbb{R}}&\left|\frac{p_n(x)w(x)(|x|+a_n/n)^{\rho}}{(x-x_{j,n})p_n'(x_{j,n})w(x_{j,n})(|x_{j,n}|+a_n/n)^{\rho}}\right| \\ &= \max_{x\in\mathbb{R}}\left|l_{jn}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right|w^{-1}(x_{j,n})\left(|x_{j,n}|+\frac{a_n}{n}\right)^{\rho} \sim 1.\end{aligned} \tag{4.4}$$

(4) *(See [19, Theorem 2.5(a)]) Let* $w \in \mathcal{F}(C^2+)$ *and* $\rho > -1/2$. *For* $1 \leqslant j \leqslant n$ *we have*

$$\left|p_n' w\right|(x_{j,n})\left(\left|x_{j,n}\right| + \frac{a_n}{n}\right)^{\rho} \sim \varphi_n^{-1}(x_{j,n})\left|a_n^2 - x_{j,n}^2\right|^{-1/4}. \tag{4.5}$$

Lemma 4.2 (see [19, Theorem 2.2]). *Let* $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$ *and* $\rho > -1/2$. *For the zeros* $x_{j,n} = x_{j,n,\rho}$, *one has the following:*

(1) *For* $n \geqslant 1$ *and* $1 \leqslant j \leqslant n-1$,

$$\begin{gathered} x_{j,n} - x_{j+1,n} \sim \varphi_n(x_{j,n}), \\ \varphi_n(x_{j,n}) \sim \varphi_n(x_{j+1,n}) \quad (\textit{see } ([19, \textit{Lemma } A.1(A.3)])). \end{gathered} \tag{4.6}$$

(2) *For the minimum positive zero* $x_{[n/2],n}$ ($[n/2]$ *is the largest integer* $\leqslant n/2$), *one has*

$$x_{[n/2],n} \sim a_n n^{-1}, \tag{4.7}$$

and for large enough n,

$$1 - \frac{x_{1,n}}{a_n} \sim \delta_n. \tag{4.8}$$

(3) *(See [19, Lemma 4.7])* $b_n = \gamma_{n-1}/\gamma_n \sim a_n \sim x_{1,n}$, *where* γ_n *is defined by* (1.3).

Lemma 4.3. *Let* $w \in \mathcal{F}(C^2+)$. *Then there exist* $C_1, C_2 > 0$ *such that*

$$\begin{aligned} \sup_{x\in\mathbb{R}}\left|p_{n,\rho}(x)w(x)\right|\left(|x| + \frac{a_n}{n}\right)^{\rho}\Phi^{1/4}(x) &\leqslant C_1 \sup_{x\in\mathbb{R}}\left|p_{n,\rho}(x)w(x)\right|\left(|x| + \frac{a_n}{n}\right)^{\rho}\Phi_n^{1/4}(x) \\ &\leqslant C_2 a_n^{-1/2}. \end{aligned} \tag{4.9}$$

Proof. The first inequality follows from Lemma 3.4. We show the second inequality. Noting (4.8), from Lemma 4.1 (4.1) we have

$$\begin{aligned} C &\geqslant \sup_{|x|\leqslant x_{1,n}}\left|p_{n,\rho}(x)w(x)\right|\left(|x| + \frac{a_n}{n}\right)^{\rho}\left|x^2 - a_n^2\right|^{1/4} \\ &\sim \sup_{|x|\leqslant x_{1,n}}\left|p_{n,\rho}(x)w(x)\right|\left(|x| + \frac{a_n}{n}\right)^{\rho} a_n^{1/2}\,\Phi_n^{1/4}(x). \end{aligned} \tag{4.10}$$

From (4.2) we see that

$$\begin{aligned} a_n^{-1/2} &\geqslant C \sup_{|x|>x_{1,n}} \left|p_{n,\rho}(x)w(x)\right| \left(|x| + \frac{a_n}{n}\right)^{\rho} \delta_n^{1/4} \\ &\sim C \sup_{|x|>x_{1,n}} \left|p_{n,\rho}(x)w(x)\right| \left(|x| + \frac{a_n}{n}\right)^{\rho} \Phi_n^{1/4}(x). \end{aligned} \tag{4.11}$$

Therefore we have the result. □

Proof of Proposition 3.7. We recall the definition of $X_n(\nu, f; x)$:

$$\begin{aligned} X_n(\nu, f; x) &= \sum_{j=1}^{n} f(x_{j,n}) l_{jn}^{\nu}(x) \sum_{i=0}^{\nu-2} e_i(\nu, j, n)(x - x_{j,n})^i \\ &= \sum_{i=0}^{\nu-2} \sum_{j=1}^{n} f(x_{j,n}) l_{jn}^{\nu}(x)(x - x_{j,n})^i e_i(\nu, j, n). \end{aligned} \tag{4.12}$$

Using Lemma 2.3, we may estimate, for $i = 0, 1, \ldots, \nu - 2$ and $1 \leqslant j \leqslant n$,

$$A_{i,j}(x) := \left(\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right)^{\nu} \left|f(x_{j,n})\right| \left|l_{jn}(x)\right|^{\nu} \left|x - x_{j,n}\right|^i \left(\frac{n}{\sqrt{a_{2n}^2 - x_{j,n}^2}}\right)^i. \tag{4.13}$$

Let $(x_{m+1,n} + x_{m,n})/2 < x \leqslant x_{m,n}$ or $x_{m,n} \leqslant x < (x_{m-1,n} + x_{m,n})/2$. For simplicity, we assume $x > 0$, and let $x_{0,n} := x_{1,n} + \delta\varphi_n(x_{1,n})$ for a fixed $\delta > 0$ small enough. Then we can assume that there exists a constant $c > 0$ such that

$$x_{0,n} := x_{1,n} + \delta\varphi_n(x_{1,n}) < a_n - c\delta_n. \tag{4.14}$$

Assume that $x < x_{0,n}$. Then we first estimate $A_{i,m}(x)$. By Lemma 4.1(3) and the definition of $\varphi_n(x)$, we have

$$\left(\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right)^{\nu} |l_{mn}(x)|^{\nu} \leqslant C\Phi^{3\nu/4}(x)\left(w(x_{m,n})\left(|x_{m,n}| + \frac{a_n}{n}\right)^{\rho}\right)^{\nu}, \tag{4.15}$$

$$|x - x_{m,n}|^i \left(\frac{n}{\sqrt{a_{2n}^2 - x_{m,n}^2}}\right)^i \leqslant C\left(\frac{n\varphi_n(x_{m,n})}{\sqrt{a_{2n}^2 - x_{m,n}^2}}\right)^i \leqslant C\left(\frac{\sqrt{1 - |x_{m,n}|/a_{2n}}}{\sqrt{1 - |x_{m,n}|/a_n}}\right)^i, \tag{4.16}$$

and by Remark 3.6

$$\left|f(x_{m,n})\right| \leqslant C(f)\Phi^{3\nu/4}(x_{m,n})\left(w(x_{m,n})\left(|x_{m,n}| + \frac{a_n}{n}\right)^{\rho}\right)^{-\nu}. \tag{4.17}$$

