Research Article

Global Attractor for Doubly Nonlinear Parabolic Equation

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Our aim in this paper is to study the long-time behavior for a class of doubly nonlinear parabolic equations. First we show that the problem has a unique solution. Then we prove that the semigroup corresponding to the problem is norm-to-weak continuous in L^q and H_0^1 . Finally we establish the existence of global attractor of the problem in L^q and H_0^1 .

1. Introduction

We study the long-time behavior (in terms of attractors) of the solution of the following problem:

$$\begin{aligned} \frac{\partial \alpha(u)}{\partial t} - \Delta u + f(u) &= g(x), \quad x \in \Omega, \\ u(x, t)|_{\partial\Omega} &= 0, \\ u(x, 0) &= u_0(x). \end{aligned} \tag{1.1}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 1$) and $g(x)$ is a given function in $L^2(\Omega)$.

Such equations appear, for example, in the study of gas filtration (so-called porous medium equation [1]). It has been extensively studied when α is linear [1–5], and the existence of attractors has been proved in [6–11] (for $\alpha = I$).

Our aim in this paper is to extend the result of [1–4] to the more general equation (1.1). We make the following assumptions:

$$\alpha(s) \in \mathcal{C}^1(\mathbb{R}), \quad \alpha(0) = 0, \quad \alpha'(s) \geq \alpha_0, \quad \alpha_0 > 0, \quad s \in \mathbb{R}, \quad (1.2)$$

$$\alpha_1|s|^{r+2} - \alpha_3 \leq \alpha(s)s \leq \alpha_2|s|^{r+2} + \alpha_3, \quad \alpha_1, \alpha_2 > 0, \alpha_3 \geq 0, \quad r \geq 0, \quad (1.3)$$

$$\alpha'(s) \leq \alpha_4|s|^r + \alpha_5, \quad \alpha_4 > 0, \quad \alpha_5 \geq 0, \quad s \in \mathbb{R}, \quad (1.4)$$

$$f(s) \in \mathcal{C}(\mathbb{R}), \quad \gamma_1|s|^q - \gamma_3 \leq f(s)s \leq \gamma_2|s|^q + \gamma_3, \quad s \in \mathbb{R}, \gamma_1, \gamma_2 > 0, \gamma_3 \geq 0, \quad q \geq r + 2. \quad (1.5)$$

There exists a constant $C_0 \geq 0$, such that

$$C_0\alpha(s) + f(s) \text{ is increasing.} \quad (1.6)$$

By hypotheses (1.2)–(1.5), α and f are nonlinear functions with polynomial growth of arbitrary order. Here α is more general than in [1–4] (where α is linear growth), which is an essential difficulty in proving the existence of global attractor. To problem (1.1), the key points are to obtain the continuous and compactness of semigroup. By using Legendre transform and the asymptotic a priori estimate method introduced in [10, 11], we show the existence of global attractor.

This paper is organized as follows. In Section 2, we recall some basic concepts about the global attractor. In Section 3, we show the uniqueness of solution and norm-to-weak continuous semigroup for (1.1). In Section 4, we verify the asymptotic compactness of the semigroup $S(t)$ in $L^q(\Omega)$ and prove the existence of the $(L^{r+2}(\Omega), L^q(\Omega))$ -global attractor under the hypotheses (1.2)–(1.6). Finally, in Section 5, we prove the existence of the $(L^{r+2}(\Omega), H_0^1(\Omega))$ -global attractor for $N \leq 2$.

Throughout this paper we use the following notation: $H = L^2(\Omega)$, and the norms in $H_0^1(\Omega)$ and $L^p(\Omega)$ ($1 \leq p \leq \infty$) are denoted by $\|u\|^2 = \int_{\Omega} |\nabla u|^2 dx$ and $|u|_p^p = \int_{\Omega} |u|^p dx$, respectively; $\Omega(u \geq M) = \{x \in \Omega : u(x) \geq M\}$ and $\Omega(u \leq -M) = \{x \in \Omega : u(x) \leq -M\}$; $m(\Omega)$ or $|\Omega|$ denotes Lebesgue measure of Ω ; sometimes for special differentiation, we denote the different positive constants by $c_1, c_2, \dots$

2. Preliminaries

In this section, we recall some basic concepts about the global attractors.

Definition 2.1 (see [6–8]). Let $\{S(t)\}_{t \geq 0}$ be a semigroup on a Banach space X . A subset $\mathcal{A} \subset X$ is called a global attractor for the semigroup if $\mathcal{A}$ is compact in X and enjoys the following properties:

- (1) $\mathcal{A}$ is an invariant set, that is, $S(t)\mathcal{A} = \mathcal{A}$ for any $t \geq 0$;
- (2) $\mathcal{A}$ attracts all bounded sets of X , that is, for any bounded subset B of X

$$\text{dist}(S(t)B, \mathcal{A}) \longrightarrow 0, \quad \text{as } t \longrightarrow \infty, \quad (2.1)$$

where $\text{dist}(A, B)$ is the Hausdorff semidistance of the two sets A and B :

$$\text{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_X. \quad (2.2)$$

And a subset B_0 of X is called a bounded absorbing set of the semigroup $\{S(t)\}_{t \geq 0}$ in X , if for any bounded B of X , there exists some $T_B \geq 0$, such that $S(t)B \subset B_0$ for any $t \geq T_B$.

