Research Article

Approximation of Solutions of Nonlinear Integral Equations of Hammerstein Type with Lipschitz and Bounded Nonlinear Operators

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Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm and $F, K : E \rightarrow E$ be Lipschitz accretive maps with $D(K) = R(F) = E$. Suppose that the Hammerstein equation $u + KF u = 0$ has a solution. An explicit *iteration method* is shown to converge strongly to a solution of the equation. No invertibility assumption is imposed on K and the operator F is not restricted to be angle-bounded. Our theorems are significant improvements on important recent results (e.g., (Chiume and Djitte, 2012)).

1. Introduction

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to have a *Gâteaux differentiable* norm if the limit

$$\lim_{t \rightarrow 0^+} \frac{\|x + ty\| - \|x\|}{t} \quad (1.1)$$

exists for each $x, y \in S$. The normed space E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Furthermore, E is said to be *uniformly smooth* if the limit exists uniformly for $(x, y) \in S \times S$.

Let E be a normed linear space with dimension greater than or equal to 2. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_E(t) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = t \right\}. \quad (1.2)$$

In terms of the modulus of smoothness, the space E is called uniformly smooth if and only if $\lim_{t \rightarrow 0^+} (\rho_E(t)/t) = 0$. E is called *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$, $t > 0$. L_p (and ℓ_p) spaces, $1 < p < +\infty$ are *q-uniformly smooth*. In particular, L_p is 2-uniformly smooth if $2 \leq p < +\infty$ and *p-uniformly smooth* if $1 < p < 2$. It is easy to see that for $1 < q < +\infty$, every *q-uniformly smooth* real Banach space is uniformly smooth and thus has a uniformly Gâteaux differentiable norm.

Let E be a real normed linear space with dual E^* . For $q > 1$, we denote by J_q the generalized duality mapping from E to 2^{E^*} defined by

$$J_q(x) := \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\| \cdot \|f^*\|, \|f^*\|^q = \|x\|^{q-1} \right\}, \quad (1.3)$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. J_2 is denoted by J . If E^* is strictly convex, then J_q is single-valued. We note that $J_q(x) = \|x\|^{q-2} J(x)$, for $x \neq 0$. It is known that if E is a real Banach space with uniform Gâteaux differentiable norm, then the duality map J is norm-to-weak* uniformly continuous on bounded subsets of E .

A map G with domain $D(G)$ in a normed linear space E is said to be *strongly accretive* if there exists a constant $k > 0$ such that for every $x, y \in D(G)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Gx - Gy, j_q(x - y) \rangle \geq k \|x - y\|^q. \quad (1.4)$$

If $k = 0$, G is said to be *accretive*. If E is a Hilbert space, accretive operators are called *monotone*.

The accretive mappings were introduced independently in 1967 by Browder [1] and Kato [2]. Interest in such mappings stems mainly from their firm connection with equations of evolution. It is known (see, e.g., Zeidler [3]) that many physically significant problems can be modelled by initial-value problems of the form

$$u'(t) + Au(t) = 0, \quad u(0) = u_0, \quad (1.5)$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave, or the Schrödinger equations. If in (1.5), $u(t)$ is independent of t , then (1.5) reduces to

$$Au = 0, \quad (1.6)$$

whose solutions correspond to the equilibrium points of the system (1.5). Consequently, considerable research efforts have been devoted, especially within the past 40 years or so, to methods of finding approximate solutions (when they exist) of (1.6). An early fundamental

result in the theory of accretive operators, due to Browder [1], states that the initial-value problem (1.5) is solvable if A is locally Lipschitzian and accretive on E . Utilizing the existing result for (1.5), Browder [1] proved that if A is locally Lipschitzian and accretive on E , then A is m -accretive that is, $R(I + A) = E$. Clearly, a consequence of this is that the following equation

$$u + Au = 0 \quad (1.7)$$

has a solution.

One important generalization of (1.7) is the so-called *equation of Hammerstein type* (see e.g., Hammerstein [4]), where a nonlinear integral equation of Hammerstein type is one of the form:

$$u(x) + \int_{\Omega} \kappa(x, y) f(y, u(y)) dy = h(x), \quad (1.8)$$

where dy is a σ -finite measure on the measure space Ω ; the real kernel κ is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear, and h is a given function on Ω . If we now define an operator K by

$$Kv(x) := \int_{\Omega} \kappa(x, y) v(y) dy; \quad x \in \Omega, \quad (1.9)$$

and the so-called *superposition* or *Nemytskii* operator by $Fu(y) := f(y, u(y))$ then, the integral equation (1.8) can be put in operator theoretic form as follows:

$$u + KFu = 0, \quad (1.10)$$

where without loss of generality, we have taken $h \equiv 0$.