Therefore, we have

$$\begin{aligned}
A_{i,m}(x) &\leqslant C(f)\Phi^{3\nu/4}(x)\Phi^{3\nu/4}(x_{m,n})\left(\frac{\sqrt{1-|x_{m,n}|/a_{2n}}}{\sqrt{1-|x_{m,n}|/a_n}}\right)^i \\
&= C(f)\Phi^{3\nu/4}(x)\left(\max\left\{1-\frac{|x_{m,n}|}{a_n},\delta_n\right\}\right)^{3\nu/4}\left(\frac{\sqrt{1-|x_{m,n}|/a_{2n}}}{\sqrt{1-|x_{m,n}|/a_n}}\right)^i \\
&= C(f)\Phi^{3\nu/4}(x)\left(1-\frac{|x_{m,n}|}{a_n}\right)^{3\nu/4-i/2}\left(\sqrt{1-\frac{|x_{m,n}|}{a_{2n}}}\right)^i \\
&\leqslant C(f).
\end{aligned} \tag{4.18}$$

Next, we estimate $\sum_{1\leqslant j\leqslant n, j\neq m} A_{i,j}(x)$. For $1 \leqslant j \leqslant n$, $j \neq m$, we have, by Lemma 4.3 and Lemma 4.1(4),

$$\begin{aligned}
&\left(\Phi^{3/4}(x)w(x)(|x|+a_n/n)^\rho\right)^\nu |l_{jn}(x)|^\nu |x-x_{j,n}|^i \\
&\quad = \Phi^{\nu/2}(x)\left|\frac{p_n(x)\Phi^{1/4}(x)w(x)(|x|+a_n/n)^\rho}{p_n'(x_{j,n})w(x_{j,n})(|x_{j,n}|+a_n/n)^\rho}\right|^\nu \frac{\left(w(x_{j,n})(|x_{j,n}|+a_n/n)^\rho\right)^\nu}{|x-x_{j,n}|^{\nu-i}} \\
&\quad = \Phi^{\nu/2}(x)a_n^{-\nu/2}\varphi_n^\nu(x_{j,n})\left|a_n^2-x_{j,n}^2\right|^{\nu/4}\frac{\left(w(x_{j,n})(|x_{j,n}|+a_n/n)^\rho\right)^\nu}{|x-x_{j,n}|^{\nu-i}}.
\end{aligned} \tag{4.19}$$

Then, since we know from Remark 3.6 that

$$|f(x_{j,n})| \leqslant C(f)\Phi^{3\nu/4}(x_{j,n})\left(w(x_{j,n})\left(|x_{j,n}|+\frac{a_n}{n}\right)^\rho\right)^{-\nu}, \tag{4.20}$$

we have, by Lemma 3.4,

$$\begin{aligned}
\sum_{j\neq m} A_{i,j}(x) &\leqslant C(f)\sum_{j\neq m}\Phi_n^{\nu/2}(x)\Phi_n^{3\nu/4}(x_{j,n})\varphi_n^\nu(x_{j,n}) \\
&\quad\times\left(1-\frac{|x_{j,n}|}{a_n}\right)^{\nu/4}\left(\frac{1}{|x-x_{j,n}|}\right)^{\nu-i}\left(\frac{n}{\sqrt{a_{2n}^2-x_{j,n}^2}}\right)^i.
\end{aligned} \tag{4.21}$$

By the definition of $\varphi_n(x)$ and by Lemma 3.3(a) $a_{2n} \sim a_n$, we have

$$\varphi_n^\nu(x_{j,n})\left(\frac{n}{\sqrt{a_{2n}^2-x_{j,n}^2}}\right)^i \leqslant C\left(\frac{a_n}{n}\right)^{\nu-i}\left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\nu-i/2}\left(1-\frac{|x_{j,n}|}{a_n}\right)^{-\nu/2}. \tag{4.22}$$

By Lemma 4.2(1), we have

$$\begin{aligned}
\frac{1}{|x-x_{j,n}|} &\leqslant C\frac{1}{\sum_{j\leqslant s\leqslant m+1 \text{ or } m-1\leqslant s\leqslant j}\varphi_n(x_{s,n})} \\
&\leqslant C\frac{n/a_n}{\sum_{j\leqslant s\leqslant m+1 \text{ or } m-1\leqslant s\leqslant j}(1-|x_{s,n}|/a_{2n})/(1-|x_{s,n}|/a_n+\delta_n)^{1/2}} \\
&\leqslant C\frac{n/a_n}{\sum_{j\leqslant s\leqslant m+1 \text{ or } m-1\leqslant s\leqslant j}(1-|x_{s,n}|/a_n)/(1-|x_{s,n}|/a_n+\delta_n)^{1/2}} \\
&\leqslant C\frac{n/a_n}{|m-j|(1-\max\{|x|,|x_{j,n}|\}/a_n)^{1/2}}.
\end{aligned} \tag{4.23}$$

Here, we note from (4.14) and (4.8) that

$$\Phi_n(x)\sim\left(1-\frac{|x|}{a_n}\right),\qquad \Phi_n(x_{j,n})\sim\left(1-\frac{|x_{j,n}|}{a_n}\right). \tag{4.24}$$

