Research Article

Explicit Evaluations of Cubic and Quartic Theta-Functions

Nipen Saikia

*Department of Mathematics, Rajiv Gandhi University, Rono Hills,
 Doimukh 791112, Arunachal Pradesh, India*

Correspondence should be addressed to Nipen Saikia, nipennak@yahoo.com

Received 24 February 2012; Accepted 8 April 2012

Academic Editors: H.-J. Kreowski and W. Liu

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We find explicit values of cubic and quartic theta-functions and their quotients by parameterizations. In the process, we also find some transformation formulas of these theta-functions.

1. Introduction

For any complex number a , define

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \quad \text{and } (a; q)_\infty := \prod_{k=1}^{\infty} (1 - aq^{k-1}). \quad (1.1)$$

Ramanujan's general theta-function $f(a, b)$ is given by

$$f(a, b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad (1.2)$$

where $|ab| < 1$. If we set $a = qe^{2iz}$, $b = qe^{-2iz}$, and $q = e^{\pi i\tau}$, where z is complex and $\text{Im}(\tau) > 0$, then $f(a, b) = \vartheta_3(z, \tau)$, where $\vartheta_3(z, \tau)$ [1, page 464] denotes one of the classical theta-functions in its standard notation.

We also define the following three special cases of $f(a, b)$:

$$\begin{aligned}\phi(q) &:= f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; q^2)_{\infty} (q^2; q^2)_{\infty}}{(q; q^2)_{\infty} (-q^2; q^2)_{\infty}}, \\ \psi(q) &:= f(q, q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, \\ f(-q) &:= f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}.\end{aligned}\tag{1.3}$$

If $q = e^{2\pi iz}$ with $\text{Im}(z) > 0$, then $f(-q) = q^{-1/24} \eta(z)$, where $\eta(z)$ denotes the classical Dedekind eta-function.

In his famous paper [2] and [3, pages 23–39], Ramanujan offered 17 elegant series for $1/\pi$ and remarked that 14 of these series belong to the “corresponding theories” in which the base q in classical theory of elliptic functions is replaced by one or other of the functions:

$$q_r := q_r(x) = \exp\left(-\pi \csc\left(\frac{\pi}{r}\right) \frac{{}_2F_1(1/r, (r-1)/r, 1, 1-x)}{{}_2F_1(1/r, (r-1)/r, 1, x)}\right),\tag{1.4}$$

where $r = 3, 4$, and 6 , where ${}_2F_1$ denotes the Gaussian hypergeometric function. In the classical theory, the variable $q = q_2$. Ramanujan did not offer any proof of these 14 series for $1/\pi$ or any of his theorems in the “corresponding” or “alternative” theories. In 1987, J. M. Borwein and P. B. Borwein [4] proved the formulas for $1/\pi$. However, in his second notebook [5, Vol. II], Ramanujan recorded, without proof, some of his theorems in alternative theories which were first proved by Berndt et al. [6] in 1995. These theories are now known as the theory of signature r , where $r = 3, 4$, and 6 . In particular, the theories of signature 3 and 4 are called cubic and quartic theories, respectively. An account of this work may also be found in Berndt’s book [7].

In Ramanujan’s cubic theory, the theta-functions $a(q)$, $b(q)$, and $c(q)$ are defined by

$$\begin{aligned}a(q) &= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, & b(q) &= \sum_{m,n=-\infty}^{\infty} w^{m-n} q^{m^2+mn+n^2}, \\ c(q) &= \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2},\end{aligned}\tag{1.5}$$

where $w = \exp(2\pi i/3)$. These theta-functions were first introduced by J. M. Borwein and P. B. Borwein [8], who also proved that

$$a^3(q) = b^3(q) + c^3(q).\tag{1.6}$$

Cubic theta-functions $b(q)$ and $c(q)$ are related with the Dedekind eta-function by [7, page 109, Lemma 5.1]:

$$b(q) = \frac{f^3(-q)}{f(-q^3)}, \quad c(q) = \frac{3q^{1/3}f^3(-q^3)}{f(-q)}. \quad (1.7)$$

The Borwein brothers [8, (2.2)] also established the following three transformation formulas:

$$a(e^{-2\pi t}) = \frac{1}{t\sqrt{3}} a(e^{-2\pi/3t}), \quad (1.8)$$

$$b(e^{-2\pi t}) = \frac{1}{t\sqrt{3}} c(e^{-2\pi/3t}), \quad (1.9)$$

$$c(e^{-2\pi t}) = \frac{1}{t\sqrt{3}} b(e^{-2\pi/3t}), \quad (1.10)$$

where $\operatorname{Re}(t) > 0$. Cooper [9] also found alternate proofs of (1.8)–(1.10).

In quartic theory, Berndt et al. [6] (see also [7, page 146, (9.7)]) established a “transfer” principle of Ramanujan by which formulas in this theory can be derived from those of the classical theory. Taking place of $a(q)$, $b(q)$, and $c(q)$ in cubic theory is the functions $A(q)$, $B(q)$, and $C(q)$ [10], defined by

$$\begin{aligned} A(q) &= \phi^4(q) + 16q\psi^4(q^2), & B(q) &= \phi^4(q) - 16q\psi^4(q^2), \\ C(q) &= 8\sqrt{q}\phi^2(q)\psi^2(q^2), \end{aligned} \quad (1.11)$$

which also satisfy the equality:

$$A^2(q) = B^2(q) + C^2(q). \quad (1.12)$$

Berndt et al. [10] used (1.12) to establish the inversion formula:

$$z_4 := {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; x\right) = \sqrt{A(q)}, \quad (1.13)$$

where $q := q_4$ is given by (1.4). Therefore, they were able to prove the theorems in the quartic theory directly.

The quartic analogues of (1.7) are given by [10, page 139, Theorem 3.1]

$$B(q) = \left(\frac{f^2(-q)}{f(-q^2)}\right)^4, \quad C(q) = 8\sqrt{q}\left(\frac{f^2(-q^2)}{f(-q)}\right)^4. \quad (1.14)$$

While proving the explicit values of $\phi(q)$ and $\psi(q)$ recorded by Ramanujan in his notebooks, Berndt [7], explicitly determined the value of cubic theta-function $a(e^{-2\pi})$ [7, page 328, Corollary 3], namely,

$$\frac{a(e^{-2\pi})}{\phi^2(e^{-\pi})} = \frac{1}{(12)^{1/8} \sqrt{\sqrt{3}-1}}, \quad (1.15)$$

where $\phi(e^{-\pi}) = \pi^{1/4}/\Gamma(3/4)$ is classical [1]. Certain quotients of $A(q)$, $B(q)$, and $C(q)$ were also evaluated by Berndt et al. [10] while deriving the series for $1/\pi$ associated with the theory of signature 4.

In this paper, we find several new explicit values of cubic and quartic theta-functions and their quotients by parameterizations. In the process, we also find some transformation formulas of these theta-functions.

We now define some parameters of Dedekind eta-function $f(q)$ and Ramanujan's theta-functions $\phi(q)$ and $\psi(q)$. For positive real numbers n and k , define

$$r_{k,n} := \frac{f(-q)}{k^{1/4} q^{(k-1)/24} f(-q^k)}, \quad q = e^{-2\pi\sqrt{n/k}}, \quad (1.16)$$

$$r'_{k,n} := \frac{f(q)}{k^{1/4} q^{(k-1)/24} f(q^k)}, \quad q = e^{-\pi\sqrt{n/k}}. \quad (1.17)$$

The parameters $r_{k,n}$ and $r'_{k,n}$ are defined by Yi [11]. She also evaluated several explicit values of $r_{k,n}$ and $r'_{k,n}$ by using eta-function identities and transformation formulas.

In his lost notebook [12, page 212], Ramanujan defined

$$\lambda_n = \frac{1}{3\sqrt{3}} \frac{f^6(q)}{\sqrt{q} f^6(q^3)}, \quad q = e^{-\pi\sqrt{n/3}}. \quad (1.18)$$

Closely related to λ_n is the parameter μ_n defined by Ramanathan [13] as

$$\mu_n = \frac{1}{3\sqrt{3}} \frac{f^6(-q)}{\sqrt{q} f^6(-q^3)}, \quad q = e^{-2\pi\sqrt{n/3}}. \quad (1.19)$$

From the definitions of $r_{k,n}$, μ_n , $r'_{k,n}$, and μ_n , we note that $r'_{3,n} = \lambda_n$ and $r_{3,n}^6 = \mu_n$. Ramanujan [12] also provided a list of eleven recorded values of λ_n and ten unrecorded values of λ_n . All 21 values of λ_n and several new were established by Berndt et al. [14]. Yi [11], and Baruah and Saikia [15, 16] also found several new values of parameters λ_n and μ_n .

In [11], Yi also introduced the following two parameterizations $h_{k,n}$ and $h'_{k,n}$ along with $r_{k,n}$ and $r'_{k,n}$:

$$h_{k,n} := \frac{\phi(q)}{k^{1/4}\phi(q^k)}, \quad q = e^{-\pi\sqrt{n/k}}, \quad (1.20)$$

$$h'_{k,n} := \frac{\phi(-q)}{k^{1/4}\phi(-q^k)}, \quad q = e^{-2\pi\sqrt{n/k}}, \quad (1.21)$$

where k and n are positive real numbers. Employing modular transformation formulas and theta-function identities, Yi evaluated several many explicit values of $h_{k,n}$ and $h'_{k,n}$ to find explicit values of $\phi(q)$ and their quotients.

