Research Article

The Existence of Positive Solutions for a Nonlinear Sixth-Order Boundary Value Problem

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By using the Krein-Rutman theorem and bifurcation methods, we discuss the existence of positive solutions for the boundary value problems of a sixth-order ordinary differential equation.

1. Introduction

In recent years, the following boundary value problems for sixth-order ordinary differential equations have been studied extensively (see, e.g., [1–7] and the references therein):

$$\begin{aligned} u^{(6)} + Au^{(4)} + Bu'' + Cu - f(t, u) &= 0, \quad 0 < x < L, \\ u(0) = u(L) = u''(0) = u''(L) = u^{(4)}(0) &= u^{(4)}(L) = 0, \end{aligned} \quad (1.1)$$

where A , B , and C are some given real constants and $f(x, u)$ is a continuous function on R^2 . The boundary value problems were motivated by the study for stationary solutions of the sixth-order parabolic differential equations:

$$\frac{\partial u}{\partial t} = \frac{\partial^6 u}{\partial x^6} + A \frac{\partial^4 u}{\partial x^4} + B \frac{\partial^2 u}{\partial x^2} + f(x, u). \quad (1.2)$$

This equation arose in the formation of the spatial periodic patterns in bistable systems and is also a model for describing the behavior of phase fronts in materials that are undergoing a transition between the liquid and solid state. When $f(x, u) = u - u^3$, it was studied by [4, 5].

In [2], the existence and multiplicity results of nontrivial solutions for (1.1) were proved using a minimization theorem and Clarks theorem [6], respectively, when $C = 1$ and $f(x, u) = u^3$. The authors studied also the homoclinic solutions for (1.1) when $C = -1$ and $f(x, u) = -a(x)u|u|^\sigma$, where $a(x)$ is a positive periodic function and σ is a positive constant, by the mountain-pass theorem and concentration-compactness arguments. In [3], by variational tools, including Brezis-Nirenbergs linking theorems, Gyulov et al. studied also the existence and multiplicity of nontrivial solutions of BVP (1.1). In [7], using the fixed point index theory of cone mapping, the authors gave some results for existence and multiplicity of positive solutions of BVP (1.1).

On the other hand, in [8, 9], by the Krein-Rutman theorem and the global bifurcation techniques, Ma et al. were concerned with the existence of positive solutions of the following fourth-order boundary value problem:

$$\begin{aligned} u^{(4)}(t) &= f(t, u(t), u''(t)), \quad t \in (0, 1), \\ u(0) &= u(1) = u''(0) = u''(1) = 0, \end{aligned} \quad (1.3)$$

where $f : [0, 1] \times [0, +\infty) \times (-\infty, 0] \rightarrow [0, +\infty)$ is continuous and satisfies some conditions.

Inspired by the works of the above papers, in this paper, we consider the following nonlinear sixth-order boundary value problem:

$$\begin{aligned} -u^{(6)}(t) &= f(t, u(t), u''(t), u^{(4)}(t)), \quad t \in (0, 1), \\ u(0) &= u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0, \end{aligned} \quad (1.4)$$

under the following assumptions on the nonlinear term.

(A₁) $f : [0, 1] \times [0, \infty) \times (-\infty, 0] \times [0, \infty) \rightarrow [0, \infty)$ is continuous and there exist functions $a, b, c, d, m, n \in C([0, 1], [0, \infty))$ with $a(t) + b(t) + c(t) > 0$ and $d(t) + m(t) + n(t) > 0$ on $[0, 1]$ such that

$$f(t, u, p, q) = a(t)u - b(t)p + c(t)q + o(|u, p, q|), \quad \text{as } |u, p, q| \rightarrow 0, \quad (1.5)$$

uniformly for $t \in [0, 1]$, and

$$f(t, u, p, q) = d(t)u - m(t)p + n(t)q + o(|u, p, q|), \quad \text{as } |(u, p, q)| \rightarrow \infty, \quad (1.6)$$

uniformly for $t \in [0, 1]$. Here $|(u, p, q)| := \sqrt{u^2 + p^2 + q^2}$.

(A₂) $f(t, u, p, q) > 0$ for $t \in [0, 1]$ and $(u, p, q) \in ([0, \infty) \times (-\infty, 0] \times [0, \infty)) \setminus \{(0, 0, 0)\}$.