Thus, we have, for $j\neq m$,

$$\begin{aligned}
&\Phi_n^{\nu/2}(x)\Phi_n^{3\nu/4}(x_{j,n})\varphi_n^{\nu}(x_{j,n})\left(1-\frac{|x_{j,n}|}{a_n}\right)^{\nu/4}\left(\frac{1}{|x-x_{j,n}|}\right)^{\nu-i}\left(\frac{n}{\sqrt{a_{2n}^2-x_{j,n}^2}}\right)^{i} \\
&\quad\leqslant C\left(1-\frac{|x|}{a_n}\right)^{\nu/2}\left(1-\frac{|x_{j,n}|}{a_n}\right)^{\nu/2}\left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\nu-i/2} \\
&\qquad\times\left(\frac{1}{|m-j|(1-\max\{|x|,|x_{j,n}|\}/a_n)^{1/2}}\right)^{\nu-i} \\
&\quad\leqslant C\left(\frac{1}{|m-j|}\right)^{\nu-i}.
\end{aligned} \tag{4.25}$$

Therefore, we have

$$\sum_{\substack{1\leqslant j\leqslant n,\\ j\neq m}}A_{i,j}(x)\leqslant C(f)\sum_{j\neq m}\left(\frac{1}{|m-j|}\right)^{\nu-i}\leqslant C(f), \tag{4.26}$$

because of $\nu-i\geqslant 2$.

Remark 4.4. If we consider the estimate of $\sum_{i=1}^{n}A_{i,\nu-1}(x)$ with (4.13), then we obtain

$$\sum_{i=1}^{n}A_{i,\nu-1}(x)\leqslant C(f)\log n. \tag{4.27}$$

We continue the proof Proposition 3.7. We need to estimate $\sum_{j=1}^{n} A_{i,j}(x)$ for $|x| > x_{0,n}$. Now, suppose $x > x_{0,n}$. Then similarly to (4.23), we have

$$\frac{1}{|x_{0,n} - x_{j,n}|} \leqslant C \frac{1}{\sum_{1\leqslant s\leqslant j} \varphi_n(x_{s,n})} \leqslant C \frac{n/a_n}{j(1 - x_{0,n}/a_n)^{1/2}} \leqslant C \frac{1}{j\delta_n^{1/2}} \frac{n}{a_n}. \tag{4.28}$$

Similarly to (4.21), we have using (4.22)

$$\begin{aligned}
\sum A_{i,j}(x) &\leqslant C(f) \sum_j \Phi^{\nu/2}(x) \Phi_n^{3\nu/4}(x_{j,n}) \varphi_n^{\nu}(x_{j,n}) \\
&\quad \times \left(1 - \frac{|x_{j,n}|}{a_n}\right)^{\nu/4} \left(\frac{1}{|x_{0,n} - x_{j,n}|}\right)^{\nu-i} \left(\frac{n}{\sqrt{a_{2n}^2 - x_{j,n}^2}}\right)^{i} \\
&\leqslant C(f) \sum_j \Phi_n^{\nu/2}(x) \left(1 - \frac{|x_{j,n}|}{a_n}\right)^{\nu/2} \left(1 - \frac{|x_{j,n}|}{a_{2n}}\right)^{\nu-i/2} \left(\frac{1}{j\delta_n^{1/2}}\right)^{\nu-i} \\
&\leqslant C(f) \sum_j \frac{1}{j^{\nu-i}} C(f),
\end{aligned} \tag{4.29}$$

because of $\nu - i \geqslant 2$ and $\Phi^{\nu/2}(x)\delta_n^{-(\nu-i)/2} \leqslant \delta_n^{\nu/2}\delta_n^{-(\nu-i)/2} \leqslant C$. Consequently we achieve the result. □

Remark 4.5. The above proof implies the following: there exists a constant $C > 0$ such that

$$\sum_{j=1}^{n} |h_{j,n,\rho}(\nu; x)| \leqslant \sum_{j=1}^{n} |l_{jn}(x)|^{\nu} \sum_{i=0}^{\nu-2} |e_i(\nu, k, n)| \, |x - x_{j,n}|^{i} \leqslant C, \quad x \in \mathbb{R}. \tag{4.30}$$

Proof of Proposition 3.8. We use the same method to prove of Proposition 3.7. So, we let $(x_{m+1,n} + x_{m,n})/2 < x \leqslant x_{m,n}$ or $x_{m,n} \leqslant x < (x_{m-1,n} + x_{m,n})/2$. For simplicity, we assume $x > 0$, and let $x_{0,n} := x_{1,n} + \delta\varphi_n(x_{1,n})$ for a fixed $\delta > 0$ small enough and there exists a constant $\delta_1 > 0$ such that $x_{0,n} := x_{1,n} + \delta\varphi_n(x_{1,n}) < a_n - \delta_1\delta_n$. Assume that $x < x_{0,n}$. Since $f(0) = 0$, we may leave out the term with $x_{j,n} = 0$. Hence we consider only the term of $x_{j,n} \neq 0$. Noting Assumption 3.1, (3.2), and Lemma 2.7, we estimate $\sum_{j=1}^{n} B_{\nu-1,j}(x)$, where

$$\begin{aligned}
B_{\nu-1,j}(x) &:= \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n}\right)^{\rho}\right)^{\nu} \\
&\quad \times |f(x_{j,n})| |l_{jn}(x)|^{\nu} |x - x_{j,n}|^{\nu-1} \left\{\frac{T(a_n)}{a_n} + |Q'(x_{j,n})| + \frac{1}{|x_{j,n}|}\right\} \left(\frac{n}{a_{2n} - |x_{j,n}|}\right)^{\nu-2}.
\end{aligned} \tag{4.31}$$

First we estimate $B_{\nu-1,m}(x)$. By Lemma 4.2, (4.16), and the definition of $\varphi_n(x)$, we have

$$\begin{aligned}|x-x_{m,n}|^{\nu-1}\left(\frac{n}{\sqrt{a_{2n}^2-x_{m,n}^2}}\right)^{\nu-2} &\leqslant C\varphi_n(x_{m,n})\left(\frac{\sqrt{1-|x_{m,n}|/a_{2n}}}{\sqrt{1-|x_{m,n}|/a_n}}\right)^{\nu-2}\\ &\leqslant C\frac{a_n}{n}\left(1-\frac{|x_{m,n}|}{a_{2n}}\right)^{1/2}\left(\frac{\sqrt{1-|x_{m,n}|/a_{2n}}}{\sqrt{1-|x_{m,n}|/a_n}}\right)^{\nu-1}.\end{aligned} \tag{4.32}$$