Motivated by Yi's work, for any positive real numbers k and n , Baruah and Saikia [17] defined the parameters $g_{k,n}$ and $g'_{k,n}$ by

$$g_{k,n} := \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)}, \quad q = e^{-\pi\sqrt{n/k}}, \quad (1.22)$$

$$g'_{k,n} := \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)}, \quad q = e^{-\pi\sqrt{n/k}}. \quad (1.23)$$

In [17], they proved many properties of the parameterizations $g_{k,n}$ and $g'_{k,n}$ and established their relationship with Yi's parameters $r_{k,n}$, $r'_{k,n}$, $h_{k,n}$, $h'_{k,n}$, and Weber-Ramanujan class-invariants G_n and g_n , where G_n and g_n defined by

$$G_n := 2^{-1/4}q^{-1/24}(-q; q^2)_\infty, \quad g_n := 2^{-1/4}q^{-1/24}(q; q^2)_\infty; \quad q = e^{-\pi\sqrt{n}}. \quad (1.24)$$

They also found several values of the parameters $g_{k,n}$ and $g'_{k,n}$.

In Section 2, we record some known values of above parameters, which will be used in this paper.

In Sections 3 and 4, we deal with explicit evaluations of cubic theta-functions and their quotients. In Sections 5 and 6, we find explicit values of the quartic theta-functions and their quotients.

2. Explicit Values of Parameters

Lemma 2.1. *If $r_{k,n}$ is as defined in (1.16), then*

$$r_{1,1} = 1, \quad r_{2,1} = 1, \quad r_{2,2} = 2^{1/8}, \quad r_{2,3} = (1 + \sqrt{2})^{1/6}, \quad r_{2,4} = 2^{1/8}(1 + \sqrt{2})^{1/8},$$

$$r_{2,5} = \sqrt{\frac{1 + \sqrt{5}}{2}}, \quad r_{2,6} = 2^{1/24}(\sqrt{3} + 1)^{1/4}, \quad r_{2,7} = \left(\frac{\sqrt{2} + 1 + \sqrt{2\sqrt{2} - 1}}{2}\right)^{1/2},$$

$$r_{2,8} = 2^{3/16} (1 + \sqrt{2})^{1/4}, \quad r_{2,9} = (\sqrt{2} + \sqrt{3})^{1/3},$$

$$r_{2,10} = \left(\frac{1}{2} (1 + \sqrt{5}) \left(\sqrt{\sqrt{5} + 1} + \sqrt{2} \right) \right)^{1/4},$$

$$r_{2,12} = (1 + \sqrt{2})^{5/24} \left(2(1 + \sqrt{2} + \sqrt{6}) \right)^{1/8},$$

$$r_{2,16} = 2^{1/8} (1 + \sqrt{2})^{1/4} \left(4 + \sqrt{2 + 10\sqrt{2}} \right)^{1/8},$$

$$r_{2,18} = \frac{(1 + \sqrt{3})^{1/3} (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{1/3}}{2^{11/24}},$$

$$r_{2,20} = \frac{(1 + \sqrt{5})^{5/8} (2 + 3\sqrt{2} + \sqrt{5})^{1/8}}{\sqrt{2}},$$

$$r_{2,27} = (1 + \sqrt{2})^{5/18} \left\{ \sqrt{2} + \sqrt{2} (1 + \sqrt{2})^{1/3} + (1 + \sqrt{2})^{2/3} \right\}^{1/3},$$

$$r_{2,32} = 2^{3/16} (1 + \sqrt{2})^{1/4} \left(16 + 15 \cdot 2^{1/4} + 12\sqrt{2} + 9 \cdot 2^{3/4} \right)^{1/8},$$

$$r_{2,36} = \frac{\{2(1 + 35\sqrt{2} - 28\sqrt{3})\}^{1/8}}{(\sqrt{3} - \sqrt{2})^{2/3}},$$

$$r_{2,49} = \frac{1 + \sqrt{7 + 2\sqrt{14}}}{2\sqrt{2}} + \frac{\sqrt{\sqrt{14} + \sqrt{7 + 2\sqrt{14}}}}{2}, \quad r_{2,50} = \frac{2^{5/8}}{5^{1/4} - 1},$$

$$r_{2,72} = \frac{(\sqrt{2} + \sqrt{3})^{1/3} (-\sqrt{2} + 4 + 2\sqrt{3} + 3^{3/4}(\sqrt{3} + 1))^{1/3}}{2^{13/48} (\sqrt{2} - 1)^{5/12}}, \quad r_{2,3/2} = \frac{(1 + \sqrt{3})^{1/4}}{2^{7/24}},$$

$$r_{2,5/2} = \frac{(\sqrt{\sqrt{5} + 1} + \sqrt{2})^{1/4}}{2^{1/4}}, \quad r_{2,7/2} = \frac{(3 + \sqrt{7})^{1/4}}{2^{3/8}}, \quad r_{2,9/2} = \frac{(1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{1/3}}{2^{13/24}},$$

$$r_{2,25/2} = \frac{5^{1/4} + 1}{2^{5/8}}, \quad r_{2,27/2} = \frac{(1 + \sqrt{3})^{1/12} (1 - \sqrt{3} + 2^{2/3}\sqrt{3})^{1/3}}{2^{3/8} (2^{1/3} - 1)^{1/3}},$$

$$r_{2,63/2} = \frac{(7 - 2\sqrt{3} + \sqrt{21} + (3 + \sqrt{3})\sqrt{3 + 16\sqrt{21} - 27\sqrt{7}})^{1/3}}{2^{13/24} (\sqrt{3} - 1)^{2/3} (3 - \sqrt{7})^{1/12}},$$

$$r_{2,9/4} = \frac{(-1 + 35\sqrt{2} + 28\sqrt{3})^{1/8}}{2^{1/8} (\sqrt{2} + \sqrt{3})^{1/3}},$$

$$\begin{aligned}
r_{2,9/8} &= \frac{2^{5/48}(\sqrt{2}-1)^{5/12}(\sqrt{3}-1)^{1/3}(\sqrt{3}-\sqrt{2})^{1/3}}{(-1-\sqrt{2}+\sqrt{3}+3^{3/4}\sqrt{2}-\sqrt{6})^{1/3}}, \\
r_{3,3} &= 3^{1/12}(3+2\sqrt{3})^{1/12} = \frac{3^{1/8}(1+\sqrt{3})^{1/6}}{2^{1/12}}, \quad r_{3,4} = \sqrt{\frac{\sqrt{3}+1}{\sqrt{2}}}, \\
r_{3,5} &= \left(\frac{\sqrt{5}+1}{2}\right)^{5/6} = \left(\frac{11+5\sqrt{5}}{2}\right)^{1/6}, \quad r_{3,7} = \left(\frac{\sqrt{3}+\sqrt{7}}{2(2-\sqrt{3})}\right)^{1/4}, \\
r_{3,8} &= (\sqrt{2}+1)^{1/3}(\sqrt{2}+\sqrt{3})^{1/4}, \quad r_{3,9} = 3^{1/6}(1+2^{1/3}+2^{2/3})^{1/3} = \frac{3^{1/6}}{(2^{1/3}-1)^{1/3}}, \\
r_{3,18} &= 3^{1/6}(1+\sqrt{2})^{5/18}\left(2+\sqrt{2}(1+\sqrt{2})^{1/3}+(1+\sqrt{2})^{2/3}\right)^{1/3}, \\
r_{3,25} &= \frac{1}{2}\left(1+\sqrt[3]{10}+\sqrt{5+2\sqrt[3]{10}+\sqrt[3]{10^2}}\right), \\
r_{3,49} &= \frac{3+\sqrt[3]{2^2\sqrt[3]{7}}+\sqrt[3]{2\sqrt[3]{7^2}}+\sqrt{49+13\sqrt[3]{2^2\sqrt[3]{7}}+8\sqrt[3]{2\sqrt[3]{7^2}}}}{2\sqrt{3}}, \\
r_{4,4} &= 2^{5/16}(1+\sqrt{2})^{1/4}, \quad r_{4,5} = \left(\frac{1+\sqrt{5}+\sqrt{2}+\sqrt{1+\sqrt{5}}}{2}\right)^{1/2}, \\
r_{4,8} &= 2^{1/4}(1+\sqrt{2})^{3/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}, \quad r_{4,9} = \frac{1}{2}(1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}), \\
r_{4,7} &= (8+3\sqrt{7})^{1/4}, \quad r_{4,9} = \frac{1}{2} + \frac{3^{1/4}}{\sqrt{2}} + \frac{\sqrt{3}}{2}, \quad r_{4,25} = \frac{1}{2}(3+\sqrt[4]{5}+\sqrt{5}+\sqrt[4]{5^3}) = \frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}, \\
r_{4,49} &= \frac{1}{4}\left(\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}+\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}\right)^2, \\
r_{5,5} &= (25+10\sqrt{5})^{1/6} = \sqrt{\frac{5+\sqrt{5}}{2}}, \quad r_{6,6} = \frac{3^{1/8}\sqrt{\sqrt{3}+1}(1+\sqrt{3}+\sqrt{2}3^{3/4})^{1/3}}{2^{13/24}}.
\end{aligned} \tag{2.1}$$

For values of $r_{4,7}$, $r_{4,9}$, and $r_{4,49}$ see [18]. For remaining values we refer to [11] or [17].