Definition 2.2 (see [11]). Let X be a Banach space and let $\{S(t)\}_{t \geq 0}$ be a family of operator in X . We say that $\{S(t)\}_{t \geq 0}$ is norm-to-weak continuous semigroup in X , if $\{S(t)\}_{t \geq 0}$ satisfies

- (1) $S(0) = I$ (the identity);
- (2) $S(t)S(s) = S(t+s), \forall t, s \geq 0$;
- (3) $S(t_n)x_n \rightharpoonup S(x)x$ if $t_n \rightarrow t$ and $x_n \rightarrow x$ in X .

Definition 2.3 (see [11]). A set $\mathcal{A} \subset X$, which is invariant, closed in X , compact in Z and attracts the bounded subsets of X in the topology of Z , is called an (X, Z) -global attractor.

Definition 2.4 (see [11]). Let $\{S(t)\}_{t \geq 0}$ be a semigroup on Banach space X . $\{S(t)\}_{t \geq 0}$ is called (X, Z) -asymptotically compact, if for any bounded (in X) sequence $\{x_n\}_{n=1}^\infty \subset X$ and $t_n \geq 0$, $t_n \rightarrow \infty$ as $n \rightarrow \infty$, $\{S(t_n)x_n\}_{n=1}^\infty$ has a convergence subsequence with respect to the topology of Z .

Definition 2.5 (see [11]). Let $\{S(t)\}_{t \geq 0}$ be a semigroup on Banach space X . A bounded subset B_0 of Z is called an (X, Z) -bounded absorbing set, if for any bounded set B of X , there exists some $t_B \geq 0$, such that $S(t)B \subset B_0$ for any $t \geq t_B$.

Theorem 2.6 (see [11]). Let X be a Banach space and let $\{S(t)\}_{t \geq 0}$ be a norm-to-weak continuous semigroup on X . Then $\{S(t)\}_{t \geq 0}$ has a global attractor in X provided that the following conditions hold:

- (1) $\{S(t)\}_{t \geq 0}$ has a bounded absorbing set B_0 in X ;
- (2) $\{S(t)\}_{t \geq 0}$ is asymptotically compact in X .

Theorem 2.7 (see [11]). Let $\{S(t)\}_{t \geq 0}$ be a norm-to-weak continuous semigroup on X . One assumes that $S(t)X \subset L^p(\Omega)$. Then $\{S(t)\}_{t \geq 0}$ has an $(X, L^p(\Omega))$ -global attractor provided that the following conditions hold:

- (1) $\{S(t)\}_{t \geq 0}$ has an $(X, L^p(\Omega))$ -bounded absorbing set $B_0 \subset L^p(\Omega)$;
- (2) there is a $q(1 \leq q \leq p)$ such that $\{S(t)\}_{t \geq 0}$ is $(X, L^q(\Omega))$ -asymptotically compact;
- (3) for any $\varepsilon > 0$ and for any bounded $B \subset X$, there exist constants $M = M(\varepsilon, B)$ and $T = T(\varepsilon, B)$, such that

$$\int_{\Omega(|S(t)u_0| \geq M)} |S(t)u_0|^p dx < \varepsilon \quad \text{for any } u_0 \in B \text{ and } t \geq T. \quad (2.3)$$

Lemma 2.8 (see [11]). Let $\{S(t)\}_{t \geq 0}$ be a semigroup on $L^p(\Omega)$ ($p \geq 1$) and suppose that $\{S(t)\}_{t \geq 0}$ has a bounded absorbing set in $L^p(\Omega)$. Then for any $\varepsilon > 0$ and any bounded subset $B \subset L^p(\Omega)$, there exist positive constants $T = T(B)$ and $M = M(\varepsilon)$ such that

$$m(\Omega(|S(t)u_0| \geq M)) \leq C\varepsilon \quad \text{for any } u_0 \in B \text{ and } t \geq T, \quad (2.4)$$

where the positive constant C is independent of B, T , and ε .

3. Uniqueness of Solution and Norm-to-Weak Continuous Semigroup

The existence of weak solution for (1.1) can be obtained by the standard Faedo-Galerkin approximation method. Here we only state the result.

Lemma 3.1. Assume that $g(x) \in L^2(\Omega)$, α and f satisfying (1.3)–(1.5), $u_0(x) \in L^{r+2}(\Omega)$. Then for any initial data $u_0(x) \in L^{r+2}(\Omega)$, there exists solution $u(x, t)$ for (1.1) which satisfies

$$u(x, t) \in C\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right) \cap L^q(0, T; L^q(\Omega)). \quad (3.1)$$

We now show that the solution is uniqueness and continuous dependent on initial conditions.

Theorem 3.2. Assume that $g(x) \in L^2(\Omega)$, $u_0(x) \in L^{r+2}(\Omega)$, α and f satisfying (1.3)–(1.6). Then there exists a unique solution of (1.1).