(A₃) There exist constants $a_0, b_0, c_0 \in [0, \infty)$ satisfying $a_0^2 + b_0^2 + c_0^2 > 0$ and $f(t, u, p, q) \geq a_0 u - b_0 p + c_0 q + o(|(u, p, q)|)$ for $(t, u, p, q) \in [0, 1] \times [0, \infty) \times (-\infty, 0] \times [0, \infty)$.

The existence of positive solution for (1.4) is proved using Krein-Rutman theorem [10] and the Global Bifurcation Theory [11]. The idea of this work comes from [8, 9].

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, we discuss the existence of positive solution of the problem (1.4).

2. Preliminaries

In this section, we will make some preliminaries which are needed to show our main results. Let us assume that

(A₄) $A, B, C \in C([0, 1], [0, \infty))$ with $A(t) > 0$ or $B(t) > 0$ or $C(t) > 0$ with $t \in [0, 1]$.

Definition 2.1. We say μ is a generalized eigenvalue of the linear problem

$$\begin{aligned} -u^{(6)} &= \mu \left(A(t)u - B(t)u'' + C(t)u^{(4)} \right), \quad t \in (0, 1), \\ u(0) &= u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0, \end{aligned} \quad (2.1)$$

if (2.1) has nontrivial solutions.

Let $e(t) := \sin \pi t, t \in [0, 1]$ and $Y = C[0, 1]$ be the Banach space equipped with the norm $\|u\|_\infty = \max_{0 \leq t \leq 1} |u(t)|$. X is defined as

$$\begin{aligned} X &= \left\{ u \in C^4[0, 1] \mid u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0, \right. \\ &\quad \left. \exists \varepsilon \in [0, 1], \text{ s.t. } -\varepsilon e(t) \leq u^{(4)}(t) \leq \varepsilon e(t), t \in [0, 1] \right\}. \end{aligned} \quad (2.2)$$

For any $u \in X$, we have

$$u(t) = \int_0^1 \int_0^1 H(t, \tau) H(\tau, s) u^{(4)}(s) ds d\tau, \quad (2.3)$$

where

$$H(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \quad (2.4)$$

Based on (2.3) and $(1/\pi^4)e(t) = \int_0^1 \int_0^1 H(t, \tau) H(\tau, s) e(s) ds d\tau$, we come to

$$-\frac{\varepsilon}{\pi^4} e(t) \leq u(t) \leq \frac{\varepsilon}{\pi^4} e(t), \quad t \in [0, 1]. \quad (2.5)$$

Since $-\varepsilon/\pi^4 < \varepsilon$, the norm of $u \in X$ can be defined as

$$\|u\|_X := \inf \left\{ \varepsilon \mid -\varepsilon e(t) \leq u^{(4)}(t) \leq \varepsilon e(t), t \in [0, 1] \right\}. \quad (2.6)$$

It is not difficult to verify that $(X, \|\cdot\|_X)$ is a Banach space. Let

$$P := \left\{ u \in X \mid u^{(4)}(t) \geq 0, u''(t) \leq 0, u(t) \geq 0, t \in [0, 1] \right\}. \quad (2.7)$$

Then the cone P is normal and has a nonempty interior $\overset{\circ}{P}$.

Lemma 2.2. For $u \in X$, then $\|u\|_\infty \leq \|u'\|_\infty \leq \|u''\|_\infty \leq \|u'''\|_\infty \leq \|u^{(4)}\|_\infty \leq \|u\|_X$.

Proof. (1) By $u(0) = u(1)$, there is a $\xi \in (0, 1)$ such that $u'(\xi) = 0$, and so $-u'(t) = \int_t^\xi u''(s)ds$, $t \in [0, \xi]$. Hence $|u'(t)| \leq \int_t^\xi |u''(s)|ds \leq \int_0^1 |u''(s)|ds \leq \|u''\|_\infty$. By $u''(0) = u''(1)$, there is a $\eta \in (0, 1)$, which makes $u'''(\eta) = 0$, thereby we come to $-u'''(t) = \int_t^\eta u^{(4)}(s)ds$, $t \in [0, \eta]$. And we have $|u'''(t)| \leq \int_t^\eta |u^{(4)}(s)|ds \leq \int_0^1 |u^{(4)}(s)|ds \leq \|u^{(4)}\|_\infty$.