We also note that

$$r_{k,1} = 1, \quad r_{k,1/n} = \frac{1}{r_{k,n}}, \quad r_{k,n} = r_{n,k}. \tag{2.2}$$

Lemma 2.2. *One has*

$$\begin{aligned}
 \text{(i)} \quad & h_{1,1} = 1, \\
 \text{(ii)} \quad & h_{2,2} = \sqrt{2\sqrt{2} - 2}, \\
 \text{(iii)} \quad & h_{3,3} = \left(2\sqrt{3} - 3\right)^{1/4} = \frac{3^{1/8}\sqrt{\sqrt{3} - 1}}{2^{1/4}}, \\
 \text{(iv)} \quad & h_{4,4} = \frac{2^{3/4}}{\sqrt[4]{2} + 1}, \\
 \text{(v)} \quad & h_{5,5} = \sqrt{5 - 2\sqrt{5}}, \\
 \text{(vi)} \quad & h_{6,6} = \frac{2^{3/4}3^{1/8}\left((\sqrt{2} - 1)(\sqrt{3} - 1)\right)^{1/6}}{\left(-4 + 3\sqrt{2} + 3^{5/4} + 2\sqrt{3} - 3^{3/4} + 2\sqrt{2} \cdot 3^{3/4}\right)^{1/3}}.
 \end{aligned} \tag{2.3}$$

We refer to [19, page 19, Theorem 5.4] or [11, page 150, Theorem 9.2.4] for proofs of the above assertions.

Lemma 2.3. *One has*

$$\begin{aligned}
 \text{(i)} \quad & h'_{1,1} = 1, \\
 \text{(ii)} \quad & h'_{2,2} = 2^{1/16}(\sqrt{2} - 1)^{1/4}, \\
 \text{(iii)} \quad & h'_{3,3} = \frac{2^{1/3}3^{1/8}(\sqrt{3} - 1)^{1/6}}{\left(1 + \sqrt{3} + \sqrt{2}\sqrt[4]{3^3}\right)^{1/3}}, \\
 \text{(iv)} \quad & h'_{4,4} = \frac{2^{1/4}}{\left(16 + 15\sqrt[4]{2} + 12\sqrt{2} + 9\sqrt[4]{2^3}\right)^{1/8}}, \\
 \text{(v)} \quad & h'_{5,5} = \frac{1}{2}(\sqrt[4]{5} - 1)\sqrt{5 + \sqrt{5}}, \\
 \text{(vi)} \quad & h'_{6,6} = \frac{2^{1/4}3^{1/8}(\sqrt{2} - 1)^{1/12}(\sqrt{3} + 1)^{1/6}(-1 - \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{1/3}}{\left(2 - 3\sqrt{2} + 3^{5/4} + 3^{3/4}\right)^{1/3}}, \\
 \text{(vii)} \quad & h'_{3,1} = 2^{-1/4}\sqrt{\sqrt{3} - 1}.
 \end{aligned} \tag{2.4}$$

For proofs (i)–(vi), see [19, page 21, Theorem 5.6] or [11, page 152, Theorem 9.2.6]. For proof of (vii), see [19, page 15, Theorem 4.11] or [11, page 145, Theorems 9.1.10].

Lemma 2.4. *One has*

$$\begin{aligned}
 \text{(i)} \quad & g'_{1,1} = 1, \\
 \text{(ii)} \quad & g'_{2,2} = 2^{3/8}, \\
 \text{(iii)} \quad & g'_{3,3} = \frac{3^{1/3} \left(1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4}\right)^{1/3} \left(1 + \sqrt{3}\right)^{1/6}}{\sqrt{2}}, \\
 \text{(iv)} \quad & g'_{4,4} = 2^{3/8} \left(1 + \sqrt{2}\right)^{1/2}, \\
 \text{(v)} \quad & g'_{5,5} = \frac{\left(5 + \sqrt{5}\right)^{1/2} \left(5^{1/4} + 1\right)}{2}, \\
 \text{(vi)} \quad & g'_{6,6} = \frac{3^{1/8} \left(1 + \sqrt{3}\right)^{5/6} \left(1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4}\right)^{2/3}}{2^{29/24}}, \\
 \text{(vii)} \quad & g'_{9,9} = \frac{a + (2(b - 2c))^{1/3} + (2(b + 2c))^{1/3}}{2},
 \end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
 a &= 2 + \sqrt{2} \cdot 3^{1/4} + 2\sqrt{3} + \sqrt{2} \cdot 3^{3/4}, & b &= 82 + 45\sqrt{2} + 48\sqrt{3} + 25\sqrt{2} \cdot 3^{3/4}, \\
 c &= \sqrt{3 \left(88 + 47\sqrt{2} \cdot 3^{1/4} + 50\sqrt{3} + 27\sqrt{2} \cdot 3^{3/4}\right)}.
 \end{aligned} \tag{2.6}$$

For proofs we refer to [17, page 1781, Theorem 6.7].

3. Theorems on Explicit Evaluation of $a(q)$, $b(q)$, and $c(q)$

In this section, we present some general formulas for the explicit evaluations of cubic theta-functions and their quotients by parameterizations given in Section 1. In the process, we also establish some transformation formulas of quotients of cubic theta-functions.

Theorem 3.1. *For any positive real number n , one has*

$$\frac{b\left(e^{-2\pi\sqrt{n/3}}\right)}{c\left(e^{-2\pi\sqrt{n/3}}\right)} = r_{3,n}^4 = \mu_n^{2/3}, \tag{3.1}$$

where $r_{k,n}$ and μ_n are as defined in (1.16) and (1.19), respectively.

Proof. Using the definitions of $b(q)$ and $c(q)$ from (1.7), one has

$$\frac{3b(q)}{c(q)} = \left(\frac{f(-q)}{q^{1/12}f(-q^3)} \right)^4. \quad (3.2)$$

Setting $q = e^{-2\pi\sqrt{n/3}}$ and then employing the definitions of $r_{k,n}$ and μ_n , we finish the proof. $\square$

Remark 3.2. Replacing n by $1/n$ in Theorem 3.1 and noting that $r_{3,1/n} = 1/r_{3,n}$ from (2.2), we also have

$$\frac{b\left(e^{-2\pi\sqrt{n/3}}\right)}{c\left(e^{-2\pi\sqrt{n/3}}\right)} = \frac{c\left(e^{-2\pi/\sqrt{3n}}\right)}{b\left(e^{-2\pi/\sqrt{3n}}\right)}. \quad (3.3)$$

Thus, if we know the value of one quotient of (3.3), then the other quotient follows readily.

From Theorem 3.1 and (1.6), the following theorem is apparent.

Theorem 3.3. *One has*

$$\frac{a\left(e^{-2\pi\sqrt{n/3}}\right)}{c\left(e^{-2\pi\sqrt{n/3}}\right)} = \left(r_{3,n}^{12} + 1\right)^{1/3}. \quad (3.4)$$

Theorem 3.4. *For any positive real number n , one has*

$$\frac{b\left(e^{-2\pi\sqrt{n}}\right)}{c\left(e^{-2\pi\sqrt{n}/3}\right)} = \frac{r_{9,n}}{\sqrt{3}}. \quad (3.5)$$

Proof. From the definitions $b(q)$ and $c(q)$ in (1.7), we observe that

$$\frac{b(q^3)}{c(q)} = \frac{f(-q)}{3q^{1/3}f(-q^9)}. \quad (3.6)$$

Setting $q = e^{-2\pi\sqrt{n}/3}$ in (3.6) and then employing the definition of $r_{k,n}$, we arrive at the desired result. $\square$

Remark 3.5. Noting that $r_{9,1/n} = 1/r_{9,n}$ from (2.2) and using Theorem 3.4, we find that

$$\frac{3b\left(e^{-2\pi\sqrt{n}}\right)}{c\left(e^{-2\pi\sqrt{n}/3}\right)} = \frac{c\left(e^{-2\pi/3\sqrt{n}}\right)}{b\left(e^{-2\pi/\sqrt{n}}\right)}. \quad (3.7)$$

Now, from (3.7), it is obvious that if we know the value of one quotient then the other quotient can easily be evaluated.

In the next theorem, we give a relation between $c(q)$ and the parameter $h_{k,n}$ as defined in (1.21).

Theorem 3.6. *For any positive real number n , one has*

$$\frac{c\left(e^{-8\pi\sqrt{n/3}}\right)}{c\left(e^{-2\pi\sqrt{n/3}}\right)} = \frac{1}{4} \left(1 - \sqrt{3} \left(h'_{3,n}\right)^2\right). \quad (3.8)$$

Proof. From [10, page 111, Lemma 5.5], we note that

$$1 - 4 \frac{c(q^4)}{c(q)} = \left(\frac{\phi(-q)}{\phi(-q^3)} \right)^2. \quad (3.9)$$

Now applying the definition of $h'_{k,n}$, with $k = 3$, in (3.9), we complete the proof. $\square$

The next theorem connects $a(q)$ with the parameter $r_{k,n}$ defined in (1.16).

Theorem 3.7. *For any positive real number n , one has*

$$a^{12}\left(e^{-2\pi\sqrt{n/3}}\right) = \frac{27\left(r_{3,n}^{12} + 1\right)^4 e^{-2\pi\sqrt{n/3}} f^{24}\left(-e^{-2\pi\sqrt{n/3}}\right)}{r_{3,n}^{36}}. \quad (3.10)$$

Proof. From [20, page 196, (2.9)], we note that

$$27qf^{24}(-q) = a^{12}(q)(1 - \alpha(q))^3 \alpha(q), \quad (3.11)$$

where $\alpha(q) = c^3(q)/a^3(q)$.