Since here $|Q'(x_{m,n})|+1/|x_{m,n}|\geqslant C$ for some positive constant C, we know that

$$\left(\frac{T(a_n)}{a_n}+\left|Q'(x_{m,n})\right|+\frac{1}{|x_{m,n}|}\right)\leqslant C\max\left\{1,\frac{T(a_n)}{a_n}\right\}\left(\left|Q'(x_{m,n})\right|+\frac{1}{|x_{m,n}|}\right), \tag{4.33}$$

and by (3.25)

$$\left|f(x_{m,n})\right|\leqslant C(f)\Phi^{3\nu/4}(x_{m,n})\left(w(x_{m,n})\left(|x_{m,n}|+\frac{a_n}{n}\right)^{\rho}\right)^{-\nu}\left(\left|Q'(x_{m,n})\right|+\frac{1}{|x_{m,n}|}\right)^{-1}. \tag{4.34}$$

Therefore, using (3.3) and (4.15), we have

$$\begin{aligned}B_{\nu-1,m}(x) &\leqslant C(f)\max\left\{1,\frac{T(a_n)}{a_n}\right\}\Phi_n^{3\nu/4}(x)\Phi_n^{3\nu/4}(x_{m,n})\frac{a_n}{n}\left(1-\frac{|x_{m,n}|}{a_{2n}}\right)^{1/2}\\ &\quad\times\left(\frac{\sqrt{1-|x_{m,n}|/a_{2n}}}{\sqrt{1-|x_{m,n}|/a_n}}\right)^{\nu-1}\\ &\leqslant C(f)\varepsilon_n\Phi_n^{3\nu/4}(x)\Phi_n^{3\nu/4}(x_{m,n})\left(1-\frac{|x_{m,n}|}{a_{2n}}\right)^{1/2}\left(\frac{\sqrt{1-|x_{m,n}|/a_{2n}}}{\sqrt{1-|x_{m,n}|/a_n}}\right)^{\nu-1}.\end{aligned} \tag{4.35}$$

Noting (4.24), we have

$$B_{\nu-1,m}(x)\leqslant C(f)\varepsilon_n\Phi_n^{3\nu/4}(x)\left(1-\frac{|x_{m,n}|}{a_n}\right)^{(\nu+2)/4}\left(1-\frac{|x_{m,n}|}{a_{2n}}\right)^{\nu/2}\leqslant C(f)\varepsilon_n. \tag{4.36}$$

Next, we estimate $\sum_{j\neq m}B_{\nu-1,j}(x)$. Noting (4.19) and (4.22), we have

$$\begin{aligned}&\left(\Phi_n^{3/4}(x)w(x)\left(|x|+\frac{a_n}{n}\right)^{\rho}\right)^{\nu}\left|l_{jn}(x)\right|^{\nu}|x-x_{j,n}|^{\nu-1}\\ &\quad=\Phi_n^{\nu/2}(x)a_n^{-\nu/2}\varphi_n^{\nu}(x_{j,n})\left|a_n^2-x_{j,n}^2\right|^{\nu/4}\frac{\left(w(x_{j,n})\left(|x_{j,n}|+a_n/n\right)^{\rho}\right)^{\nu}}{|x-x_{j,n}|},\\ &\varphi_n^{\nu}(x_{j,n})\left(\frac{n}{\sqrt{a_{2n}^2-x_{j,n}^2}}\right)^{\nu-2}\leqslant C\left(\frac{a_n}{n}\right)^2\left(1-\frac{|x_{j,n}|}{a_n}\right)^{-\nu/2}\left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\nu/2+1}.\end{aligned} \tag{4.37}$$

Then by (4.33) and (3.25) (noting (4.34)), using the notation (3.3) and (4.24), we have

$$\begin{aligned}
B_{\nu-1,j}(x) &\leqslant C(f)\max\left\{1,\frac{T(a_n)}{a_n}\right\}\left(\frac{a_n}{n}\right)^2\Phi_n^{\nu/2}(x)\Phi_n^{3\nu/4}(x_{j,n})a_n^{-\nu/2}\left|a_n^2-x_{j,n}^2\right|^{\nu/4}\\
&\quad\times\frac{1}{|x-x_{j,n}|}\left(1-\frac{|x_{j,n}|}{a_n}\right)^{-\nu/2}\left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\nu/2+1}\\
&\leqslant C(f)\varepsilon_n\frac{a_n}{n}\Phi_n^{\nu/2}(x)\Phi_n^{3\nu/4}(x_{j,n})\left(1-\frac{|x_{j,n}|}{a_n}\right)^{-\nu/4}\left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\nu/2+1}\frac{1}{|x-x_{j,n}|}\\
&\leqslant C(f)\varepsilon_n\frac{a_n}{n}\left(1-\frac{|x|}{a_n}\right)^{\nu/2}\left(1-\frac{|x_{j,n}|}{a_n}\right)^{\nu/2}\left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\nu/2+1}\frac{1}{|x-x_{j,n}|}.
\end{aligned}\tag{4.38}$$

Therefore, since we know from (4.23) that

$$\frac{1}{|x-x_{j,n}|}\leqslant C\frac{n/a_n}{|m-j|\left(1-\max\{|x|,|x_{j,n}|\}/a_n\right)^{1/2}},\tag{4.39}$$

noting (4.24), we have

$$\begin{aligned}
\sum_{j\neq m}B_{\nu-1,j}(x) &\leqslant C(f)\varepsilon_n\left(1-\frac{|x|}{a_n}\right)^{\nu/2}\sum_{j\neq m}\frac{\left(1-|x_{j,n}|/a_n\right)^{\nu/2}\left(1-|x_{j,n}|/a_{2n}\right)^{\nu/2+1}}{|m-j|\left(1-\max\{|x|,|x_{j,n}|\}/a_n\right)^{1/2}}\\
&\leqslant C(f)\varepsilon_n\left(1-\frac{|x|}{a_n}\right)^{(\nu-1)/2}\sum_{j\neq m}\frac{\left(1-|x_{j,n}|/a_n\right)^{(\nu-1)/2}\left(1-|x_{j,n}|/a_{2n}\right)^{\nu/2+1}}{|m-j|}\\
&\leqslant C(f)\varepsilon_n\sum_{j\neq m}\frac{1}{|m-j|}\leqslant C(f)\varepsilon_n\log n.
\end{aligned}\tag{4.40}$$