(2) Because of $u(0) = 0$, we come to $|u(t)| \leq \int_0^t |u'(s)|ds \leq \|u'\|_\infty$. Correspondingly, $\|u\|_\infty \leq \|u'\|_\infty$ and we can obtain $\|u''\|_\infty \leq \|u'''\|_\infty$ for the same reason.

(3) By the definition of X , we know $|u^{(4)}(t)| \leq \varepsilon e(t) \leq \varepsilon$, that is, $|u^{(4)}|_\infty \leq \varepsilon$. Correspondingly, we come to $|u^{(4)}|_\infty \leq \|u\|_X$.

Based on the combination of (1), (2) and (3), the conclusion can be reached and the lemma is thus proved. $\square$

For any $u \in X$, define a linear operator $T : X \rightarrow Y$ by

$$Tu(t) := \int_0^1 \int_0^1 \int_0^1 H(t, x)H(x, \tau)H(\tau, s) \left[A(s)u(s) - B(s)u''(s) + C(s)u^{(4)}(s) \right] ds d\tau dx. \quad (2.8)$$

Theorem 2.3. Assume that (A_4) holds, let $r(t)$ be the spectral radius of T , then Problem (2.1) has an algebraically simple eigenvalue, $\mu_1(A, B, C) = (r(t))^{-1}$, with a positive eigenfunction $\varphi(\cdot) \in \overset{\circ}{P}$. Moreover, there is no other eigenvalue with a positive eigenfunction.

Proof. It is easy to check that Problem (2.1) is equivalent to the integral equation $u(t) = \mu Tu(t)$.

We define $T : X \rightarrow X$.

In fact, for $u \in X$, we have

$$(Tu)^{(4)}(t) = \int_0^1 H(t, s) \left[A(s)u(s) - B(s)u''(s) + C(s)u^{(4)}(s) \right] ds. \quad (2.9)$$

Combining this with the fact $[A(s)u(s) - B(s)u''(s) + C(s)u^{(4)}(s)] \in Y$, it can be concluded that

$$-c_0\omega(t) \leq (Tu)^{(4)}(t) \leq c_0\omega(t), \quad t \in [0, 1], \quad (2.10)$$

where

$$\begin{aligned} c_0 &:= \|A\|_\infty \|u_\infty\| + \|B\|_\infty \|u''_\infty\| + \|C\|_\infty \|u^{(4)}_\infty\|, \\ \omega(t) &= \int_0^1 \int_0^1 H(t, \tau) H(\tau, s) ds d\tau = \frac{t}{24} (t-1) (t^2 - t - 1). \end{aligned} \quad (2.11)$$

Thus, there exist corresponding constants ρ_1, ρ_2 , which make

$$\rho_1 e(t) \leq \omega(t) \leq \rho_2 e(t), \quad t \in [0, 1]. \quad (2.12)$$

Consequently, we obtain $Tu \in X$, thus $T(X) \subseteq X$. The assertion is proved.

If $u \in P$, then $[A(s)u(s) - B(s)u''(s) + C(s)u^{(4)}(s)] \geq 0$, $s \in [0, 1]$, and correspondingly,

$$\begin{aligned} (Tu)^{(4)}(t) &= \int_0^1 H(t, s) [A(s)u(s) - B(s)u''(s) + C(s)u^{(4)}(s)] ds \geq 0, \\ (Tu)''(t) &= \int_0^1 H(t, \tau) (-Tu)^{(4)}(\tau) d\tau \leq 0, \\ (Tu)(t) &= \int_0^1 \int_0^1 H(t, x) H(x, \tau) (Tu)^{(4)}(\tau) d\tau dx \geq 0. \end{aligned} \quad (2.13)$$

So, $Tu \in P$, and correspondingly $T(P) \subseteq P$.

Because $T(X) \subset C^6[0, 1] \cap X$, and $C^6[0, 1] \cap X$ is compactly embedded in X , thus we obtain that $T : X \rightarrow X$ is completely continuous.

Next, we will prove that $T : X \rightarrow X$ is strongly positive.

(1) For any $u \in P \setminus \{0\}$, if $A(t) > 0$ on $[0, 1]$, then there exists a constant $k > 0$ such that

$$Tu(t) \geq k \int_0^1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) H(\tau, s) u(s) ds d\tau dx := T_1 u(t). \quad (2.14)$$

It is easy to verify that there exists $r_1 > 0$, such that $T_1 u(t) \geq r_1 e(t)$ on $[0, 1]$. Thus $Tu(t) \geq r_1 e(t)$.