Setting $q = e^{-2\pi\sqrt{n/3}}$ and then applying (3.3) in (3.11), we obtain

$$27e^{-2\pi\sqrt{n/3}} f^{24}\left(-e^{-2\pi\sqrt{n/3}}\right) = a^{12}\left(e^{-2\pi\sqrt{n/3}}\right) \left(1 - \frac{1}{r_{3,n}^{12} + 1}\right)^3 \left(\frac{1}{r_{3,n}^{12} + 1}\right), \quad (3.12)$$

which on simplification gives the required result. $\square$

Theorem 3.8. *One has*

$$a\left(e^{-3n\pi}\right) = \frac{1}{3} \left\{ a\left(e^{-n\pi}\right) + 2b\left(e^{-n\pi}\right) \right\}. \quad (3.13)$$

Proof. From [7, page 93, (2.8)], one has

$$b(q) = \frac{1}{2} \left\{ 3a\left(q^3\right) - a(q) \right\}. \quad (3.14)$$

Setting $q = e^{-n\pi}$ in (3.14), we readily complete the proof. $\square$

Theorem 3.9. For any positive real number n , one has

$$\begin{aligned} \text{(i)} \quad b(e^{-n\pi}) &= \frac{f^3(-e^{-n\pi})}{f(-e^{-3n\pi})}, \\ \text{(ii)} \quad b(-e^{-n\pi}) &= \frac{f^3(e^{-n\pi})}{f(e^{-3n\pi})}. \end{aligned} \quad (3.15)$$

Proof. Setting $q = e^{-n\pi}$ and $q = -e^{-n\pi}$ in (1.7), we readily arrive at (i) and (ii), respectively. $\square$

Theorem 3.10. For all positive real numbers n , one has

$$\begin{aligned} \text{(i)} \quad b\left(e^{-2\pi\sqrt{n/3}}\right) &= 3^{1/4} e^{-\pi\sqrt{n}/6\sqrt{3}} f^2\left(-e^{-2\pi\sqrt{n/3}}\right) r_{3,n}, \\ \text{(ii)} \quad b\left(-e^{-\pi\sqrt{n/3}}\right) &= 3^{1/4} e^{-\pi\sqrt{n}/12\sqrt{3}} f^2\left(e^{-\pi\sqrt{n/3}}\right) r'_{3,n}, \end{aligned} \quad (3.16)$$

where the parameters $r_{3,n}$ and $r'_{3,n}$ are defined in (1.16) and (1.17), respectively.

Proof. We rewrite $b(q)$ in (1.7) as

$$b(q) = f^2(-q) q^{1/12} \frac{f(-q)}{q^{1/12} f(-q^3)}. \quad (3.17)$$

Setting $q = e^{-2\pi\sqrt{n/3}}$ and employing the definition of $r_{3,n}$, we arrive at (i). To prove (ii), we replace q by $-q$ in (3.17) and then use the definition of $r'_{3,n}$. $\square$

Theorem 3.11. For all positive real number n , we have

$$\begin{aligned} \text{(i)} \quad c(e^{-n\pi}) &= 3e^{-n\pi/3} \frac{f^3(-e^{-3n\pi})}{f(-e^{-n\pi})}, \\ \text{(ii)} \quad c(-e^{-n\pi}) &= -3e^{-n\pi/3} \frac{f^3(e^{-3n\pi})}{f(e^{-n\pi})}. \end{aligned} \quad (3.18)$$

Proof. It follows readily from (1.7) with $q = e^{-n\pi}$ and $q = -e^{-n\pi}$. $\square$

Theorem 3.12. For all positive real number n , one has

$$c\left(e^{-2\pi\sqrt{n/3}}\right) = \frac{3^{3/4} e^{-\pi\sqrt{n}/2\sqrt{3}} f^2\left(-e^{-2\pi\sqrt{n/3}}\right)}{r_{3,n}}. \quad (3.19)$$

Proof. We set $q = e^{-2\pi\sqrt{n/3}}$ in (1.7) and then employ the definition of the parameter $r_{k,n}$ to finish the proof. $\square$

4. Explicit Values of $a(q)$, $b(q)$, and $c(q)$

In this section, we find explicit values of cubic theta-functions and their quotients by using the results established in the previous section.

Theorem 4.1. *One has*

$$(i) \quad \frac{b\left(e^{-2\pi/\sqrt{3}}\right)}{c\left(e^{-2\pi/\sqrt{3}}\right)} = 1,$$

$$(ii) \quad \frac{b\left(e^{-2\pi\sqrt{2/3}}\right)}{c\left(e^{-2\pi\sqrt{2/3}}\right)} = \left(1 + \sqrt{2}\right)^{2/3},$$

$$(iii) \quad \frac{b\left(e^{-2\pi}\right)}{c\left(e^{-2\pi}\right)} = \frac{3^{1/2}\left(1 + \sqrt{3}\right)^{2/3}}{2^{1/3}},$$

$$(iv) \quad \frac{b\left(e^{-4\pi/\sqrt{3}}\right)}{c\left(e^{-4\pi/\sqrt{3}}\right)} = \left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right)^2,$$

$$(v) \quad \frac{b\left(e^{-2\pi\sqrt{5/3}}\right)}{c\left(e^{-2\pi\sqrt{5/3}}\right)} = \left(\frac{1 + \sqrt{5}}{2}\right)^{10/3},$$

$$(vi) \quad \frac{b\left(e^{-2\pi\sqrt{7/3}}\right)}{c\left(e^{-2\pi\sqrt{7/3}}\right)} = \frac{\sqrt{3} + \sqrt{7}}{2(2 - \sqrt{3})},$$

$$(vii) \quad \frac{b\left(e^{-4\pi\sqrt{2/3}}\right)}{c\left(e^{-4\pi\sqrt{2/3}}\right)} = \left(1 + \sqrt{2}\right)^{4/3}(\sqrt{2} + \sqrt{3}),$$

$$(viii) \quad \frac{b\left(e^{-2\pi\sqrt{3}}\right)}{c\left(e^{-2\pi\sqrt{3}}\right)} = \frac{3^{2/3}}{(2^{1/3} - 1)^{4/3}},$$

$$(ix) \quad \frac{b\left(e^{-6\pi\sqrt{2/3}}\right)}{c\left(e^{-6\pi\sqrt{2/3}}\right)} = 3^{2/3}\left(1 + \sqrt{2}\right)^{10/9}\left(2 + \sqrt{2}\left(1 + \sqrt{2}\right)^{1/3} + \left(1 + \sqrt{2}\right)^{2/3}\right)^{4/3},$$

$$(x) \quad \frac{b\left(e^{-2\pi\sqrt{13/3}}\right)}{c\left(e^{-2\pi\sqrt{13/3}}\right)} = \left(\frac{\sqrt{11 + \sqrt{13}} + \sqrt{3 + \sqrt{13}}}{2\sqrt{2}}\right)^4,$$

$$\begin{aligned}
\text{(xi)} \quad \frac{b\left(e^{-10\pi/\sqrt{3}}\right)}{c\left(e^{-10\pi/\sqrt{3}}\right)} &= \frac{1}{16} \left(1 + \sqrt[3]{10} + \sqrt{5 + 2\sqrt[3]{10} + \sqrt[3]{10^2}}\right)^4, \\
\text{(xi)} \quad \frac{b\left(e^{-14\pi/\sqrt{3}}\right)}{c\left(e^{-14\pi/\sqrt{3}}\right)} &= \left(\frac{3 + \sqrt[3]{2^2\sqrt[3]{7}} + \sqrt[3]{2}\sqrt[3]{7^2} + \sqrt{49 + 13\sqrt[3]{2^2\sqrt[3]{7}} + 8\sqrt[3]{2}\sqrt[3]{7^2}}}{2\sqrt{3}}\right)^4.
\end{aligned}
\tag{4.1}$$

Proof. It follows directly from Theorem 3.1 and the corresponding values of $r_{3,n}$ listed in Lemma 2.1. $\square$

More values can be calculated by employing Theorem 3.1 and the corresponding values of μ_n evaluated in [15, 16].

Theorem 4.2. *One has*

$$\begin{aligned}
\text{(i)} \quad \frac{a\left(e^{-2\pi/\sqrt{3}}\right)}{c\left(e^{-2\pi/\sqrt{3}}\right)} &= \sqrt[3]{2}, \\
\text{(ii)} \quad \frac{a\left(e^{-2\pi\sqrt{2/3}}\right)}{c\left(e^{-2\pi\sqrt{2/3}}\right)} &= 2^{1/3} (2 + \sqrt{2})^{1/3}, \\
\text{(iii)} \quad \frac{a\left(e^{-2\pi}\right)}{c\left(e^{-2\pi}\right)} &= \left(\frac{3^{3/2}(1 + \sqrt{3})^2}{2} + 1\right)^{1/3}, \\
\text{(iv)} \quad \frac{a\left(e^{-4\pi/\sqrt{3}}\right)}{c\left(e^{-4\pi/\sqrt{3}}\right)} &= \left(\left(\frac{(1 + \sqrt{3})}{\sqrt{2}}\right)^6 + 1\right)^{1/3}, \\
\text{(v)} \quad \frac{a\left(e^{-2\pi\sqrt{5/3}}\right)}{c\left(e^{-2\pi\sqrt{5/3}}\right)} &= \left(\left(\frac{1 + \sqrt{5}}{2}\right)^{10} + 1\right)^{1/3}, \\
\text{(vi)} \quad \frac{a\left(e^{-2\pi\sqrt{7/3}}\right)}{c\left(e^{-2\pi\sqrt{7/3}}\right)} &= \left(\left(\frac{\sqrt{3} + \sqrt{7}}{2(2 - \sqrt{3})}\right)^3 + 1\right)^{1/3}, \\
\text{(vii)} \quad \frac{a\left(e^{-4\pi\sqrt{2/3}}\right)}{c\left(e^{-4\pi\sqrt{2/3}}\right)} &= \left((1 + \sqrt{2})^4 (\sqrt{2} + \sqrt{3})^4 + 1\right)^{1/3},
\end{aligned}$$

$$\begin{aligned}
\text{(viii)} \quad \frac{a(e^{-10\pi/\sqrt{3}})}{c(e^{-10\pi/\sqrt{3}})} &= \left(\left(\frac{1 + \sqrt[3]{16} + \sqrt{5 + 2\sqrt[3]{10} + \sqrt[3]{10^2}}}{2} \right)^{12} + 1 \right)^{1/3}, \\
\text{(ix)} \quad \frac{a(e^{-6\pi/\sqrt{3}})}{c(e^{-6\pi/\sqrt{3}})} &= \left(\frac{9}{(2^{1/4} - 1)^4} + 1 \right)^{1/3}, \\
\text{(x)} \quad \frac{a(e^{-14\pi/\sqrt{3}})}{c(e^{-14\pi/\sqrt{3}})} &= \left(\left(\frac{3 + \sqrt[3]{2}\sqrt[3]{7} + \sqrt[3]{2}\sqrt[3]{7^2} + \sqrt{49 + 13\sqrt[3]{2}\sqrt[3]{7} + 8\sqrt[3]{2}\sqrt[3]{7^2}}}{2\sqrt{3}} \right) + 1 \right)^{1/3}, \\
\text{(xi)} \quad \frac{a(e^{-6\pi})}{c(e^{-6\pi})} &= \left(3^4(1 + \sqrt{2})^{10/3} \left(2 + \sqrt{2}(1 + \sqrt{2})^4 + (1 + \sqrt{2})^8 \right) + 1 \right)^{1/3}.
\end{aligned} \tag{4.2}$$