Proof. Suppose that $u(t), v(t)$ be two solutions of (1.1) with initial conditions $u_0(x), v_0(x)$; then

$$\frac{\partial(\alpha(u) - \alpha(v))}{\partial t} - \Delta(u - v) + f(u) - f(v) = 0, \quad (3.2)$$

that is

$$\frac{\partial(\alpha(u) - \alpha(v))}{\partial t} - \Delta(u - v) + (C_0\alpha(u) + f(u)) - (C_0\alpha(v) + f(v)) = C_0(\alpha(u) - \alpha(v)). \quad (3.3)$$

We define the sign function by

$$\text{sign}(\tau) = \begin{cases} 1 & \text{if } \tau > 0, \\ 0 & \text{if } \tau = 0, \\ -1 & \text{if } \tau < 0. \end{cases} \quad (3.4)$$

Multiplying (3.3) by $\text{sign}(u - v)$ and integrating in Ω , we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\alpha(u) - \alpha(v)| dx - \int_{\Omega} \Delta(u - v) \text{sign}(u - v) \\ & \quad + \int_{\Omega} [(C_0 \alpha(u) + f(u)) - (C_0 \alpha(v) + f(v))] \text{sign}(\alpha(u) - \alpha(v)) dx \\ & = C_0 \int_{\Omega} |\alpha(u) - \alpha(v)| dx. \end{aligned} \quad (3.5)$$

Using (1.6), we get

$$\int_{\Omega} [(C_0 \alpha(u) + f(u)) - (C_0 \alpha(v) + f(v))] \text{sign}(\alpha(u) - \alpha(v)) dx \geq 0. \quad (3.6)$$

Since $\text{sign}(u - v) = \lim_{\varepsilon \rightarrow 0^+} (u - v) / (\varepsilon + |u - v|)$, by dominated convergence theorem, we have

$$\begin{aligned} - \int_{\Omega} \Delta(u - v) \text{sign}(u - v) dx &= - \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} \Delta(u - v) \frac{u - v}{\varepsilon + |u - v|} dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} \nabla(u - v) \nabla \left(\frac{u - v}{\varepsilon + |u - v|} \right) dx \geq 0. \end{aligned} \quad (3.7)$$

So

$$\frac{d}{dt} \int_{\Omega} |\alpha(u) - \alpha(v)| dx \leq C_0 \int_{\Omega} |\alpha(u) - \alpha(v)| dx. \quad (3.8)$$

By Gronwall inequality, we get

$$\int_{\Omega} |\alpha(u(t)) - \alpha(v(t))| dx \leq \int_{\Omega} |\alpha(u_0) - \alpha(v_0)| dx. \quad (3.9)$$

From (1.2), we have

$$\int_{\Omega} |u(t) - v(t)| dx \leq \frac{1}{\alpha_0} \int_{\Omega} |\alpha(u_0) - \alpha(v_0)| dx, \quad (3.10)$$

which gives continuous dependence on initial conditions and uniqueness of solution in $L^1(\Omega)$.

By Theorem 3.2, we can define the operator semigroup $\{S(t)\}_{t \geq 0}$ in $L^1(\Omega)$ as the following:

$$S(t)u_0 : \mathbb{R}^+ \times L^{r+2}(\Omega) \longrightarrow L^1(\Omega), \quad (3.11)$$

which is continuous in $L^1(\Omega)$.

Since α is a continuous increasing function with $\alpha(0) = 0$, we define for $t \in \mathbb{R}$,

$$\psi(t) = \int_0^t \alpha(\tau) d\tau. \quad (3.12)$$

Then the Legendre transform ψ^* is defined by

$$\psi^*(\tau) = \sup_{s \in \mathbb{R}} \{\tau s - \psi(s)\}. \quad (3.13)$$

Note that

$$\psi^*(\tau) \geq 0, \quad \psi^*(\alpha(\tau)) + \psi(\tau) = \tau \alpha(\tau), \quad \psi^*(\alpha(\tau)) \leq \tau \alpha(\tau). \quad (3.14)$$

□

Theorem 3.3. Assume that the conditions (1.2)–(1.6) are satisfied, $g(x) \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t \geq 0}$ is norm-to-weak continuous in $L^q(\Omega)$ and $H_0^1(\Omega)$.

Proof. Let $u_{m0}(x) \rightarrow u_0(x)$ in $L^{r+2}(\Omega)$, $u_m(t), u(t)$ be the solutions of (1.1) corresponding to initial data $u_{m0}(x), u_0(x)$. In (1.1), replace $u(t)$ by $u_m(t)$. Multiplying (1.1) by $u_m(t)$ and integrating in Ω , we get

$$\frac{d}{dt} \int_{\Omega} \psi^*(\alpha(u_m(t))) dx + |\nabla u_m|_2^2 + (f(u_m), u_m) = (g, u_m). \quad (3.15)$$

Applying Young inequality, we have

$$\left| \int_{\Omega} g(x) u_m dx \right| \leq \frac{\gamma_1}{2} |u_m|_q^q + c \int_{\Omega} |g(x)|^{q'} dx. \quad \left(\frac{1}{q} + \frac{1}{q'} = 1 \right). \quad (3.16)$$

So

$$\frac{d}{dt} \int_{\Omega} \psi^*(\alpha(u_m(t))) dx + |\nabla u_m|_2^2 + \frac{\gamma_1}{2} |u_m|_q^q \leq \gamma_3 |\Omega| + c \int_{\Omega} |g|^{q'} dx. \quad (3.17)$$