Finally, we estimate $\sum_{j=1}^{n}B_{\nu-1,j}(x)$ for $x>x_{0,n}$. Suppose $x>x_{0,n}$. Then similar to the above computations, we have

$$\begin{aligned}
B_{\nu-1,j}(x) &\leqslant C(f)\varepsilon_n\frac{a_n}{n}\Phi^{\nu/2}(x)\Phi_n^{3\nu/4}(x_{j,n})\left(1-\frac{|x_{j,n}|}{a_n}\right)^{-\nu/4}\left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\nu/2+1}\frac{1}{|x_{0,n}-x_{j,n}|}\\
&\leqslant C(f)\varepsilon_n\frac{a_n}{n}\Phi^{\nu/2}(x)\left(1-\frac{|x_{j,n}|}{a_n}\right)^{\nu/2}\left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\nu/2+1}\frac{1}{|x_{0,n}-x_{j,n}|}.
\end{aligned}\tag{4.41}$$

Then, since we know by (4.28) that

$$\frac{1}{|x_{0,n}-x_{j,n}|}\leqslant C\frac{1}{j\delta_n^{1/2}}\frac{n}{a_n},\tag{4.42}$$

we have

$$B_{\nu-1,j}(x) \leqslant C(f)\varepsilon_n \Phi^{\nu/2}(x)\left(1-\frac{|x_{j,n}|}{a_n}\right)^{\nu/2}\left(1-\frac{|x_{j,n}|}{a_{2n}}\right)^{\nu/2+1}\frac{1}{j\delta_n^{1/2}}. \tag{4.43}$$

Then, since $\Phi^{\nu/2}(x)\delta_n^{-1/2} < C$ for $x > x_{0,n}$, we have

$$\sum_{j=1}^{n} B_{\nu-1,j}(x) \leqslant C(f)\varepsilon_n \sum_{j=1}^{n}\frac{1}{j} \leqslant C(f)\varepsilon_n \log n. \tag{4.44}$$

Therefore, the result is proved. □

Proof of Proposition 3.9. By Lemma 2.3 we see that, for a constant $C_1 > 0$,

$$\begin{aligned}
|Z_n(l,\nu,f;x)| &\leqslant C_1 \sum_{j=1}^{n}\sum_{s=1}^{l}\left|f^{(s)}(x_{j,n})\right|\left|l_{jn}(x)\right|^{\nu}\sum_{i=s}^{\nu-1}\left(\frac{n}{\sqrt{a_{2n}^2-x_{j,n}^2}}\right)^{i-s}|x-x_{j,n}|^i \\
&\leqslant C_1\frac{a_n}{n}\sum_{j=1}^{n}\sum_{s=1}^{l}\left|f^{(s)}(x_{j,n})\right|\left|l_{jn}(x)\right|^{\nu}\sum_{i=s}^{\nu-1}\left(\frac{n}{\sqrt{a_{2n}^2-x_{j,n}^2}}\right)^{i}|x-x_{j,n}|^i \\
&\leqslant C_1\frac{a_n}{n}\sum_{s=1}^{l}\sum_{i=0}^{\nu-1}\sum_{j=1}^{n}\left|f^{(s)}(x_{j,n})\right|\left|l_{jn}(x)\right|^{\nu}|x-x_{j,n}|^i\left(\frac{n}{\sqrt{a_{2n}^2-x_{j,n}^2}}\right)^{i} \\
&=: \frac{a_n}{n}\sum_{s=1}^{l}\sum_{i=0}^{\nu-1}C_{i,s}(x),
\end{aligned} \tag{4.45}$$

where,

$$C_{i,s}(x) := \sum_{j=1}^{n}\left|f^{(s)}(x_{j,n})\right|\left|l_{jn}(x)\right|^{\nu}|x-x_{j,n}|^i\left(\frac{n}{\sqrt{a_{2n}^2-x_{j,n}^2}}\right)^{i}. \tag{$*$}$$

We set

$$X_{i,s,n}(\nu,f;x) := \sum_{j=1}^{n} f^{(s)}(x_{j,n})l_{jn}^{\nu}(x)e_i(\nu,j,n)(x-x_{j,n})^i, \quad i=0,1,\ldots,\nu-1, \tag{4.46}$$

and hereafter we wrote $(*)$ as $|X|_{i,s,n}(\nu,f;x)$. Now, we repeat the proof of Proposition 3.7 by exchanging $f(x)$ with $f^{(s)}(x)$, $s = 1,2,\ldots,\nu-1$, and we note (3.23) (and (3.26)). Then, for $0 \leqslant i \leqslant \nu-2$ we obtain

$$\left\{\Phi_n^{3/4}(x)w(x)\left(|x|+\frac{a_n}{n}\right)\right\}^{\nu}|X|_{i,s,n}(\nu,f;x) \leqslant C_1 C(f). \tag{4.47}$$

For $i = \nu - 1$, we use the estimate for $l_{mn}(x)$ in the proof of Proposition 3.7; furthermore we use Remark 4.4. Then we have

$$\left\{\Phi_n^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)\right\}^{\nu} |X|_{\nu-1,s,n}(\nu, f; x) \leqslant C_1 C(f) \log n. \tag{4.48}$$

Consequently, we have

$$\left\{\Phi_n(x)^{3/4}w(x)\left(|x| + \frac{a_n}{n}\right)\right\}^{\nu} \left|Z_n(l, \nu, f; x)\right| \leqslant C_1 C(f) \frac{a_n \log n}{n}. \tag{4.49}$$

Similarly, we have

$$\left\{\Phi(x)^{3/4}w(x)\left(|x| + \frac{a_n}{n}\right)\right\}^{\nu} |W_n(l, \nu, G; x)| \leqslant C_1 C(G) \frac{a_n \log n}{n}. \tag{4.50}$$