Proof. It follows easily from (3.3) and the corresponding values of $r_{3,n}$ listed in Lemma 2.1. $\square$

Theorem 4.3. *One has*

$$\begin{aligned}
\text{(i)} \quad \frac{b(e^{-2\pi})}{c(e^{-2\pi/3})} &= \frac{1}{\sqrt{3}}, \\
\text{(ii)} \quad \frac{b(e^{-2\pi\sqrt{2}})}{c(e^{-2\pi\sqrt{2}/3})} &= \frac{(\sqrt{3} + \sqrt{2})^{1/3}}{\sqrt{3}}, \\
\text{(iii)} \quad \frac{b(e^{-2\pi\sqrt{3}})}{c(e^{-2\pi/\sqrt{3}})} &= \left(\frac{1}{3(\sqrt[3]{2} - 1)} \right)^{1/3}, \\
\text{(iv)} \quad \frac{b(e^{-4\pi})}{c(e^{-4\pi/3})} &= \frac{1}{2\sqrt{3}}(1 + \sqrt{2} \cdot 3^{1/4} + \sqrt{3}), \\
\text{(v)} \quad \frac{b(e^{-2\sqrt{5}\pi})}{c(e^{-2\sqrt{5}\pi/3})} &= \frac{1}{\sqrt{3}}(104 + 60\sqrt{3} + 45\sqrt{5} + 26\sqrt{15})^{1/6}.
\end{aligned} \tag{4.3}$$

Proof. It follows from Theorem 3.4 and the corresponding values of $r_{9,n}$ in listed in Lemma 2.1. $\square$

Theorem 4.4. *One has*

$$\begin{aligned}
 \text{(i)} \quad & \frac{c(e^{-8\pi/\sqrt{3}})}{c(e^{-2\pi/\sqrt{3}})} = \frac{1}{4} \left(\frac{\sqrt{2} + \sqrt{3} - 3}{\sqrt{2}} \right), \\
 \text{(ii)} \quad & \frac{c(e^{-8\pi})}{c(e^{-2\pi})} = \frac{1}{4} \left(1 - \sqrt{3} \left(\frac{2^{2/3} 3^{1/4} (\sqrt{3} - 1)^{1/3}}{(1 + \sqrt{3} + \sqrt{2} \sqrt[4]{3^3})^{2/3}} \right) \right).
 \end{aligned} \tag{4.4}$$

Proof. We set $n = 1$ and 3 in Theorem 3.6 and then employ the values of $h'_{3,1}$ and $h'_{3,3}$ from Lemma 2.3(vii) and (iii), respectively, to finish the proof. $\square$

For the remaining part of this paper, we set $a := \phi(e^{-\pi}) = \pi^{1/4}/\Gamma(3/4)$.

Lemma 4.5. *One has*

$$\begin{aligned}
 \text{(i)} \quad & f(-e^{-\pi}) = a 2^{-3/8} e^{\pi/24}, \\
 \text{(ii)} \quad & f(e^{-\pi}) = a 2^{-1/4} e^{\pi/24}, \\
 \text{(iii)} \quad & f(-e^{-2\pi}) = a 2^{-1/2} e^{\pi/12}, \\
 \text{(iv)} \quad & f(-e^{-3\pi}) = \frac{a e^{\pi/4} (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{1/3}}{3^{3/8} 2^{17/24} (1 + \sqrt{3})^{1/6}}, \\
 \text{(v)} \quad & f(-e^{-4\pi}) = a 2^{-7/8} e^{\pi/6}, \\
 \text{(vi)} \quad & f(-e^{-6\pi}) = a 2^{-7/12} 3^{-3/8} e^{\pi/4} (\sqrt{3} - 1)^{1/4}, \\
 \text{(vii)} \quad & f(-e^{-12\pi}) = \frac{a e^{\pi/2}}{2^{5/24} 3^{3/8} \sqrt{1 + \sqrt{3}} (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{1/3}}, \\
 \text{(viii)} \quad & f(-e^{-\pi/3}) = \frac{a 2^{7/24} 3^{1/8} e^{\pi/72}}{\sqrt{1 + \sqrt{3}} (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{1/3}}, \\
 \text{(ix)} \quad & f(-e^{-2\pi/3}) = a 2^{-7/12} 3^{1/8} e^{\pi/36} (\sqrt{3} - 1)^{1/6}, \\
 \text{(x)} \quad & f(e^{-2\pi}) = a 2^{-13/16} e^{\pi/12} (\sqrt{2} + 1)^{1/4}, \\
 \text{(xi)} \quad & f(e^{-3\pi}) = a 2^{-1/3} 3^{-3/8} e^{\pi/8} (\sqrt{3} + 1)^{1/6}, \\
 \text{(xii)} \quad & f(e^{-6\pi}) = \frac{a e^{\pi/4} (2 - 3\sqrt{2} + 3^{5/4} + 3^{3/4})^{1/3}}{2^{15/16} 3^{3/8} (\sqrt{2} - 1)^{1/12} (\sqrt{3} + 1)^{1/6}}.
 \end{aligned} \tag{4.5}$$

For a proof of the lemma, we refer to [7, page 326, Entry 6] and [11, page 125–129].

Theorem 4.6. *One has*

$$\begin{aligned}
 \text{(i)} \quad b(e^{-\pi}) &= \frac{a^2 3^{3/8} (1 + \sqrt{3})^{1/6}}{2^{5/12} (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{1/3}}, \\
 \text{(ii)} \quad b(e^{-2\pi}) &= \frac{a^2 3^{3/8}}{2^{11/12} (\sqrt{3} - 1)^{1/6}}, \\
 \text{(iii)} \quad b(e^{-4\pi}) &= a^2 2^{-29/12} 3^{3/8} (1 + \sqrt{3})^{1/2} (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{1/3}, \\
 \text{(iv)} \quad b(e^{-\pi/3}) &= \frac{a^2 2^{5/4} 3^{3/8}}{(1 + \sqrt{3})^{3/2} (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})}, \\
 \text{(v)} \quad b(e^{-2\pi/3}) &= \frac{a^2 3^{1/8} (\sqrt{3} - 1)^{1/3}}{2^{13/12} (\sqrt{3} + 1)^{1/6}}, \\
 \text{(vi)} \quad b(-e^{-\pi}) &= \frac{a^2 3^{3/8}}{2^{5/12} (\sqrt{3} + 1)^{1/6}}, \\
 \text{(vii)} \quad b(-e^{-2\pi}) &= \frac{a^2 3^{3/8} (\sqrt{2} + 1)^{3/4} (\sqrt{2} - 1)^{1/12} (\sqrt{3} + 1)^{1/6}}{2^{3/2} (2 - 3\sqrt{2} + 3^{5/4} + 3^{3/4})^{1/3}}.
 \end{aligned} \tag{4.6}$$

Proof. To prove (i)–(v), we set $n = 1, 2, 4, 1/3$, and $2/3$, respectively, in Theorem 3.9(i) and use the corresponding values of $f(\pm e^{-\pi n})$ from Lemma 4.5.

To prove (vi) and (vii), we set $n = 1$ and 2 , respectively, in Theorem 3.9(ii) and then use the corresponding values $f(\pm e^{-\pi n})$ from Lemma 4.5. $\square$

Theorem 4.7. *One has*

$$\begin{aligned}
 \text{(i)} \quad c(e^{-4\pi/3}) &= \frac{a^2 3^{7/8} (1 + \sqrt{3})^{1/6}}{2^{17/12} (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{1/3}}, \\
 \text{(ii)} \quad c(e^{-2\pi/3}) &= a^2 2^{-13/12} 3^{7/8} (\sqrt{3} + 1)^{1/6}, \\
 \text{(iii)} \quad c(e^{-\pi/3}) &= 2^{-17/12} 3^{7/8} a^2 (1 + \sqrt{3})^{1/2} (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{1/3}, \\
 \text{(iv)} \quad c(e^{-4\pi}) &= \frac{a^2 3^{3/8}}{2^{1/4} (1 + \sqrt{3})^{3/2} (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})},
 \end{aligned}$$

$$\begin{aligned}
\text{(v)} \quad c(e^{-2\pi}) &= \frac{a^2(\sqrt{3}-1)^{1/3}}{3^{3/8}2^{13/12}(\sqrt{3}+1)^{1/6}}, \\
\text{(vi)} \quad c(e^{-\pi}) &= \frac{a^2(1+\sqrt{3}+\sqrt{2} \cdot 3^{3/4})}{3^{1/8}2^{7/4}(1+\sqrt{3})^{1/2}}, \\
\text{(vii)} \quad c(e^{-8\pi}) &= \frac{1}{4} \left(1 - \sqrt{3} \left(\frac{2^{2/3} \cdot 3^{1/4}(\sqrt{3}-1)^{1/3}}{(1+\sqrt{3}+\sqrt{2}+\sqrt[4]{3^2})^{2/3}} \right) \right) \times \frac{a^2(\sqrt{3}-1)^{1/3}}{3^{3/8} \cdot 2^{13/12}(\sqrt{3}+1)^{1/6}}.
\end{aligned} \tag{4.7}$$

Proof. To prove (i)–(v), we set $t = 1/2, 1, 2, 1/6$, and $1/3$, respectively, in (1.9) and then apply the corresponding values of $b(e^{-n\pi})$ from Theorem 4.6.