Integrating from 0 to T , we obtain

$$\begin{aligned} & \int_{\Omega} \psi^*(\alpha(u_m(T))) dx + \int_0^T |\nabla u_m|_2^2 dt + \frac{\gamma_1}{2} \int_0^T |u_m|_q^q dt \\ & \leq \int_{\Omega} \psi^*(\alpha(u_{m0})) dx + cT \\ & \leq \int_{\Omega} u_{m0} \alpha(u_{m0}) dx + \alpha_3 |\Omega| + cT \\ & \leq \alpha_2 |u_{m0}|_{r+2}^{r+2} + cT + \alpha_3 |\Omega|. \end{aligned} \quad (3.18)$$

$u_{m0} \rightarrow u_0$ in $L^{r+2}(\Omega)$, so there existence $M > 0$, such that $|u_{m0}|_{r+2}^{r+2} \leq M$. $\{u_m(t)\}$ Is bounded in $L^2(0, T; H_0^1(\Omega))$ and $L^q(0, T; L^q(\Omega))$. Therefore, there exists weak convergent subsequence $\{u_{m_k}(t)\}$ of $\{u_m(t)\}$ in $L^2(0, T; H_0^1(\Omega))$ and $L^q(0, T; L^q(\Omega))$. Let $u_m(t) \rightharpoonup v(t)$ obviously, $v(t)$ be a solution of (1.1) satisfying the initial value condition $v(0) = u_0(x)$. By the unique of solution for (1.1), we have $u(t) = v(t)$, that is, $u_{m_k} \rightharpoonup u(t)$ in $L^2(0, T; H_0^1(\Omega))$ and $L^q(0, T; L^q(\Omega))$. By Definition 2.2, Theorem 3.3 holds. $\square$

Remark 3.4. The semigroup $\{S(t)\}_{t \geq 0}$ is norm-to-weak continuous in $L^2(\Omega)$.

4. $(L^{r+2}(\Omega), L^q(\Omega))$ -Global Attractor

By Theorem 3.3, we can define operator semigroup $\{S(t)\}_{t \geq 0}$ as the following:

$$\begin{aligned} S(t)u_0 : \mathbb{R}^+ \times L^{r+2}(\Omega) &\longrightarrow L^q(\Omega), \\ S(t)u_0 : \mathbb{R}^+ \times L^{r+2}(\Omega) &\longrightarrow H_0^1(\Omega). \end{aligned} \quad (4.1)$$

Theorem 4.1. Assume that the conditions (1.2)–(1.6) are satisfied, $g(x) \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t \geq 0}$ exists bounded absorbing sets in $L^q(\Omega)$ and $H_0^1(\Omega)$; that is, for arbitrary bounded set $B \subset L^{r+2}(\Omega)$, there exist $M > 0$, $T(B) > 0$, $\forall t \geq T$, $\forall u_0(x) \in B$. We have

$$|u(t)|_q^q + |\nabla u|_2^2 \leq M. \quad (4.2)$$

Proof. Let $u(t)$ be the solution of (1.1) with initial date $u_0(x)$; taking scalar product with $u(t)$ in (1.1), we deduce that

$$\frac{d}{dt} \int_{\Omega} \psi^*(\alpha(u)) dx + |\nabla u|_2^2 + (f(u), u) = (g, u). \quad (4.3)$$

By (1.5), we obtain

$$\frac{d}{dt} \int_{\Omega} \psi^*(\alpha(u)) dx + |\nabla u|_2^2 + \gamma_1 |u|_q^q - \gamma_3 |\Omega| \leq (g, u). \quad (4.4)$$

Using Young inequality, we get

$$|(g, u)| \leq \frac{\gamma_1}{2} \int_{\Omega} |u|^q dx + c_1 \int_{\Omega} |g|^q dx. \quad (4.5)$$

By (4.4), we have

$$\frac{d}{dt} \int_{\Omega} \psi^*(\alpha(u)) dx + |\nabla u|_2^2 + \frac{\gamma_1}{2} |u|_q^q \leq \gamma_3 |\Omega| + c_1 \int_{\Omega} |g|^q dx. \quad (4.6)$$

Since $q \geq r + 2$, by (1.3), there exist $c_2, c_3 > 0$ such that

$$\frac{\gamma_1}{4} \int_{\Omega} |u|^q dx \geq c_3 \int_{\Omega} \psi^*(\alpha(u)) dx - c_2. \quad (4.7)$$

Hence,

$$\frac{d}{dt} \int_{\Omega} \psi^*(\alpha(u)) dx + |\nabla u|_2^2 + \frac{\gamma_1}{4} |u|_q^q + c_3 \int_{\Omega} \psi^*(\alpha(u)) dx \leq c_4. \quad (4.8)$$

We get

$$\frac{d}{dt} \int_{\Omega} \psi^*(\alpha(u)) dx + c_3 \int_{\Omega} \psi^*(\alpha(u)) dx \leq c_4. \quad (4.9)$$

By the Gronwall lemma, we have

$$\int_{\Omega} \psi^*(\alpha(u(t))) \leq e^{-c_3 t} \int_{\Omega} \psi^*(\alpha(u_0)) dx + \frac{c_4}{c_3}. \quad (4.10)$$