□

Proof of Proposition 3.10. Let $P \in \mathcal{P}_m$ be fixed. From Proposition 3.9 we see

$$\begin{aligned}
&\left\|\left(\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right)^{\nu} \{L_n(\nu, P) - P\}\right\|_{L_\infty(\mathbb{R})} \\
&\quad = \left\|\left(\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right)^{\nu} \{L_n(\nu, P) - L_n(\nu - 1, \nu, P)\}\right\|_{L_\infty(\mathbb{R})} \\
&\quad = \left\|\left(\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right)^{\nu} Z_n(\nu - 1, \nu, P)\right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.51}$$

□

Proof of Proposition 3.11. Let $P \in \mathcal{P}_m$ with $P(0) = 0$. Then P satisfies the condition (A-1). By Proposition 3.8, we see

$$\left\|\left(\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right)^{\nu} Y_n(\nu, P)\right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.52}$$

So, from Propositions 3.10 and 3.8 (noting (3.3)) we have

$$\begin{aligned}
&\left\|\left(\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right)^{\nu} (X_n(\nu, P) - P)\right\|_{L_\infty(\mathbb{R})} \\
&\quad \leqslant C_1 \left\|\left(\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right)^{\nu} (L_n(\nu, P) - P)\right\|_{L_\infty(\mathbb{R})} \\
&\qquad + \left\|\left(\Phi^{3/4}(x)w(x)\left(|x| + \frac{a_n}{n}\right)^{\rho}\right)^{\nu} Y_n(\nu, P)\right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.53}$$

□

Proof of Theorem 3.12. Since f satisfies (3.22), we see

$$\lim_{|x|\to\infty} |f(x)| w(x)^{\nu-\delta} = 0. \tag{4.54}$$

For a given $\varepsilon > 0$, there exists a polynomial $P \in \mathcal{P}_m$ with $P(0) = 0$ such that

$$\sup_{x\in\mathbb{R}} |f(x) - P(x)| \left\{ \Phi^{-3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} < \varepsilon. \tag{4.55}$$

In fact, by [21, Theorem 1.4], there exists a polynomial $R \in \mathcal{P}_m$ such that

$$\sup_{x\in\mathbb{R}} |f(x) - R(x)| w^{\nu-\delta}(x) < \frac{\varepsilon}{2}. \tag{4.56}$$

Let $P(x) := R(x) - R(0)$. Noting that

$$\left\{ \Phi^{-3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} \leqslant C w^{\nu-\delta}(x), \quad x \in \mathbb{R} \text{ for some } C > 0 \tag{4.57}$$

and $f(0) = 0$, we have

$$\begin{aligned} &\sup_{x\in\mathbb{R}} |f(x) - P(x)| \left\{ \Phi^{-3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right\}^{\nu} \\ &\quad \leqslant \sup_{x\in\mathbb{R}} |f(x) - P(x)| w^{\nu-\delta}(x) \\ &\quad \leqslant \sup_{x\in\mathbb{R}} |f(x) - R(x)| w^{\nu-\delta}(x) + |f(0) - R(0)| w^{\nu-\delta}(0) < \varepsilon; \end{aligned} \tag{4.58}$$

that is, we have (4.55). Here, we know that

$$L_n(\nu, f)(x) - f(x) = X_n(\nu, f - P) + (X_n(\nu, P) - P) + (P - f) + Y_n(\nu, f). \tag{4.59}$$

Therefore, by Propositions 3.7, 3.11, and 3.8, we have for n large enough

$$\begin{aligned}
&\left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} (L_n(\nu, f) - f) \right\|_{L_\infty(\mathbb{R})} \\
&\quad \leqslant C \left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} X_n(\nu, f - P) \right\|_{L_\infty(\mathbb{R})} \\
&\qquad + \left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} (X_n(\nu, P) - P) \right\|_{L_\infty(\mathbb{R})} \\
&\qquad + \left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} (P - f) \right\|_{L_\infty(\mathbb{R})} \\
&\qquad + \left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} Y_n(\nu, f) \right\|_{L_\infty(\mathbb{R})} \\
&\quad =: I_n + II_n + III_n + IV_n.
\end{aligned} \tag{4.60}$$

Here we see that, by Proposition 3.7 with $\varepsilon := C(f - P)$ (constant depending only on $f - P$) and (4.55),

$$I_n \leqslant C\varepsilon, \qquad III_n \leqslant \varepsilon, \tag{4.61}$$

and for $n \geqslant n_0$ large enough, we have

$$\begin{aligned}
II_n &\leqslant C\varepsilon, \quad (\text{by Proposition 3.9}), \\
IV_n &\leqslant C\varepsilon, \quad (\text{by Proposition 3.6}).
\end{aligned} \tag{4.62}$$

Consequently, noting (3.3), we have

$$\lim_{n \to \infty} \{ I_n + II_n + III_n + IV_n \} = 0. \tag{4.63}$$

Then we have Theorem 3.12. □

Proof of Theorem 3.13. From Theorem 3.12 and Proposition 3.9, we have

$$\begin{aligned}
&\left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} (L_n(l, \nu, f) - f) \right\|_{L_\infty(\mathbb{R})} \\
&\quad \leqslant C_1 \left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} (L_n(\nu, f) - f) \right\|_{L_\infty(\mathbb{R})} \\
&\qquad + \left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} Z_n(l, \nu, f) \right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.64}$$

□

Proof of Corollary 3.14. From Theorem 3.12 and Proposition 3.8, we have

$$\begin{aligned}
&\left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} (X_n(\nu, f) - f) \right\|_{L_\infty(\mathbb{R})} \\
&\quad \leqslant C_1 \left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} (L_n(\nu, f) - f) \right\|_{L_\infty(\mathbb{R})} \\
&\qquad + \left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} Y_n(\nu, f) \right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.65}$$

□

Proof of Theorem 3.15. From Theorem 3.13 and Proposition 3.10 we have

$$\begin{aligned}
&\left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} \left(\widetilde{L}_n(l, \nu, f \oplus G) - f \right) \right\|_{L_\infty(\mathbb{R})} \\
&\quad \leqslant C \left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} (L_n(l, \nu, f) - f) \right\|_{L_\infty(\mathbb{R})} \\
&\qquad + \left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} W_n(l, \nu, G) \right\|_{L_\infty(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.66}$$