Interest in (1.10) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Greens functions can, as a rule, be transformed into the form (1.10). Among these, we mention the problem of the forced oscillations of finite amplitude of a pendulum (see, e.g., Pascali and Sburlan [5], Chapter IV).

Example 1.1. We consider the problem of the pendulum.

$$\begin{aligned} \frac{d^2 v(t)}{dt^2} + a^2 \sin v(t) &= z(t), \quad t \in [0, 1], \\ v(0) &= v(1) = 0, \end{aligned} \quad (1.11)$$

where the driving force z is odd. The constant a , ($a \neq 0$) depends on the length of the pendulum and gravity. Since Green's function of the problem:

$$v''(t) = 0, \quad v(0) = v(1) = 0 \quad (1.12)$$

is given by:

$$k(t, s) := \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases} \quad (1.13)$$

it follows that problem (1.11) is equivalent to the nonlinear integral equation

$$v(t) = - \int_0^1 k(t, s) [z(s) - a^2 \sin v(s)] ds, \quad t \in [0, 1]. \quad (1.14)$$

If we set

$$g(t) := \int_0^1 k(t, s) z(s) ds, \quad u(t) := v(t) + g(t), \quad t \in [0, 1], \quad (1.15)$$

then

$$v = u - g \quad (1.16)$$

and (1.14) can be written as

$$u(t) + \int_0^1 k(t, s) a^2 \sin(g(s) - u(s)) ds = 0, \quad (1.17)$$

which is in the Hammerstein equation form:

$$u(t) + \int_0^1 k(t, s) f(s, u(s)) ds = 0, \quad (1.18)$$

where $f(t, s) = a^2 \sin(g(t) - s)$, $t, s \in [0, 1]$.

Equations of Hammerstein type play a crucial role in the theory of optimal control systems and in automation and network theory (see, e.g., Dolezal [6]).

Several existence and uniqueness theorems have been proved for equations of the Hammerstein type (see, e.g., Brézis and Browder [7–9], Browder [1], Browder et al. [10], Browder and Gupta [11], Chepanovich [12], and De Figueiredo and Gupta [13]).

For the iterative approximation of solutions of (1.6) and (1.7), the *monotonicity/accretivity* of A is crucial. The Mann iteration scheme (see, e.g., Mann [14]) has successfully been employed (see, e.g., the recent monographs of Berinde [15] and Chidume [16]). The recurrence formulas used involved K^{-1} which is also assumed to be strongly monotone, and this, apart from limiting the class of mappings to which such iterative schemes are applicable, is also not convenient in applications. Part of the difficulty is the fact that the composition of two monotone operators need not to be monotone. In the special case in which the operators are defined on subsets D of E which are compact, Brézis and Browder [7] proved the strong convergence of a suitably defined *Galerkin approximation* to a solution of (1.10) (see also Brézis and Browder [9]). In fact, they proved the following Theorem.

Theorem BB (Brézis and Browder). *Let H be a separable real Hilbert space and C be a closed subspace of H . Let $K : H \rightarrow C$ be a bounded continuous monotone operator and $F : C \rightarrow H$ be an angle-bounded and weakly compact mapping. For a given $f \in C$, consider the Hammerstein equation*

$$(I + KF)u = f, \quad (1.19)$$

and its n th Galerkin approximation given by

$$(I + K_n F_n)u_n = P^* f, \quad (1.20)$$

where $K_n = P_n^ K P_n : H \rightarrow C_n$ and $F_n = P_n F P_n^* : C_n \rightarrow H$, where the symbols have their usual meanings (see, [5]). Then, for each $n \in \mathbb{N}$, the Galerkin approximation (1.20) admits a unique solution u_n in C_n and $\{u_n\}$ converges strongly in H to the unique solution $u \in C$ of (1.19).*

Theorem BB is a special case of the actual theorem of Brézis and Browder in which the Banach space is a separable real Hilbert space. The main theorem of Brézis and Browder is proved in an arbitrary separable real Banach space.

We observe that the Galerkin method of Brézis and Browder is not *iterative*. The first attempt to construct an *iterative method* for the approximation of a solution of a Hammerstein equation, as far as we know, was made by Chidume and Zegeye [17] who constructed a sequence in the cartesian product $E \times E$ and proved the convergence of the sequence to a solution of the Hammerstein equation. In subsequent papers, [18, 19], these authors were able to construct explicit coupled algorithms *in the original space E* which converge strongly to a solution of the equation. Following this, Chidume and Djitte studied this explicit coupled algorithms and proved several strong convergence theorems (see, [20–22]).

Recently, Chidume and Djitte [20] introduced and studied a *coupled explicit iterative process* (see Theorem CD below) to approximate solutions of nonlinear equations of Hammerstein-type in real Hilbert spaces when the operators K and F are *bounded and maximal monotone*. They proved the following theorem.