Therefore,

$$\int_{\Omega} \psi^*(\alpha(u_0)) dx \leq \int_{\Omega} u_0 \alpha(u_0) dx. \quad (4.11)$$

We obtain

$$\int_{\Omega} \psi^*(\alpha(u(t))) \leq \alpha_2 |u_0|_{r+2}^{r+2} e^{-c_3 t} + \frac{c_4}{c_3} + \alpha_3 |\Omega| \leq c_5. \quad (4.12)$$

It follows from (4.8) that there exists $T_1 > 0, \forall t \geq T_1$; we get

$$\int_t^{t+1} |\nabla u|_2^2 ds + \int_t^{t+1} |u|_q^q ds \leq c_6. \quad (4.13)$$

Multiplying (1.1) by u_t and integrating over Ω , we get

$$\int_{\Omega} \alpha'(u) u_t^2 dx + \frac{d}{dt} \int_{\Omega} (|\nabla u|^2 + F(u)) dx = (g, u_t) \leq \frac{1}{2} \int_{\Omega} \alpha'(u) u_t^2 dx + \frac{1}{2\alpha_0} |g(x)|_2^2, \quad (4.14)$$

where $F(u) = \int_0^u f(s) ds$. Now by (1.2) we get

$$\frac{d}{dt} \int_{\Omega} (|\nabla u|^2 + F(u)) dx \leq \frac{1}{2\alpha_0} |g(x)|_2^2. \quad (4.15)$$

It follows from (1.5) that there exist $\gamma'_1, \gamma'_2 > 0, \gamma'_3 \geq 0$ such that

$$\gamma'_1 |s|^q - \gamma'_3 \leq F(s) \leq \gamma'_2 |s|^q + \gamma'_3. \quad (4.16)$$

Using the uniform Gronwall Lemma that there exists $T_2 > 0, \forall t \geq T_2$, we have

$$\int_{\Omega} (|\nabla u|^2 + F(u)) dx \leq c_7. \quad (4.17)$$

By (4.16), $\forall t \geq T = \max\{T_1, T_2\}$, we obtain

$$|\nabla u|_2^2 + |u|_q^q \leq c_8. \quad (4.18)$$

Therefor, the semigroup $\{S(t)\}$ exists bounded absorbing set in $H_0^1(\Omega)$ and $L^q(\Omega)$; it follows from Theorem 2.6 that we have the following. $\square$

Theorem 4.2. *Assume that the conditions (1.2)–(1.6) are satisfied, $g(x) \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t \geq 0}$ has a $(L^{r+2}(\Omega), L^2(\Omega))$ -global attractor, which is nonempty, compact, invariant in $L^2(\Omega)$ and attracts every bounded subset of $L^{r+2}(\Omega)$ with respect to $L^2(\Omega)$ norm.*

In the following, we will give the asymptotic a priori estimate of $\{S(t)\}_{t \geq 0}$ with respect to $L^q(\Omega)$ -norm, which plays a crucial role in the proof of the $(L^{r+2}(\Omega), L^q(\Omega))$ -global attractor.

Lemma 4.3. *Assume that the conditions (1.2)–(1.6) are satisfied, $g(x) \in L^2(\Omega)$, $q \geq \max\{2r, r+2\}$, $\forall B \subset L^{r+2}(\Omega)$. For any $\varepsilon > 0$, there exist positive constants $M(= M(\varepsilon))$ and $T(= T(\varepsilon))$ such that*

$$\int_{\Omega(|u(t)| \geq M)} |u(t)|^q dx < c\varepsilon, \quad \text{for any } t \geq T \text{ and } u_0(x) \in B. \quad (4.19)$$

Proof. By (1.3) and (1.5), we find that there exists $M_1 > 0, \forall |u| \geq M_1$ such that

$$f(u)u \geq \frac{\gamma_1}{2} |u|^q, \quad \frac{\alpha_1}{2} |u|^{r+1} \leq |\alpha(u)| \leq 2\alpha_2 |u|^{r+1}. \quad (4.20)$$

Letting $M_2 = \max\{1, \alpha_1/(2)|M_1|^{r+1}\}$, when $|\alpha(u)| \geq M_2$, then $|u| \geq M_1$. Multiplying (1.1) with $|(\alpha(u) - M_2)_+|^{(q/(r+1))-2}(\alpha(u) - M_2)_+$, we get

$$\begin{aligned} & \frac{r+1}{q} \frac{d}{dt} \int_{\Omega} |(\alpha(u) - M_2)_+|^{q/(r+1)} dx + \int_{\Omega} \nabla u \nabla \left(|(\alpha(u) - M_2)_+|^{q/(r+1)-2} (\alpha(u) - M_2)_+ \right) dx \\ & \quad + \int_{\Omega} f(u) |(\alpha(u) - M_2)_+|^{q/(r+1)-2} (\alpha(u) - M_2)_+ dx \\ & = \int_{\Omega} g(x) |(\alpha(u) - M_2)_+|^{q/(r+1)-2} (\alpha(u) - M_2)_+ dx, \end{aligned} \quad (4.21)$$

where $(\alpha(u) - M_2)_+$ denote the positive part of $(\alpha(u) - M_2)$, that is:

$$(\alpha(u) - M_2)_+ = \begin{cases} \alpha(u) - M_2, & \alpha(u) \geq M_2, \\ 0, & \alpha(u) < M_2. \end{cases} \quad (4.22)$$