To prove (vi), we set $n = 1$ in Theorem 3.11 and use the corresponding values of $f(-e^{-n\pi})$ from Lemma 4.5. At last, (vii) follows from Theorems 4.7(v) and 4.4(ii). $\square$

Remark 4.8. Setting $t = 1/2$ in (1.10) and then employing the value of $c(e^{-\pi})$ from Theorem 4.7(vi), we can also evaluate $b(e^{-4\pi/3})$.

Theorem 4.9. *One has*

$$\begin{aligned}
\text{(i)} \quad a(e^{-2\pi}) &= \frac{a^2(10+6\sqrt{3})^{1/3}}{2(3+2\sqrt{3})^{1/4}}, \\
\text{(ii)} \quad a(e^{-2\pi/3}) &= \frac{a^2\sqrt{3}(10+6\sqrt{3})^{1/3}}{2(3+2\sqrt{3})^{1/4}}, \\
\text{(iii)} \quad a(e^{-6\pi}) &= \frac{a^2}{3} \left(3^{1/4}(1+\sqrt{2})^{1/6} + \frac{(10+6\sqrt{3})^{1/3}}{2(3+2\sqrt{3})^{1/4}} \right), \\
\text{(iv)} \quad a(e^{-2\pi/9}) &= \sqrt{3}a^2 \left\{ \left(3^{1/4}(1+\sqrt{2})^{1/6} + \frac{(10+6\sqrt{3})^{1/3}}{2(3+2\sqrt{3})^{1/4}} \right) \right\}.
\end{aligned} \tag{4.8}$$

Proof. To prove (i), we set $n = 3$ in Theorem 3.7 and use $f(-e^{-2\pi})$ from Lemma 4.5 and the values of $r_{3,3}$ from Lemma 2.1.

To prove (ii), we set $t = 1$ in (1.8) and then employ Theorem 4.9(i).

To prove (iii), we set $n = 2$ in Theorem 3.8 and then employ the values of $a(e^{-2\pi})$ and $b(e^{-2\pi})$ from Theorems 4.9(i) and 4.6(ii), respectively.

To prove (iv), we set $t = 3$ in (1.8) and use the value of $a(e^{-6\pi})$. $\square$

5. Theorems on Explicit Evaluations of $A(q)$, $B(q)$, and $C(q)$

In this section, we use the parameters $r_{k,n}$, $h_{k,n}$, and $g'_{k,n}$ defined in (1.16), (1.20), and (1.23), respectively, to establish some formulas for the explicit evaluations of quartic theta-functions and their quotients.

Theorem 5.1. *For any positive real number n , one has*

$$\frac{B(e^{-\pi\sqrt{2n}})}{C(e^{-\pi\sqrt{2n}})} = r_{2,n}^{12} = g_{2n}^{12}. \quad (5.1)$$

Proof. Employing the definition of $B(q)$ and $C(q)$ given in (1.14), we find that

$$\frac{B(q)}{C(q)} = \left(\frac{f(-q)}{2^{1/4} q^{1/24} f(-q^2)} \right)^{12}. \quad (5.2)$$

Setting $q = e^{-2\pi\sqrt{n/2}}$ in (5.2) and then using the definition of $r_{k,n}$, we arrive at the first equality. Second equality readily follows from (1.24) and (5.2). $\square$

Remark 5.2. From Theorem 5.1 and (2.2), we have the following transformation formula:

$$\frac{B(e^{-\pi\sqrt{2/n}})}{C(e^{-\pi\sqrt{2/n}})} = \frac{C(e^{-\pi\sqrt{2n}})}{B(e^{-\pi\sqrt{2n}})}. \quad (5.3)$$

Thus, if we know the value of one of the quotient of (5.3), then the other one follows immediately.

Theorem 5.3. *One has*

$$\frac{B(e^{-2\pi\sqrt{n}})}{C(e^{-\pi\sqrt{n}})} = \frac{r_{4,n}^4}{2}. \quad (5.4)$$

Proof. Theorem follows easily from (1.14) and the definition of $r_{k,n}$ with $k = 4$. $\square$

Remark 5.4. Using the fact that $r_{4,1/n} = 1/r_{4,n}$ in Theorem 5.3, we have the following transformation formula

$$\frac{4B(e^{-2\pi/\sqrt{n}})}{C(e^{-\pi/\sqrt{n}})} = \frac{C(e^{-\pi\sqrt{n}})}{B(e^{-2\pi\sqrt{n}})}. \quad (5.5)$$

Hence, if we know one quotient of (5.5) then the other quotient follows immediately.

Lemma 5.5. *One has*

$$\begin{aligned}
 \text{(i)} \quad \phi\left(-e^{-2n\pi}\right) &= \frac{a}{2^{1/8}n^{1/4}h'_{n,n}} = \frac{ar_{2,2n^2}}{n^{1/4}2^{1/4}r_{n,n}}, \\
 \text{(ii)} \quad \phi\left(e^{-n\pi}\right) &= \frac{a}{n^{1/4}h_{n,n}} = \frac{aG_{n^2}^2}{n^{1/4}r_{n,n}}, \\
 \text{(iii)} \quad \psi\left(e^{-n\pi}\right) &= \frac{a2^{-5/8}e^{n\pi/8}}{n^{1/4}g'_{n,n}} = \frac{a2^{-3/4}e^{n\pi/8}}{n^{1/4}r_{2,n^2/2}r_{n,n}}, \\
 \text{(iv)} \quad \psi\left(e^{-\pi/n}\right) &= \frac{an^{1/4}2^{-3/4}r_{2,2n^2}e^{n\pi/8}}{r_{n,n}},
 \end{aligned} \tag{5.6}$$

where the parameters $r_{k,n}$, $h_{k,n}$, $h'_{k,n}$, $g'_{k,n}$, and G_n are as defined in (1.16), (1.20), (1.21), (1.23), and (1.24), respectively.

For proofs of (i) and (ii), we refer to [11, page 150] or [19]. For proofs of (iii) and (iv), we refer to [17, Theorem 6.2(ii)] and [17, Theorem 6.3(ii)], respectively.

Theorem 5.6. *For any positive real number n , one has*

$$\begin{aligned}
 \text{(i)} \quad B\left(e^{-2\pi n}\right) &= \frac{a^4}{\sqrt{2}n h_{n,n}^4} = \frac{a^4 r_{2,2n^2}^4}{2nr_{n,n}^4}, \\
 \text{(ii)} \quad B\left(e^{-2\pi/n}\right) &= \frac{a^4 n r_{2,2/n^2}^4}{2r_{n,n}^4},
 \end{aligned} \tag{5.7}$$

where $h'_{k,n}$ is as defined in (1.21).

Proof. From [21, page 39, Entry 24(iii)], we note that

$$\phi(-q) = \frac{f^2(-q)}{f(-q^2)}. \tag{5.8}$$

Employing (5.8) in (1.14), we obtain

$$B(q) = \phi^4(-q). \tag{5.9}$$

Setting $q = e^{-2\pi n}$ in (5.9) and then employing Lemma 5.5(i), we arrive at (i).

To prove (ii), we replace n by $1/n$ in (i) and employ the result $r_{1/n,1/n} = r_{n,n}$, which is easily derivable from (2.2). $\square$

Theorem 5.7. *One has*

$$\begin{aligned} \text{(i)} \quad B(-e^{-n\pi}) &= \frac{a^4}{nh_{n,n}^4} = \frac{aG_{n^2}^8}{nr_{n,n}^4}, \\ \text{(ii)} \quad B(-e^{-\pi/n}) &= \frac{na^4}{h_{n,n}^4} = \frac{a^4nG_{n^2}^8}{r_{n,n}^4}, \end{aligned} \quad (5.10)$$

where $h'_{k,n}$ is as defined in (1.21).

Proof. Replacing q by $-q$ in (5.9) and setting $q = e^{-n\pi}$, we have

$$B(-e^{-n\pi}) = \phi^4(e^{-n\pi}), \quad (5.11)$$

Employing Lemma 5.5(ii) in (5.11), we finish the proof of (i).

To prove (ii), we replace n by $1/n$ in (i) and use the results $h_{n,n} = h_{1/n,1/n}$ [19] and $G_{1/n} = G_n$. $\square$

Remark 5.8. The following transformation formula is apparent from Theorem 5.7(i) and (ii),

$$n^2 B(-e^{-n\pi}) = B(-e^{-\pi/n}). \quad (5.12)$$

Theorem 5.9. *For any positive real number n , one has*

$$C(e^{-n\pi}) = \frac{\sqrt{2}a^4 e^{n\pi/2}}{ng_{n,n}^4}, \quad (5.13)$$

where $g'_{k,n}$ is as defined in (1.23).