For any $u \in P \setminus \{0\}$, if $B(t) > 0$ on $[0, 1]$, then there exists a constant $k_1 > 0$ such that

$$\begin{aligned} Tu(t) &\geq \int_0^1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) H(\tau, s) (-B(s)u''(s)) ds d\tau dx \\ &\geq -k_1 \int_0^1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) H(\tau, s) u''(s) ds d\tau dx \\ &= k_1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) u(s) d\tau dx := T_2 u(t). \end{aligned} \quad (2.15)$$

Then there exists $r_2 > 0$, which makes $T_2 u(t) \geq r_2 e(t)$ on $[0, 1]$. Thus, $Tu(t) \geq r_2 e(t)$.

For any $u \in P \setminus \{0\}$, if $C(t) > 0$ on $[0, 1]$, similarly, we can verify that there exists $r_3 > 0$, which makes $Tu(t) \geq r_3 e(t)$ on $[0, 1]$.

Hence we obtain $Tu(t) \geq r e(t)$, for all $t \in [0, 1]$, in which $r = \min\{r_1, r_2, r_3\}$.

(2) Thanks to the definition of T , we come to

$$(Tu)^{(4)}(t) = \int_0^1 H(t, s) [A(s)u(s) - B(s)u''(s) + C(s)u^{(4)}(s)] ds. \quad (2.16)$$

For any $u \in P \setminus \{0\}$, if $A(t) > 0$ on $[0, 1]$, there exist $k > 0$ and $r_4 > 0$ such that

$$(Tu)^{(4)}(t) \geq \int_0^1 H(t, s) A(s) u(s) ds \geq k \int_0^1 H(t, s) u(s) ds \geq r_4 e(t). \quad (2.17)$$

For any $u \in P \setminus \{0\}$, if $B(t) > 0$ on $[0, 1]$, there exist $k_1 > 0$ and $r_5 > 0$ such that

$$\begin{aligned} (Tu)^{(4)}(t) &\geq \int_0^1 H(t, s) [-B(s)u''(s)] ds \\ &\geq k_1 \int_0^1 H(t, s) (-u''(s)) ds \\ &= k_1 u(s) \\ &\geq r_5 e(t). \end{aligned} \quad (2.18)$$

To $u \in P \setminus \{0\}$, if $C(t) > 0$ on $[0, 1]$, there exist $k_2 > 0$ and $r_6 > 0$ such that

$$\begin{aligned} (Tu)^{(4)}(t) &\geq \int_0^1 H(t, s) C(s) u^{(4)}(s) ds \\ &\geq k_2 \int_0^1 H(t, s) u^{(4)}(s) ds \\ &= -k_2 u''(s) \\ &\geq r_6 e(t). \end{aligned} \quad (2.19)$$

Hence we obtain $(Tu)^{(4)}(t) \geq r' e(t)$, for all $t \in [0, 1]$, where $r' = \min\{r_4, r_5, r_6\}$.

(3) It is easy to come to

$$(Tu)''(t) = \int_0^1 \int_0^1 H(t, \tau) H(\tau, s) [A(s)u(s) - B(s)u''(s) + C(s)u^{(4)}(s)] ds d\tau, \quad (2.20)$$

for all $u \in P \setminus \{0\}$, if $A(t) > 0$ on $[0, 1]$, there exist constants $k > 0$ and $r_7 > 0$ such that

$$\begin{aligned} -(Tu)''(t) &\geq \int_0^1 \int_0^1 H(t, \tau) H(\tau, s) A(s) u(s) ds d\tau \\ &\geq k \int_0^1 \int_0^1 H(t, \tau) H(\tau, s) u(s) ds d\tau \\ &\geq r_7 e(t), \end{aligned} \quad (2.21)$$

for all $u \in P \setminus \{0\}$, if $B(t) > 0$ on $[0, 1]$, there exist constants $k_1 > 0$ and $r_8 > 0$ such that

$$\begin{aligned} -(Tu)''(t) &\geq \int_0^1 \int_0^1 H(t, \tau) H(\tau, s) [-B(s) u''(s)] ds d\tau \\ &\geq k_1 \int_0^1 \int_0^1 H(t, \tau) H(\tau, s) (-u''(s)) ds d\tau \\ &= k_1 \int_0^1 H(t, \tau) u(\tau) d\tau \\ &\geq r_8 e(t), \end{aligned} \quad (2.22)$$