Theorem CD (Chidume and Djitte, [20]). *Let H be a real Hilbert space and $F, K : H \rightarrow H$ be bounded and maximal monotone operators. Let $\{u_n\}$ and $\{v_n\}$ be sequences in H defined iteratively from arbitrary points $u_1, v_1 \in H$ as follows:*

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1), \quad n \geq 1, \\ v_{n+1} &= v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), \quad n \geq 1, \end{aligned} \quad (1.21)$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

- (1) $\lim \theta_n = 0$,
- (2) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$, $\lambda_n = o(\theta_n)$,
- (3) $\lim_{n \rightarrow \infty} ((\theta_{n-1})/\theta_n) - 1/\lambda_n \theta_n = 0$.

Suppose that $u + KF u = 0$ has a solution in H . Then, there exists a constant $d_0 > 0$ such that if $\lambda_n \leq d_0 \theta_n$ for all $n \geq n_0$ for some $n_0 \geq 1$, then the sequence $\{u_n\}$ converges to u^ , a solution of $u + KF u = 0$.*

It is known that L_p spaces, $p = 2$, are Hilbert and L_p spaces, $1 < p < \infty$, $p \neq 2$, are not. So, the results of Chidume and Djitte [20] do not cover L_p spaces, $1 < p < \infty$, $p \neq 2$.

It is our purpose in this paper to prove convergence theorems in reflexive real Banach spaces that include all L_p spaces, $1 < p < \infty$. In fact, it is proved in this paper, that an iteration process converges strongly in *reflexive real Banach spaces with uniformly Gâteaux differentiable norm*, to a solution of the Hammerstein equation (assuming existence) when the operators K and F are *Lipschitz* and *accretive*. These spaces include L_p spaces, $1 < p < \infty$. This complements the results of Chidume and Djitte [20] to provide iterative methods for the approximation of solutions of the Hammerstein equation $u + KF u = 0$ in all L_p spaces, $1 < p < \infty$. Our method of proof is different and is also of independent interest.

2. Preliminaries

In the sequel, we will need the followings results.

Lemma 2.1 (see, e.g., [23]). *Let $\{\lambda_n\}$ be a sequence of nonnegative numbers and $\{\alpha_n\} \subseteq (0, 1)$ a sequence such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let the recursive inequality*

$$\lambda_{n+1} \leq \lambda_n - 2\alpha_n \varphi(\lambda_{n+1}) + \sigma_n, \quad n = 1, 2, \dots, \quad (2.1)$$

be given where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing function such that it is positive on $(0, \infty)$, $\varphi(0) = 0$. If $\sigma_n = o(\alpha_n)$. Then $\lambda_n \rightarrow 0$, as $n \rightarrow \infty$.

Lemma 2.2. *There exists a map $\mu_n : \ell^\infty(\mathbb{N}) \rightarrow \mathbb{R}$ such that*

- (1) μ_n is linear; that is, $\mu_n(x + y) = \mu_n(x) + \mu_n(y)$ and $\mu_n(c \cdot x) = c \cdot \mu_n(x)$,
- (2) μ_n is positive: $\mu_n(x) \geq 0$ for every $x \in \ell^\infty(\mathbb{N})$ with $x_n \geq 0$, for all n ,
- (3) μ_n is normalized: $\mu_n(e) = 1$, where $e = (1, 1, \dots)$,
- (4) μ_n is shift invariant: $\mu_n(Sx) = \mu_n(x)$, for all $x \in \ell^\infty(\mathbb{N})$, where $S : \ell^\infty(\mathbb{N}) \rightarrow \ell^\infty(\mathbb{N})$ is the shift operator defined by

$$S : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, \dots), \quad (2.2)$$

the above properties on the functional μ_n imply the following:

- (5) μ_n has norm one; thus, $|\mu_n(x)| \leq \|x\|$, for all $x \in \ell^\infty(\mathbb{N})$,
- (6) μ_n extends $\lim$ on the subspace of the convergent sequences:

$$\lim_{n \rightarrow \infty} (x_n) = c \implies \mu_n(x) = c, \quad \text{where } x = (x_n), \quad (2.3)$$

- (7) for any $x = (x_n) \in \ell^\infty(\mathbb{N})$:

$$\liminf_{n \rightarrow \infty} x_n \leq \mu_n(x) \leq \limsup_{n \rightarrow \infty} x_n. \quad (2.4)$$

Remark 2.3. Functions μ_n as above are called *Banach Limits*.

Lemma 2.4 (see [24, 25]). Let $x = (x_n) \in \ell_\infty$ be such that $\mu_n(x) \leq 0$ for all Banach limits μ_n . If $\limsup_n (x_{n+1} - x_n) \leq 0$, then $\limsup_n x_n \leq 0$.

Lemma 2.5. Let E be a real normed linear space. Then, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \quad \forall j(x + y) \in J(x + y), \quad \forall x, y \in E. \quad (2.5)$$

3. Main Results

We now prove the following theorems.