Proof. From [21, page 39, Entry 24(iii)], we notice that

$$\psi(q) = \frac{f^2(-q^2)}{f(-q)}. \quad (5.14)$$

Thus, from (5.14) and (1.14), we find that

$$C(e^{-n\pi}) = 8e^{-n\pi/2} \psi^4(e^{-n\pi}), \quad (5.15)$$

Setting $q = e^{-n\pi}$ in (5.15) and employing Lemma 5.5(iii), we easily complete the proof. $\square$

Theorem 5.10. *One has*

$$C(e^{-\pi/n}) = \frac{na^4 r_{2,2n^2}^4}{r_{n,n}^4}. \quad (5.16)$$

Proof. Applying (5.14) in the definition of $C(q)$ given in (1.14) and setting $q = e^{-\pi/n}$, we find that

$$C(e^{-\pi/n}) = 8e^{-\pi/2n}\psi^4(e^{-\pi/n}), \quad (5.17)$$

Now, employing Lemma 5.5(iv) in (5.17), we finish the proof. $\square$

6. Explicit Values of Quartic Theta-Functions

In this section, we find explicit values of the quartic theta-functions $A(q)$, $B(q)$, and $C(q)$, and also their quotients by using the results established in the previous section.

Theorem 6.1. *One has*

- (i) $\frac{B(e^{-\pi\sqrt{2}})}{C(e^{-\pi\sqrt{2}})} = 1,$
- (ii) $\frac{B(e^{-2\pi})}{C(e^{-2\pi})} = 2^{3/2},$
- (iii) $\frac{B(e^{-\pi\sqrt{6}})}{C(e^{-\pi\sqrt{6}})} = 3 + 2\sqrt{2},$
- (iv) $\frac{B(e^{-\pi 2\sqrt{2}})}{C(e^{-\pi 2\sqrt{2}})} = 2^{3/2}(1 + \sqrt{2})^{3/2},$
- (v) $\frac{B(e^{-\pi\sqrt{10}})}{C(e^{-\pi\sqrt{10}})} = \left(\frac{1 + \sqrt{5}}{2}\right)^6,$
- (vi) $\frac{B(e^{-\pi 2\sqrt{3}})}{C(e^{-\pi 3\sqrt{3}})} = \sqrt{2}(\sqrt{3} + 1)^3,$
- (vii) $\frac{B(e^{-\pi\sqrt{14}})}{C(e^{-\pi\sqrt{14}})} = \left(\frac{\sqrt{2} + 1\sqrt{2\sqrt{2}-1}}{2}\right)^6,$
- (viii) $\frac{B(e^{-4\pi})}{C(e^{-4\pi})} = 2^{9/4}(1 + \sqrt{2})^3,$
- (ix) $\frac{B(e^{-\pi 3\sqrt{2}})}{C(e^{-\pi 3\sqrt{2}})} = (\sqrt{3} + \sqrt{2})^4,$

- $$\begin{aligned}
(\text{x}) \quad & \frac{B(e^{-\pi 2\sqrt{5}})}{C(e^{-\pi 2\sqrt{5}})} = \frac{1}{8} (1 + \sqrt{5})^3 \left(\sqrt{\sqrt{5} + 1} + \sqrt{2} \right)^3, \\
(\text{xi}) \quad & \frac{B(e^{-\pi 2\sqrt{6}})}{C(e^{-\pi 2\sqrt{6}})} = (1 + \sqrt{2})^{5/2} (2(1 + \sqrt{2} + \sqrt{6}))^{3/2}, \\
(\text{xii}) \quad & \frac{B(e^{-\pi 4\sqrt{2}})}{C(e^{-\pi 4\sqrt{2}})} = 2^{3/2} (1 + \sqrt{2})^3 \left(4 + \sqrt{2 + 10\sqrt{2}} \right)^{3/2}, \\
(\text{xiii}) \quad & \frac{B(e^{-6\pi})}{C(e^{-6\pi})} = 2^{-11/2} (1 + \sqrt{3})^4 (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^4, \\
(\text{xiv}) \quad & \frac{B(e^{-\pi 2\sqrt{10}})}{C(e^{-\pi 2\sqrt{10}})} = 2^{-6} (1 + \sqrt{5})^{15/2} (2 + 3\sqrt{2} + \sqrt{5})^{3/2}, \\
(\text{xv}) \quad & \frac{B(e^{-5\pi})}{C(e^{-5\pi})} = \frac{(5^{1/4} + 1)^{12}}{2^{15/2}}, \\
(\text{xvi}) \quad & \frac{B(e^{-\pi 3\sqrt{6}})}{C(e^{-\pi 6\sqrt{6}})} = (1 + \sqrt{2})^{10/3} \left\{ \sqrt{2} + \sqrt{2} (1 + \sqrt{2})^{1/3} + (1 + \sqrt{2})^{2/3} \right\}^4, \\
(\text{xvii}) \quad & \frac{B(e^{-8\pi})}{C(e^{-8\pi})} = 2^{9/4} (1 + \sqrt{2})^3 (16 + 15 \cdot 2^{1/4} + 12\sqrt{2} + 9 \cdot 2^{3/4})^{3/2}, \\
(\text{xviii}) \quad & \frac{B(e^{-\pi 6\sqrt{2}})}{C(e^{-\pi 6\sqrt{2}})} = \frac{(2(1 + 35\sqrt{2} - 28\sqrt{3}))^{3/2}}{(\sqrt{3} - \sqrt{2})^8}, \\
(\text{xix}) \quad & \frac{B(e^{-\pi 7\sqrt{2}})}{C(e^{-\pi 7\sqrt{2}})} = \left(\frac{1 + \sqrt{7 + 2\sqrt{14}}}{2\sqrt{2}} + \frac{\sqrt{\sqrt{14} + \sqrt{7 + 2\sqrt{14}}}}{2} \right)^{12}, \\
(\text{xx}) \quad & \frac{B(e^{-10\pi})}{C(e^{-10\pi})} = \frac{2^{15/2}}{(5^{1/4} - 1)^{12}}, \\
(\text{xxi}) \quad & \frac{B(e^{-12\pi})}{C(e^{-12\pi})} = \frac{\left((\sqrt{2} + \sqrt{3})(\sqrt{3} + 1)(1 + \sqrt{2} - \sqrt{3} + \sqrt{2} \cdot 3^{3/4}\sqrt{6}) \right)^4}{2^{21/4}(\sqrt{2} - 1)^5}, \\
(\text{xxii}) \quad & \frac{B(e^{-3\pi})}{C(e^{-3\pi})} = \frac{(1 + \sqrt{3} + \sqrt{2} \cdot 3^{1/4})^4}{2^{13/2}}, \\
(\text{xxiii}) \quad & \frac{B(e^{-3\pi/\sqrt{2}})}{C(e^{-3\pi/\sqrt{2}})} = \frac{(-1 + 35\sqrt{2} + 28\sqrt{3})^{3/2}}{2^{3/2}(\sqrt{2} + \sqrt{3})^4},
\end{aligned}$$

$$\begin{aligned}
(\text{xxiv}) \quad & \frac{B(e^{-\pi\sqrt{3}})}{C(e^{-\pi\sqrt{3}})} = \frac{(\sqrt{3}+1)^3}{2^{7/2}}, \\
(\text{xxv}) \quad & \frac{B(e^{-\pi 3\sqrt{2}})}{C(e^{-\pi 3\sqrt{2}})} = \frac{(1+\sqrt{3})(1-\sqrt{3}+2^{2/3} \cdot \sqrt{3})^4}{2^{9/2}(2^{1/3}-1)^4}, \\
(\text{xxvi}) \quad & \frac{B(e^{-\pi\sqrt{7}})}{C(e^{-\pi\sqrt{7}})} = \frac{(3+\sqrt{7})^3}{2^{9/2}}, \\
(\text{xxvii}) \quad & \frac{B(e^{-\pi 3\sqrt{7}})}{C(e^{-\pi 3\sqrt{7}})} = \frac{(7-2\sqrt{3}+\sqrt{21}+(3+\sqrt{3})\sqrt{3+16\sqrt{21}-27\sqrt{7}})^4}{2^{13/2}(\sqrt{3}-1)^8(3-\sqrt{7})}.
\end{aligned} \tag{6.1}$$

Proof. We employ the values of $r_{2,n}$ from Lemma 2.1 in Theorem 5.1 to finish the proof. $\square$

Theorem 6.2. *One has*

$$\begin{aligned}
(\text{i}) \quad & \frac{B(e^{-2\pi})}{C(e^{-\pi})} = \frac{1}{2}, \\
(\text{ii}) \quad & \frac{B(e^{-\pi 2\sqrt{2}})}{C(e^{-\pi\sqrt{2}})} = \left(\frac{\sqrt{2}+1}{2}\right)^{1/2}, \\
(\text{iii}) \quad & \frac{B(e^{-2\pi\sqrt{3}})}{C(e^{-\pi\sqrt{3}})} = \frac{1}{2}(\sqrt{2}+\sqrt{3}), \\
(\text{iv}) \quad & \frac{B(e^{-4\pi})}{C(e^{-2\pi})} = 2^{1/4}(\sqrt{2}+1), \\
(\text{v}) \quad & \frac{B(e^{-2\pi\sqrt{5}})}{C(e^{-\pi\sqrt{5}})} = \frac{1}{8}\left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)^2, \\
(\text{vi}) \quad & \frac{B(e^{-2\pi\sqrt{7}})}{C(e^{-\pi\sqrt{7}})} = \frac{(127+48\sqrt{7})^{1/2}}{2}, \\
(\text{vii}) \quad & \frac{B(e^{-6\pi})}{C(e^{-3\pi})} = \frac{1}{2}\left(\frac{1}{2}+\frac{3^{1/4}}{\sqrt{2}}+\frac{\sqrt{3}}{2}\right)^4,
\end{aligned}$$

$$\begin{aligned}
\text{(viii)} \quad \frac{B(e^{-10\pi})}{C(e^{-5\pi})} &= \frac{1}{2} \left(\frac{3}{2} + \frac{\sqrt{5}}{2} + \sqrt{\frac{1}{2}(5 + 3\sqrt{5})} \right)^4, \\
\text{(ix)} \quad \frac{B(e^{-14\pi})}{C(e^{-7\pi})} &= \frac{1}{2} \left(\frac{\sqrt{4 + \sqrt{7} + \sqrt{21 + 8\sqrt{7}}} + \sqrt{\sqrt{7} + \sqrt{21 + 8\sqrt{7}}}}{2} \right)^8.
\end{aligned} \tag{6.2}$$