for all $u \in P \setminus \{0\}$, if $C(t) > 0$ on $[0, 1]$, there exist constants $k_2 > 0$ and $r_9 > 0$ such that

$$\begin{aligned} -(Tu)''(t) &\geq \int_0^1 \int_0^1 H(t, \tau) H(\tau, s) C(s) u^{(4)}(s) ds d\tau \\ &\geq k_2 \int_0^1 \int_0^1 H(t, \tau) H(\tau, s) u^{(4)}(s) ds d\tau \\ &= k_2 u(t) \\ &\geq r_9 e(t). \end{aligned} \quad (2.23)$$

Hence we obtain $-(Tu)''(t) \geq r'' e(t)$, for all $t \in [0, 1]$, where $r'' = \min\{r_7, r_8, r_9\}$.

By (1), (2), and (3), we have $Tu \in \mathring{P}$.

According to Krein-Rutman theorem, we know that T has a single algebraic eigenvalue $r(T) > 0$ which corresponds to the eigenvector $\varphi(\cdot) \in \mathring{P}$. Furthermore, there is no other eigenvalues with corresponding positive eigenfunctions. Correspondingly, $\mu_1(A, B, C) = (r(t))^{-1}$ is an algebraic single eigenvalue of Problem (2.1) with a corresponding positive eigenvector $\varphi(\cdot) \in \mathring{P}$, and there is no any other eigenvalues which have corresponding positive feature vector. The theorem is thus proved. $\square$

3. Main Results

The main result of this paper is as follows.

Theorem 3.1. *Let (A_1) , (A_2) , and (A_3) hold. Assume that either*

$$\mu_1(d, m, n) < 1 < \mu_1(a, b, c) \quad \text{or} \quad \mu_1(a, b, c) < 1 < \mu_1(d, m, n). \quad (3.1)$$

Then (1.4) has at least one positive solution.

Proof. Define that $L : D(L) \rightarrow Y$ by $L(u) := -u^{(6)}$, $u \in D(L)$, where

$$D(L) = \left\{ u \in C^6[0, 1] \mid u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0 \right\}. \quad (3.2)$$

It is easy to verify that $L^{-1} : Y \rightarrow X$ is compact.

Let $\zeta, \xi \in C([0, 1] \times [0, \infty) \times (-\infty, 0] \times [0, \infty))$, and satisfy

$$\begin{aligned} f(t, u, p, q) &= a(t)u - b(t)p + c(t)q + \zeta(t, u, p, q), \\ f(t, u, p, q) &= d(t)u - m(t)p + n(t)q + \xi(t, u, p, q). \end{aligned} \quad (3.3)$$

Obviously, by the condition (A_1) , we have

$$\begin{aligned} \lim_{|(u, p, q)| \rightarrow 0} \frac{\zeta(t, u, p, q)}{|(u, p, q)|} &= 0, \quad \text{uniformly for } t \in [0, 1], \\ \lim_{|(u, p, q)| \rightarrow \infty} \frac{\xi(t, u, p, q)}{|(u, p, q)|} &= 0, \quad \text{uniformly for } t \in [0, 1]. \end{aligned} \quad (3.4)$$

Let

$$\tilde{\xi}(r) = \max\{|\xi(t, u, p, q)| \mid 0 \leq |u, p, q| \leq r, t \in [0, 1]\}, \quad (3.5)$$

It is easy to see the fact that $\tilde{\xi}(r)$ is monotone, not decreasing and

$$\lim_{r \rightarrow \infty} \frac{\tilde{\xi}(r)}{r} = 0. \quad (3.6)$$

Let us consider

$$Lu = \lambda \left[a(t)u - b(t)u'' + c(t)u^{(4)} \right] + \lambda \zeta(t, u, u'', u^{(4)}), \quad \lambda > 0, \quad (3.7)$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

It is easy to verify that (3.7) is equivalent to equation

$$\begin{aligned} u(t) &= \lambda \left\{ \int_0^1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) H(\tau, s) \left[A(s) u(s) - B(s) u''(s) + C(s) u^{(4)}(s) \right] ds d\tau dx \right\} \\ &\quad + \lambda \left\{ \int_0^1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) H(\tau, s) \zeta(t, u, u'', u^{(4)}) ds d\tau dx \right\} \\ &=: R(\lambda, u). \end{aligned} \quad (3.8)$$