□

Proof of Theorem 3.16. We use Theorems 3.12, 3.13, and Corollary 3.14, and Theorem 3.15. Then we have

$$\begin{aligned}
&\left| I_n[k, H_n(f); f] - I[k, f] \right| \\
&\quad = \int_{-\infty}^{\infty} k(x) \{ H_n(f)(x) - f(x) \} dx \\
&\quad \leqslant C \left\| \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} \{ H_n(f)(x) - f(x) \} \right\|_{L_\infty(\mathbb{R})} \\
&\qquad \times \int_{-\infty}^{\infty} \left(\Phi^{3/4}(x) w(x) \left(|x| + \frac{a_n}{n} \right)^{\rho} \right)^{\nu} k(x) dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.67}$$

□

5. Divergence Theorem

If ν is a positive odd integer, then we obtain the unboundedness of $L_n(\nu, f; x)$. We define

$$\Lambda_n(\nu, \mathbb{R}) = \max_{x \in \mathbb{R}} \sum_{k=1}^{n} |h_{kn}(x)|. \tag{5.1}$$

Theorem 5.1 (cf. [1, Theorem 2]). *Let $\nu > 0$ be an odd integer. Then there exists a constant $C > 0$ and $n_0 > 0$ such that for $n \geqslant n_0$*

$$\Lambda_n(\nu, \mathbb{R}) \geqslant C \log n. \tag{5.2}$$

Let $-\infty \leqslant a < b \leqslant \infty$, and let us define

$$\Lambda_n(\nu,(a,b)) = \max_{a\leqslant x\leqslant b} \sum_{k=1}^{n} |h_{kn}(x)|. \tag{5.3}$$

Then we will show that for $n \geqslant n_0$

$$\Lambda_n(\nu,(a,b)) \geqslant C \log n. \tag{5.4}$$

Remark 5.2. For the interpolation polynomial $L_n(\nu, f; x)$, we can see a remarkable difference between the cases of an odd number ν and of an even number ν. Let us consider any continuous function $f \in C([a,b]), 0 < a < b$. Then we can extend f to a continuous function $f^* \in C(\mathbb{R})$ which satisfies (3.22) and $f(x) = f^*(x)$, $x \in [a,b]$. Then from Theorem 3.12 for an even positive integer ν, we see

$$\left\| L_n(\nu, f^*; x) - f(x) \right\|_{L_\infty([a,b])} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.5}$$

On the other hand, the standard argument (cf, [22, Theorem 4.3]) leads us to the following. Theorem 5.1 means that there exists a certain function $f^* \in C(\mathbb{R})$ such that

$$\left\| L_n(\nu, f^*; x) - f^*(x) \right\|_{L_\infty([a,b])} \nrightarrow 0 \quad \text{as } n \longrightarrow \infty; \tag{5.6}$$

that is, for $f(x) := f^*(x)$, $a \leqslant x \leqslant b$ we see that the interpolation polynomials $L_n(\nu, f^*; x)$ do not converge to f. We also remark that the polynomial $L_n(\nu, f^*; x)$ interpolates $f \in C[a,b]$ at only $\{x_{j,n} \in [a,b], 1 \leqslant j \leqslant n\}$.

To prove the theorem we use the following lemma.

Lemma 5.3 (see [18, Theorem 11]). *For $j = 0,1,2,\ldots$, there is a polynomial $\Psi_j(x)$ of degree j such that $(-1)^j \Psi_j(-\nu) > 0$ for $\nu = 1,3,5,\ldots$, and the following relation holds. Let $0 < \varepsilon < 1$. Then one has an expression for $(1/\varepsilon)(a_n/n) \leqslant |x_{k,n}| \leqslant \varepsilon a_n$ and $s = 0,1,\ldots,(\nu-1)/2$:*

$$e_{2s}(\nu,k,n) = (-1)^s \left\{ \frac{1}{(2s)!} \right\} \Psi_s(-\nu) \beta^s(n,k) \left(\frac{n}{a_n} \right)^{2s} \left\{ 1 + \eta_{kn}(\nu,s) \right\}, \tag{5.7}$$

where for $\beta^s(n,k)$, $n = 1,2,3,\ldots$, $k = 1,2,\ldots,n$, $s = 0,1,\ldots,(\nu-1)/2$, there exist the constants C_1, C_2 such that

$$0 < C_1 \leqslant \beta^s(n,k) \leqslant C_2, \tag{5.8}$$

and $\eta_{kn}(\nu, s)$ *satisfies*

$$\left|\eta_{kn}(\nu, s)\right| \leqslant \varepsilon_n^*, \quad \varepsilon_n^* = \varepsilon_n^*(\varepsilon) \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0, \tag{5.9}$$

for k *with* $(1/\varepsilon)(a_n/n) \leqslant |x_{k,n}| \leqslant \varepsilon a_n$, *and for* $s = 0, 1, 2, \ldots$.

Proof of Theorem 5.1. To get a lower bound of $\Lambda_n(\nu, \mathbb{R})$, it suffices to consider a lower bound (5.4) of $\Lambda_n(\nu, [a, b])$ with $0 < a < b < \infty$. Let $c := a/2, d := 2b - a$ and we consider the intervals $I := [a, b], J = [c, d]$. If we consider n large enough, then we have $(1/\varepsilon)(a_n/n) \leqslant c < 2b - a \leqslant \varepsilon a_n$. See the expression

$$|h_{kn}(x)| \geqslant \left|l_{kn}^{\nu}(x) e_{\nu-1}(\nu, k, n)(x - x_{k,n})^{\nu-1}\right| - \left|l_{kn}^{\nu}(x) \sum_{i=0}^{\nu-2} e_i(\nu, k, n)(x - x_{k,n})^i\right|. \tag{5.10}$$

Then we have

$$\begin{aligned}
\Lambda_n(\nu, [a, b]) &\geqslant \max_{x \in I} \sum_{x_{kn} \in J} \left|l_{kn}^{\nu}(x) e_{\nu-1}(\nu, k, n)(x - x_{k,n})^{\nu-1}\right| \\
&\quad - \max_{x \in I} \sum_{x_{kn} \in J} \left|l_{kn}^{\nu}(x) \sum_{i=0}^{\nu-2} e_i(\nu, k, n)(x - x_{k,n})^i\right| \\
&=: \max_{x \in I} F_n(x) - \max_{x \in I} G_n(x).
\end{aligned} \tag{5.11}$$