Thus we have

$$\begin{aligned} & \int_{\Omega} \nabla u \nabla \left(|(\alpha(u) - M_2)_+|^{q/(r+1)-2} (\alpha(u) - M_2)_+ \right) dx \\ &= \int_{\Omega(\alpha(u) \geq M)} \nabla u \nabla \left(|(\alpha(u) - M_2)|^{q/(r+1)-1} \right) dx \\ &= \left(\frac{q}{r+1} - 1 \right) \int_{\Omega(\alpha(u) \geq M)} \alpha'(u) |(\alpha(u) - M_2)|^{q/(r+1)-2} |\nabla u|^2 dx \geq 0, \\ & \int_{\Omega} f(u) |(\alpha(u) - M_2)_+|^{q/(r+1)-2} (\alpha(u) - M_2)_+ dx \\ &\geq \frac{\gamma_1}{2} \int_{\Omega} |u|^{q-1} |(\alpha(u) - M_2)_+|^{q/(r+1)-1} dx \\ &\geq c \int_{\Omega} |\alpha(u)|^{(q-1)/(r+1)} |(\alpha(u) - M_2)_+|^{q/(r+1)-1} dx \\ &\geq c \int_{\Omega} |(\alpha(u) - M_2)_+|^{(2q-r-2)/(r+1)} dx, \\ & \left| \int_{\Omega} g(x) |(\alpha(u) - M_2)_+|^{q/(r+1)-2} (\alpha(u) - M_2)_+ dx \right| \\ &\leq \int_{\Omega} |g(x)| |(\alpha(u) - M_2)_+|^{q/(r+1)-1} dx \\ &\leq \frac{c}{2} \int_{\Omega} |(\alpha(u) - M_2)_+|^{(2q-r-2)/(r+1)} dx + c \int_{\Omega(\alpha(u) \geq M_2)} |g(x)|^{(2q-r-2)/(q-1)} dx. \end{aligned} \quad (4.23)$$

We obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |(\alpha(u) - M_2)_+|^{q/(r+1)} dx + c \int_{\Omega} |(\alpha(u) - M_2)_+|^{(2q-r-2)/(r+1)} dx \\ &\leq c \int_{\Omega(\alpha(u) \geq M_2)} |g(x)|^{(2q-r-2)/(q-1)} dx. \end{aligned} \quad (4.24)$$

Since $q \geq r+2$, we have

$$\frac{2q-r-2}{r+1} \geq \frac{q}{r+1}, \quad 1 < \frac{2q-r-2}{q-1} \leq 2. \quad (4.25)$$

So

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |(\alpha(u) - M_2)_+|^{q/(r+1)} dx + c \int_{\Omega} |(\alpha(u) - M_2)_+|^{q/(r+1)} dx \\ \leq c \int_{\Omega(\alpha(u) \geq M_2)} |g(x)|^{(2q-r-2)/(q-1)} dx. \end{aligned} \quad (4.26)$$

By Gronwall inequality, there exists $T(B) > 0, \forall t \geq T(B)$ such that

$$\begin{aligned} \int_{\Omega} |(\alpha(u(t)) - M_2)_+|^{q/(r+1)} dx \leq e^{-c(t-T)} \int_{\Omega} |(\alpha(u(T)) - M_2)_+|^{q/(r+1)} dx \\ + c \left(\int_{\Omega(\alpha(u) \geq M_2)} |g(x)|^{(2q-r-2)/(q-1)} dx \right) dx. \end{aligned} \quad (4.27)$$

For any $\varepsilon > 0$ there exists $\delta > 0$; for any $e \subset \Omega, m(e) < \delta$, we have

$$\int_e |g(x)|^{(2q-r-2)/(q-1)} dx < \varepsilon. \quad (4.28)$$

Letting $T(t) = \alpha(u(t))$, obviously $\{T(t)\}_{t \geq 0}$ is a semigroup in $L^1(\Omega)$. By (1.5) and (4.18), $\{T(t)\}_{t \geq 0}$ has a bounded absorbing set in $L^1(\Omega)$. Combining Lemma 2.8, for any $\varepsilon > 0$, there exist $M \geq M_2, T_1 \geq T(B)$, for any $t > T_1$, we have

$$m(\Omega(\alpha(u) \geq M)) < \min\{\delta, \varepsilon\}. \quad (4.29)$$

Hence

$$\int_{\Omega} |(\alpha(u(t)) - M_2)_+|^{q/(r+1)} dx < c\varepsilon, \quad (4.30)$$

that is

$$\int_{\Omega(\alpha(u) \geq M)} (|\alpha(u(t))| - M)^{q/(r+1)} dx < c\varepsilon. \quad (4.31)$$

By repeating the same step above and multiplying (1.1) with $|(\alpha(u) + M_2)_-|^{q/(r+1)-2}(\alpha(u) + M_2)_-$, we get

$$\int_{\Omega(\alpha(u) \leq -M)} (|\alpha(u(t))| - M)^{q/(r+1)} dx < c\varepsilon, \quad (4.32)$$

where

$$(\alpha(u) + M_2)_- = \begin{cases} \alpha(u) + M_2, & \alpha(u) \leq -M_2, \\ 0, & \alpha(u) > -M_2. \end{cases} \quad (4.33)$$