From the proof of Theorem 2.3, we know that $\hat{T} : X \rightarrow X$ is strong positive and compact:

$$\hat{T}u(t) := \int_0^1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) H(\tau, s) \left[a(s) u(s) - b(s) u''(s) + c(s) u^{(4)}(s) \right] ds d\tau dx. \quad (3.9)$$

Define $F : [0, \infty) \times X \rightarrow X$ by

$$F(\lambda, u) := \lambda \int_0^1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) H(\tau, s) \zeta(t, u, u'', u^{(4)}) ds d\tau dx, \quad (3.10)$$

then by (3.4) and Lemma 2.2, we know that when $\|u\|_X \rightarrow 0$,

$$\|F(\lambda, u)\|_X = o(\|u\|_X) \rightarrow 0 \quad \text{partly consistent to } \lambda. \quad (3.11)$$

Based on Theorem 2 in literature [11], we come to the following conclusion.

There exists an unbounded connected subset Γ for the following set:

$$D_p(\hat{T}) = \left\{ (\lambda, u) \in [0, \infty) \times P : u = \lambda \hat{T}u + F(\lambda, u), u \in \text{int } P \right\} \cup \left\{ (\mu_1(a, b, c), 0) \right\} \quad (3.12)$$

such that $(\mu_1(a, b, c), 0) \in \Gamma$.

Next, we will verify the result of this theorem.

Obviously, any solution of (3.7), such as $(1, u)$, is the solution of problem (1.4). If we want to verify Γ passing through hyperplane $\{1\} \times X$, we only need to verify that Γ connects $(\mu_1(a, b, c), 0)$ and $(\mu_1(d, m, n), \infty)$.

Let $(\eta_n, y_n) \in \Gamma$ and satisfy

$$\eta_n + \|y_n\|_X \rightarrow \infty, \quad n \rightarrow \infty. \quad (3.13)$$

Since $(0, 0)$ is the only solution of (3.7) when $\lambda = 0$ and $\Gamma \cap (\{0\} \times X) = \emptyset$, we have $\eta_n > 0$ for all $n \in N$.

Case 1 ($\mu_1(d, m, n) < 1 < \mu_1(a, b, c)$). If we want to verify that (1.4) have at least one positive solution, we only need to verify that Γ connects $(\mu_1(a, b, c), 0)$ and $(\mu_1(d, m, n), \infty)$, that is, to verify

$$(\mu_1(d, m, n), \mu_1(a, b, c)) \subseteq \{\lambda \in R \mid (\lambda, u) \in \Gamma\}. \quad (3.14)$$

The proof can be divided into the following two steps.

Step 1. If we can verify that there exists a constant $M > 0$ such that $\eta_n \subset [0, M]$, where $n \in N$, then this connects $(\mu_1(a, b, c), 0)$ and $(\mu_1(d, m, n), \infty)$.

By (3.13), when $n \rightarrow \infty$, $\|y_n\|_X \rightarrow \infty$, and we divide the two sides of equation

$$Ly_n(t) = \eta_n \left(d(t)y_n - m(t)y_n'' + n(t)y_n^{(4)} \right) + \eta_n \xi \left(t, y_n(t), y_n''(t), y_n^{(4)}(t) \right), \quad (3.15)$$

with $\|y_n\|_X$ at the same time. Let $\bar{y}_n = y_n / \|y_n\|_X$, then $\bar{y}_n$ is bounded in X , and we have already known that L^{-1} is compact and the bounded set is mapped as bcompact set, so there exists convergent subsequence in $\{\bar{y}_n(t)\}$ we might as well still mark it as $\{\bar{y}_n(t)\}$; and satisfy

$$\bar{y}_n \rightarrow \bar{y}, \quad \bar{y} \in X, \quad \|\bar{y}\|_X = 1. \quad (3.16)$$

Furthermore, because

$$\frac{\left| \xi \left(t, y_n(t), y_n''(t), y_n^{(4)}(t) \right) \right|}{\|y_n\|_X} \leq \frac{\tilde{\xi}(\|y_n\|_X)}{\|y_n\|_X}, \quad (3.17)$$

according to (3.6) and Lemma 2.2, we obtain

$$\lim_{n \rightarrow \infty} \frac{\left| \xi \left(t, y_n(t), y_n''(t), y_n^{(4)}(t) \right) \right|}{\|y_n\|_X} = 0. \quad (3.18)$$