It follows from Remark 4.5 that $\max_{a \leqslant x \leqslant b} G_n(x) \leqslant C$. Hence, it is enough to show that $\max_{x \in I} F_n(x) \geqslant C \log n$. Let $x_{k+1,n}, x_{k,n} \in J$; then by Lemma 4.2 and the definition of $\varphi_n(x_{k,n})$ there exists $0 < \alpha \leqslant \beta < \infty$ such that

$$\alpha \frac{a_n}{n} \leqslant |x_{k,n} - x_{k+1,n}| \leqslant \beta \frac{a_n}{n}. \tag{5.12}$$

For $x \in I$ we consider only $x_{k,n} \in J$ such that, for some positive integer j_k,

$$(j_k - 1)\alpha \frac{a_n}{n} \leqslant |x_{k,n} - x| \leqslant j_k \beta \frac{a_n}{n}. \tag{5.13}$$

Then for each $x \in I$ we define

$$\Gamma(x) := \left\{ j_k; x_{k,n} \in J, (j_k - 1)\alpha \frac{a_n}{n} \leqslant |x_{k,n} - x| \leqslant j_k \beta \frac{a_n}{n} \right\}. \tag{5.14}$$

Here, we will see that

$$\left\{ j; 1 \leqslant j \leqslant \frac{d-c}{2\beta}\frac{n}{a_n} - 1 \right\} \subset \Gamma(x). \tag{5.15}$$

Let

$$\begin{aligned} &x_{m+1,n} < x \leqslant x_{m,n}, \\ &x_{k(c)+1,n} < c \leqslant x_{k(c),n}, x_{k(d),n} \leqslant d < x_{k(d)-1,n}. \end{aligned} \tag{5.16}$$

Then we have

$$\begin{gathered} x - x_{m+1,n} \leqslant x_{m,n} - x_{m+1,n} \leqslant \beta\frac{a_n}{n}, \\ x - x_{m+2,n} \leqslant x_{m,n} - x_{m+2,n} \leqslant 2\beta\frac{a_n}{n}, \\ \vdots \\ x - x_{k(c),n} \leqslant x_{m,n} - x_{k(c),n} \leqslant j(c)\beta\frac{a_n}{n}, \\ x_{m,n} - x \leqslant x_{m,n} - x_{m+1,n} \leqslant \beta\frac{a_n}{n}, \\ x_{m-1,n} - x \leqslant x_{m-1,n} - x_{m+1,n} \leqslant 2\beta\frac{a_n}{n}, \\ \vdots \\ x_{k(d),n} - x \leqslant x_{k(d),n} - x_{m+1,n} \leqslant j(d)\beta\frac{a_n}{n}, \end{gathered} \tag{5.17}$$

where $j(c)$ and $j(d)$ are integers. On the other hand,

$$\frac{\left(x_{k(d),n} - x_{k(c),n}\right)}{\left(\beta a_n/n\right)} \geqslant \frac{\left(d - c - 2\beta a_n/n\right)}{\left(\beta a_n/n\right)} = \frac{d-c}{\beta}\frac{n}{a_n} - 2. \tag{5.18}$$

Therefore, we have

$$\max\left(j(c), j(d)\right) \geqslant \frac{1}{2}\left(\frac{d-c}{\beta}\frac{n}{a_n} - 2\right) = \frac{d-c}{2\beta}\frac{n}{a_n} - 1. \tag{5.19}$$

Now, we take a positive integer $N(n)$ such that

$$N(n) \leqslant \frac{d-c}{2\beta}\frac{n}{a_n} - 1 < N(n) + 1. \tag{5.20}$$

Consequently, we have the following. By Lemmas 4.2(1), Lemma 4.1(1), (2), (4), and Lemma 2.3, we have

$$\begin{aligned}
F_n(x) &= \sum_{x_{k,n}\in J} |l_{kn}(x)|^{\nu} |e_{\nu-1}(\nu,k,n)||x_{k,n}-x|^{\nu-1} \\
&= \sum_{x_{k,n}\in J} \left(\frac{|p_n(x)|w_\rho(x)}{|x-x_{k,n}||p_n'(x_{k,n})|w_\rho(x_{k,n})} \frac{w_\rho(x_{k,n})}{w_\rho(x)} \right)^{\nu} |e_{\nu-1}(\nu,k,n)||x-x_{k,n}|^{\nu-1} \\
&= \sum_{x_{k,n}\in J} \left(\frac{|p_n(x)|w_\rho(x)}{|p_n'(x_{k,n})|w_\rho(x_{k,n})} \frac{w_\rho(x_{k,n})}{w_\rho(x)} \right)^{\nu} |e_{\nu-1}(\nu,k,n)| \frac{1}{|x-x_{k,n}|} \\
&\geqslant C \sum_{j_k\in\Gamma(x)} \left(\frac{a_n^{-1/2}}{(n/a_n)a_n^{-1/2}} \frac{w_\rho(d)}{w_\rho(a)} \right)^{\nu} |e_{\nu-1}(\nu,k,n)| \frac{n}{j_k\beta a_n} \\
&\geqslant C \sum_{j_k\in\Gamma(x)} \frac{1}{j_k} \left(\frac{a_n}{n} \right)^{\nu-1} |e_{\nu-1}(\nu,k,n)|.
\end{aligned} \tag{5.21}$$

Here, using Lemma 5.3 and (5.15), we have

$$F_n(x) \geqslant C \sum_{j_k\in\Gamma(x)} \frac{1}{j_k} \left(\frac{a_n}{n} \right)^{\nu-1} \left(\frac{n}{a_n} \right)^{\nu-1} \geqslant C \sum_{1\leqslant j\leqslant N(n)} \frac{1}{j}. \tag{5.22}$$

Here, there exists $0 < \eta < 1$ such that $n^{\eta} \leqslant N(n)$. Therefore we see

$$F_n(x) \geqslant C \sum_{1\leqslant j\leqslant n^{\eta}} \frac{1}{j} \geqslant C \log n. \tag{5.23}$$

So we have (5.4), and consequently the proof of Theorem 5.1 is completed. □

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