Combining (4.31) and (4.32), we have

$$\begin{aligned}
& \int_{\Omega(|\alpha(u)| \geq 2M)} |\alpha(u(t))|^{q/(r+1)} dx \\
&= \int_{\Omega(|\alpha(u)| \geq 2M)} (|\alpha(u(t))| - M + M)^{q/(r+1)} dx \\
&\leq c \left(\int_{\Omega(|\alpha(u)| \geq 2M)} (|\alpha(u(t))| - M)^{q/(r+1)} dx + \int_{\Omega(|\alpha(u)| \geq 2M)} M^{q/(r+1)} dx \right) \\
&\leq c \left(\int_{\Omega(|\alpha(u)| \geq M)} (|\alpha(u(t))| - M)^{q/(r+1)} dx + \int_{\Omega(|\alpha(u)| \geq M)} (|\alpha(u(t))| - M)^{q/(r+1)} dx \right) \leq c\varepsilon.
\end{aligned} \tag{4.34}$$

Thanks to (4.20), we have

$$\int_{\Omega(|u(t)| \geq M)} |u(t)|^q dx < c\varepsilon. \tag{4.35}$$

From Theorem 4.2, we know that the semigroup $\{S(t)\}_{t \geq 0}$ is asymptotically compact in $L^2(\Omega)$; using Theorem 4.1, Lemma 4.3, and Theorem 2.7, we get the following. $\square$

Theorem 4.4. *Assume that the conditions (1.2)–(1.6) are satisfied, $g(x) \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t \geq 0}$ has a $(L^{r+2}(\Omega), L^q(\Omega))$ -global attractor, which is nonempty, compact, invariant in $L^q(\Omega)$ and attracts every bounded subset of $L^{r+2}(\Omega)$ with respect to $L^q(\Omega)$ norm.*

5. $(L^{r+2}(\Omega), H_0^1(\Omega))$ -Global Attractor

In this section, we want to prove the $(L^{r+2}(\Omega), H_0^1(\Omega))$ -global attractor. However, for general N -dimension space, our methods can not work, so we make the following assumption:

$$N \leq 2, \quad \alpha(s) \in C^2(\mathbb{R}), \quad f(s) \in C^1(\Omega), \quad s \in \mathbb{R}. \tag{5.1}$$

Lemma 5.1. *Assume that the conditions (1.2), (1.3), (1.5), (1.6), and (5.1) are satisfied, $g(x) \in L^2(\Omega)$. For any bounded set $B \subset L^{r+2}(\Omega)$, there exists $t^*(B) > 0$; for any $t \geq t^*$, we have*

$$\int_{\Omega} \alpha'(u) |u_t|^2 dx \leq c. \tag{5.2}$$

Proof. Multiplying (1.1) by u_t and integrating over Ω , we have

$$\int_{\Omega} \alpha'(u) |u_t|^2 dx + \frac{d}{dt} \int_{\Omega} (|\nabla u|^2 + F(u)) dx = (g(x), u_t) \leq \int_{\Omega} \frac{\alpha'(u)}{2} |u_t|^2 dx + \frac{|g(x)|_2^2}{2\alpha_0}, \tag{5.3}$$

and then

$$\int_{\Omega} \alpha'(u)|u_t|^2 + c \frac{d}{dt} \int_{\Omega} |\nabla u|^2 + F(u) dx \leq c. \quad (5.4)$$

By (4.16) and (4.18), there exists $T(B) > 0$ for any $t \geq T(B)$; we get

$$\int_{\Omega} (|\nabla u|^2 + F(u)) dx \leq c. \quad (5.5)$$

Integrating (5.4) from t to $t+1$, we obtain

$$\int_t^{t+1} \int_{\Omega} \alpha'(u)|u_t|^2 dx ds \leq c, \quad t \geq T(B). \quad (5.6)$$

By differentiating (1.1) in time t , we have

$$\alpha''(u)u_t^2 + \alpha'(u)u_{tt} - \Delta u_t + f'(u)u_t = 0. \quad (5.7)$$

Letting $r(t) = \alpha'(u)u_t^2$, we get

$$\frac{d}{dt} \int_{\Omega} r(t) dx = \int_{\Omega} \alpha''(u)u_t^3 dx + 2 \int_{\Omega} \alpha'(u)u_t u_{tt} dx, \quad (5.8)$$

that is

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} r(t) dx = \frac{1}{2} \int_{\Omega} \alpha''(u)u_t^3 dx + \int_{\Omega} \alpha'(u)u_t u_{tt} dx. \quad (5.9)$$

Multiplying (5.7) by u_t and integrating over Ω , we get

$$\int_{\Omega} \alpha''(u)u_t^3 dx + \int_{\Omega} \alpha'(u)u_{tt}u_t dx = - \int_{\Omega} |\nabla u_t|^2 dx - \int_{\Omega} f'(u)u_t^2 dx. \quad (5.10)$$