Hence there is

$$\bar{y}(t) := \int_0^1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) H(\tau, s) \bar{\eta} \left[d(s)u(s) - m(s)u''(s) + n(s)u^{(4)}(s) \right] ds d\tau dx, \quad (3.19)$$

where $\bar{\eta} = \lim_{n \rightarrow \infty} \eta_n$. Hence $L\bar{y}(t) = \bar{\eta}(d(t)\bar{y} - m(t)\bar{y}' + n(t)\bar{y}^{(4)})$. Combined with Theorem 2.3, it is easy to obtain

$$\bar{\eta} = \mu_1(d, m, n). \quad (3.20)$$

To sum up, Γ connects $(\mu_1(a, b, c), 0)$ with $(\mu_1(d, m, n), \infty)$.

Step 2. To verify the fact that arbitrary $n \in N$, there exists $m > 0$ such that $\eta_n \in [0, M]$.

Thanks to the Lemma 2.1 in [8], we only need to verify that nonlinear operator $R(\lambda, u)$ has linear function V , and there exists $(\eta, y) \in (0, \infty) \times P$ such that $\|y\|_X = 1$ and $\eta Vy \geq y$. It follows from (A_3) that there exist $a_0, b_0, c_0 \in [0, \infty)$ such that $a_0^2 + b_0^2 + c_0^2 > 0$, and

$$f(t, u, p, q) \geq a_0 u + b_0 p + c_0 q, \quad (t, u, p, q) \in [0, 1] \times [0, \infty) \times (-\infty, 0] \times [0, \infty). \quad (3.21)$$

To $u \in X$, let

$$Vu(t) = \int_0^1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) H(\tau, s) \left[a_0 u(s) - b_0 u''(s) + c_0 u^{(4)}(s) \right] ds d\tau dx. \quad (3.22)$$

Then V is the linear function of $R(\lambda, u)$.

Again, as

$$\begin{aligned} V\left(\frac{e(t)}{\pi^2}\right) &= \int_0^1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) H(\tau, s) \left[a_0 \frac{e(s)}{\pi^2} - b_0 \frac{e''(s)}{\pi^2} + c_0 \frac{e^{(4)}(s)}{\pi^2} \right] ds d\tau dx \\ &= \int_0^1 \int_0^1 \int_0^1 H(t, x) H(x, \tau) H(\tau, s) \left[\frac{a_0}{\pi^2} - b_0 + c_0 \pi^2 \right] e(s) ds d\tau dx \\ &= \frac{1}{\pi^6} \left[\frac{a_0}{\pi^2} - b_0 + c_0 \pi^2 \right] e(t), \end{aligned} \quad (3.23)$$

that is,

$$\left[\frac{a_0}{\pi^6} + \frac{b_0}{\pi^4} + \frac{c_0}{\pi^2} \right]^{-1} V\left(\frac{e(t)}{\pi^2}\right) = \frac{e(t)}{\pi^2}, \quad (3.24)$$

by the Lemma 2.1 in [8], we obtain

$$|\eta_n| \leq \left[\frac{a_0}{\pi^6} + \frac{b_0}{\pi^4} + \frac{c_0}{\pi^2} \right]^{-1}. \quad (3.25)$$

The conclusion is thus proved.

Case 2 ($\mu_1(a, b, c) < 1 < \mu_1(d, m, n)$). If $(\eta_n, y_n) \in \Gamma$, which satisfies $\lim_{n \rightarrow \infty} \eta_n + \|y_n\|_X = \infty$, and $\lim_{n \rightarrow \infty} \|y_n\|_X = \infty$, then we obtain

$$(\mu_1(a, b, c), \mu_1(d, m, n)) \subseteq \{\lambda \in R \mid (\lambda, u) \in \Gamma\}, \quad (3.26)$$

hence we have $(\{1\} \times X) \cap \Gamma \neq \emptyset$.

Similar to Case 1, the verification of Case 2 can also be divided into two steps with the conclusion that Γ connects $(\mu_1(a, b, c), 0)$ and $(\mu_1(d, m, n), \infty)$. And we come to the conclusion that Γ passes through hyperplane $\{1\} \times X$ on $R \times X$, hence (1.4) have at least one positive solution. $\square$

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