Proof. It follows easily from Theorem 5.3 and the values of $r_{4,n}$ from Lemma 2.1. $\square$

Theorem 6.3. *One has*

$$\begin{aligned}
\text{(i)} \quad B(e^{-2\pi}) &= 2^{-1/2} a^4, \\
\text{(ii)} \quad B(e^{-4\pi}) &= 2^{-7/4} a^4 (1 + \sqrt{2}), \\
\text{(iii)} \quad B(e^{-6\pi}) &= \frac{a^4 (1 + \sqrt{3} + \sqrt{2} \cdot {}^4\sqrt{3^3})^{4/3}}{2^{11/6} \cdot 3^{3/2} (\sqrt{3} - 1)^{2/3}}, \\
\text{(iv)} \quad B(e^{-8\pi}) &= 2^{-7/2} a^4 (16 + 15\sqrt[4]{2} + 12\sqrt{2} + 9\sqrt[4]{2^3})^{1/2}, \\
\text{(v)} \quad B(e^{-10\pi}) &= \frac{a^4 2^{7/2}}{5(\sqrt[4]{5} - 1)^4 (5 + \sqrt{5})^{1/2}}, \\
\text{(vi)} \quad B(e^{-12\pi}) &= \frac{a^4 (2 - 3\sqrt{2} + 3^{5/4} + 3^{3/4})^{4/3}}{3^{3/2} 2^{19/12} (\sqrt{2} - 1)^{1/3} (\sqrt{3} + 1)^{2/3} (-1 - \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{4/3}}, \\
\text{(vii)} \quad B(e^{-3\pi}) &= \frac{a^4 (1 + \sqrt{3} + \sqrt{2} \cdot 3^{1/4})^{4/3} (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^4}{2^4 \cdot 3^{3/2} (1 + \sqrt{3})^{2/3}}, \\
\text{(viii)} \quad B(e^{-5\pi}) &= \frac{a^4 (5^{1/4} + 1)^4 (3 + 2 \cdot 5^{1/4})}{5^2 \cdot 2^2 (1 + \sqrt{5})}.
\end{aligned} \tag{6.3}$$

Proof. (i)–(vi) follow readily from the first equality of Theorem 5.6(i) and the corresponding values of $h'_{n,n}$ in Lemma 2.3(i)–(vi), respectively. To prove (vii) and (viii), we employ the corresponding values of $r_{k,n}$ listed in Lemma 2.1 to the second equality of Theorem 5.6(i). $\square$

Theorem 6.4. *One has*

$$\begin{aligned}
 \text{(i)} \quad & B(e^{-\pi}) = \frac{a^4}{2}, \\
 \text{(ii)} \quad & B(e^{-\pi/2}) = \frac{a^4}{2(1+\sqrt{2})^2}, \\
 \text{(iii)} \quad & B(e^{-\pi/3}) = \frac{2^4\sqrt{3}a^4}{(1+\sqrt{3})^{10/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{8/3}}.
 \end{aligned} \tag{6.4}$$

Proof. We set $n = 2, 3$, and 6 , respectively, in Theorem 5.6(ii) and then use the corresponding values of $r_{k,n}$ from Lemma 2.1 to complete the proofs. $\square$

Theorem 6.5. *One has*

$$\begin{aligned}
 \text{(i)} \quad & B(-e^{-\pi}) = a^4, \\
 \text{(ii)} \quad & B(-e^{-2\pi}) = \frac{a^4}{8(\sqrt{2}-2)^2}, \\
 \text{(iii)} \quad & B(-e^{-3\pi}) = \frac{2a^4}{3^{3/2}(\sqrt{3}-1)^2}, \\
 \text{(iv)} \quad & B(-e^{-4\pi}) = \frac{a^4(\sqrt[4]{2}+1)^4}{32}, \\
 \text{(v)} \quad & B(-e^{-5\pi}) = \frac{a^4}{5(5-2\sqrt{5})^2}, \\
 \text{(vi)} \quad & B(-e^{-6\pi}) = \frac{a^4(-4+3\sqrt{2}+3^{5/4}+2\sqrt{3}-3^{3/4}+2\sqrt{2}\cdot 3^{3/4})^{4/3}}{2^4\cdot 3^3((\sqrt{2}-1)(\sqrt{3}-1))^{2/3}}.
 \end{aligned} \tag{6.5}$$

Proof. We employ the values of $h_{n,n}$ given in Lemma 2.2 in Theorem 5.7(i) to finish the proof. $\square$

Theorem 6.6. *One has*

$$\begin{aligned}
 \text{(i)} \quad & B(-e^{-\pi/2}) = \frac{a^4}{2(\sqrt{2}-2)^2}, \\
 \text{(ii)} \quad & B(-e^{-\pi/3}) = \frac{3a^4}{2\sqrt{3}-3},
 \end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad B\left(-e^{-\pi/4}\right) &= \frac{a^4}{2} \left(\sqrt[4]{2} + 1\right)^4, \\
\text{(iv)} \quad B\left(-e^{-\pi/5}\right) &= \frac{5a^4}{\left(5 - 2\sqrt{5}\right)^2}, \\
\text{(v)} \quad B\left(-e^{-\pi/6}\right) &= \frac{\sqrt{3} \, a^4 \left(-4 + 3\sqrt{2} + 3^{5/4} + 2\sqrt{3} - 3^{3/4} + 2\sqrt{2} \cdot 3^{3/4}\right)^{4/3}}{2^2 \sqrt{3} \left((\sqrt{2} - 1)(\sqrt{3} - 1)\right)^{2/3}}.
\end{aligned} \tag{6.6}$$

Proof. We use the values of $h_{n,n}$ from Lemma 2.2 in Theorem 5.7(ii). □

Theorem 6.7. *One has*

$$\begin{aligned}
\text{(i)} \quad C\left(e^{-\pi}\right) &= \sqrt{2}a^4, \\
\text{(ii)} \quad C\left(e^{-2\pi}\right) &= \frac{a^4}{4}, \\
\text{(iii)} \quad C\left(e^{-3\pi}\right) &= \frac{2^{5/2}a^4}{3^{7/3} \left(1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4}\right)^{4/3} \left(1 + \sqrt{3}\right)^{2/3}}, \\
\text{(iv)} \quad C\left(e^{-4\pi}\right) &= \frac{a^4}{2^3 \left(1 + \sqrt{2}\right)^2}, \\
\text{(v)} \quad C\left(e^{-5\pi}\right) &= \frac{2^{9/2}a^4}{5 \left(5 + \sqrt{5}\right)^2 \left(5^{1/4} + 1\right)^4}, \\
\text{(vi)} \quad C\left(e^{-6\pi}\right) &= \frac{2^{13/3}a^4}{3^{3/2} \left(1 + \sqrt{3}\right)^{10/3} \left(1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4}\right)^{8/3}}, \\
\text{(vii)} \quad C\left(e^{-9\pi}\right) &= \frac{a^4 \sqrt{2}}{9g_{9,9}^{'4}},
\end{aligned} \tag{6.7}$$

where $g_{9,9}'$ is given in Lemma 2.4(vii).

Proof. The proof of the theorem follows from Theorem 5.9 and the values of $g_{n,n}'$ from Lemma 2.4. □

Theorem 6.8. *One has*

$$\begin{aligned}
 \text{(i)} \quad & C\left(e^{-\pi/2}\right) = 2^{5/4} a^4 (1 + \sqrt{2}), \\
 \text{(ii)} \quad & C\left(e^{-\pi/3}\right) = 2^{-3/2} \sqrt{3} a^4 (1 + \sqrt{3})^{2/3} \left(\sqrt{2} \cdot 3^{3/4} + \sqrt{3} + 1\right)^{4/3}, \\
 \text{(iii)} \quad & C\left(e^{-\pi/4}\right) = 2^{3/2} a^4 \left(16 + 15 \cdot 2^{1/4} + 12\sqrt{2} + 9 \cdot 2^{3/4}\right)^{1/2}, \\
 \text{(iv)} \quad & C\left(e^{-\pi/5}\right) = \frac{5 \cdot 2^{9/2} a^4}{(5 + \sqrt{5})^2 (5^{1/4} - 1)^4}, \\
 \text{(v)} \quad & C\left(e^{-\pi/6}\right) = \frac{2^{17/3} 3^{1/2} a^4 \left(\left(\sqrt{2} + \sqrt{3}\right) \left(1 + \sqrt{3}\right) \left(1 + \sqrt{2} - \sqrt{3} + \sqrt{12} \cdot 3^{3/4}\right)\right)^{4/3}}{(1 + \sqrt{3})^2 (1 + \sqrt{3} + \sqrt{2} \cdot 3^{3/4})^{4/3} (\sqrt{2} - 1)^{5/3}}.
 \end{aligned} \tag{6.8}$$

Proof. We set $n = 2, 3, 4, 5$, and 6 in Theorem 5.10 and then employ the corresponding values of $r_{k,n}$ listed in Lemma 2.1. $\square$

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