Hence from (5.9), we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} r(t) dx = - \frac{1}{2} \int_{\Omega} \alpha''(u)u_t^3 dx - \int_{\Omega} |\nabla u_t|^2 dx - \int_{\Omega} f'(u)u_t^2 dx. \quad (5.11)$$

Since $N \leq 2$, from Theorem 4.1 the semigroup $\{S(t)\}_{t \geq 0}$ has a bounded absorbing set in $H_0^1(\Omega)$. Applying Sobolev embedding theorem, the semigroup $\{S(t)\}_{t \geq 0}$ is bounded in $L^\infty(\Omega)$; hence,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} r(t) dx + \int_{\Omega} |\nabla u_t|^2 dx \leq M \int_{\Omega} |u_t|^3 dx + c \int_{\Omega} |u_t|^2 dx. \quad (5.12)$$

Since $N \leq 2$, the imbedding $H^{1/3}(\Omega) \hookrightarrow L^3(\Omega)$ is compact; hence

$$|u_t|_3^3 \leq c|u_t|_{H^{1/3}(\Omega)}^3. \quad (5.13)$$

Using interpolation inequality for Sobolev space, we get

$$|u_t|_{H^{1/3}(\Omega)}^3 \leq c|\nabla u_t|_2|u_t|_2^2 \leq \frac{1}{2}|\nabla u_t|_2^2 + c|u_t|_2^4. \quad (5.14)$$

Therefore, from (5.12), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} r(t) dx + \int_{\Omega} |\nabla u_t|^2 dx &\leq c|u_t|_2^2 + c|u_t|_2^4 \\ &\leq c|u_t|_2^4 + c = c \left(\int_{\Omega} |u_t|^2 dx \right)^2 + c \\ &\leq c \left(\int_{\Omega} \alpha'(u) |u_t|^2 dx \right)^2 + c \leq c \left(\int_{\Omega} r(t) dx \right)^2 + c. \end{aligned} \quad (5.15)$$

Thanks to uniform Gronwall inequality, there exists $T(B) > 0$ for any $t \geq T(B)$; we have

$$\int_{\Omega} r(t) dx \leq c. \quad (5.16)$$

Now we prove the semigroup $\{S(t)\}_{t \geq 0}$ is asymptotically compact in $H_0^1(\Omega)$. $\square$

Lemma 5.2. Assume that the conditions (1.2), (1.3), (1.5), (1.6) and (5.1) are satisfied, $g(x) \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t \geq 0}$ is asymptotically compact in $H_0^1(\Omega)$.

Proof. Let B is a bounded absorbing set in $H_0^1(\Omega)$, $x_n \in B$, $t_n \rightarrow \infty$ as $n \rightarrow \infty$, $u_n(t) = S(t_n)x_n$. Now we will prove that there exists Cauchy sequence of $\{u_n(t)\}$ in $H_0^1(\Omega)$. By Theorem 4.2, we know that $\{S(t)\}_{t \geq 0}$ exists global attractor in $L^2(\Omega)$, therefore there exists Cauchy sequence of $\{u_n(t)\}$ in $L^2(\Omega)$, we denote by $\{u_n(t)\}$. Hence, $\forall 0 < \varepsilon < 1, \exists N, \forall n, m > N$:

$$|u_n - u_m|_2^2 < \varepsilon^2. \quad (5.17)$$

In view of (1.1), we get

$$\alpha'(u_m)u_{mt} - \alpha'(u_n)u_{nt} - \Delta(u_m - u_n) + f(u_m) - f(u_n) = 0. \quad (5.18)$$

Multiplying the above equality by $u_m - u_n$ and integrating over Ω , we obtain

$$\begin{aligned}
 & |\nabla u_m - \nabla u_n|_2^2 \\
 & \leq - \int_{\Omega} (\alpha'(u_m)u_{mt} - \alpha'(u_n)u_{nt})(u_m - u_n)dx - \int_{\Omega} (f(u_m) - f(u_n))(u_m - u_n)dx \\
 & \leq c \left(\int_{\Omega} |u_{mt}| |u_m - u_n| dx + \int_{\Omega} |u_{nt}| |u_m - u_n| dx \right) - \int_{\Omega} f'(\xi) |u_m - u_n|^2 dx \\
 & \leq c(|u_{mt}|_2 |u_m - u_n|_2 + |u_{nt}|_2 |u_m - u_n|_2) + c|u_m - u_n|_2^2.
 \end{aligned} \tag{5.19}$$

Hence, for any $n, m > N$, we get

$$|\nabla u_m - \nabla u_n|_2^2 < c\varepsilon. \tag{5.20}$$

That is to say, the semigroup $\{S(t)\}_{t \geq 0}$ is asymptotically compact in $H_0^1(\Omega)$. $\square$

Thus from Theorem 4.1 and Lemma 5.2, we have the following.

Theorem 5.3. *Assume that the conditions (1.2), (1.3), (1.5), (1.6) and (5.1) are satisfied, $g(x) \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t \geq 0}$ has a $(L^{r+2}(\Omega), H_0^1(\Omega))$ -global attractor, which is nonempty, compact, invariant in $H_0^1(\Omega)$ and attracts every bounded subset of $L^{r+2}(\Omega)$ with respect to $H_0^1(\Omega)$ norm.*

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