Theorem 3.1. Let E be a real Banach space and $F, K : E \rightarrow E$ be Lipschitz and accretive maps with $D(K) = R(F) = E$ and Lipschitz constants L_1 and L_2 , respectively. Let $\{u_n\}$ and $\{v_n\}$ be sequences in E defined iteratively from arbitrary points $u_1, v_1 \in E$ as follows:

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n \alpha_n (Fu_n - v_n) - \lambda_n \theta_n (u_n - u_1), \quad n \geq 1, \\ v_{n+1} &= v_n - \lambda_n \alpha_n (Kv_n + u_n) - \lambda_n \theta_n (v_n - v_1), \quad n \geq 1, \end{aligned} \quad (3.1)$$

where $\{\lambda_n\}$, $\{\alpha_n\}$, and $\{\theta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

- (1) $\lim \lambda_n = 0$, $\lim \alpha_n = 0$,
- (2) $\lambda_n = o(\theta_n)$, $\alpha_n = o(\theta_n)$.

Suppose that $u + KF u = 0$ has a solution in E . Then the sequences $\{u_n\}$ and $\{v_n\}$ are bounded.

Proof. Let $X := E \times E$ with the norm $\|z\|_X = (\|u\|_E^2 + \|v\|_E^2)^{1/2}$, where $z = (u, v)$. Define the sequence $\{w_n\}$ in X by: $w_n := (u_n, v_n)$. Let $u^* \in E$ be a solution of $u + KF u = 0$, set $v^* := F u^*$ and $w^* := (u^*, v^*)$. We observe that $u^* = -K v^*$. We show that the sequence $\{w_n\}$ is bounded in X .

For this, define

$$L := \max\{L_1, L_2\}, \quad \gamma_0 := \frac{\sqrt{2} - 1}{8(L + 1)(L + 2)}. \quad (3.2)$$

Since $\lambda_n = o(\theta_n)$ and $\alpha_n = o(\theta_n)$, there exists $N \in \mathbb{N}$ such that $\alpha_n/\theta_n < \gamma_0$ and $\lambda_n/\theta_n < \gamma_0$ for all $n \geq N$. Let $r > 0$ sufficiently large such that $w_1 \in B(w^*, r/2)$ and $w_N \in B(w^*, r)$. Define $B := \overline{B(w^*, r)}$.

Claim 1. w_n is in B for all $n \geq N$. The proof is by induction. By construction $w_N \in B$. Suppose that $w_n \in B$ for some $n \geq N$. We prove that $w_{n+1} \in B$. Assume for contradiction that $w_{n+1} \notin B$. Then we have $\|w_{n+1} - w^*\|_E > r$. We compute as follows:

$$\|w_{n+1} - w^*\|_X^2 = \|u_{n+1} - u^*\|_E^2 + \|v_{n+1} - v^*\|_E^2. \quad (3.3)$$

Using Lemma 2.5, We have

$$\begin{aligned}\|u_{n+1} - u^*\|_E^2 &= \|u_n - u^* - \lambda_n(\alpha_n(Fu_n - v_n) + \theta_n(u_n - u_1))\|_E^2 \\ &\leq \|u_n - u^*\|_E^2 - 2\lambda_n \langle \alpha_n(Fu_n - v_n) + \theta_n(u_n - u_1), j(u_{n+1} - u^*) \rangle.\end{aligned}\quad (3.4)$$

Observing that

$$\begin{aligned}\langle \alpha_n(Fu_n - v_n), j(u_{n+1} - u^*) \rangle &= \langle \alpha_n(Fu_{n+1} - Fu^*), j(u_{n+1} - u^*) \rangle \\ &\quad + \langle \alpha_n(Fu_n - Fu_{n+1}) + \alpha_n(v^* - v_n), j(u_{n+1} - u^*) \rangle, \\ \langle \theta_n(u_n - u_1), j(u_{n+1} - u^*) \rangle &= \langle \theta_n(u_{n+1} - u^*), j(u_{n+1} - u^*) \rangle \\ &\quad + \langle \theta_n(u_n - u_{n+1}) + \theta_n(u^* - u_1), j(u_{n+1} - u^*) \rangle,\end{aligned}\quad (3.5)$$

and using the fact that F is accretive, we obtain the following estimate:

$$\begin{aligned}\|u_{n+1} - u^*\|_E^2 &\leq \|u_n - u^*\|_E^2 - 2\lambda_n \theta_n \|u_{n+1} - u^*\|_E^2 \\ &\quad + 2\lambda_n \langle \alpha_n(Fu_{n+1} - Fu_n) + \alpha_n(v_n - v^*), j(u_{n+1} - u^*) \rangle \\ &\quad + 2\lambda_n \langle \theta_n(u_{n+1} - u_n) + \theta_n(u_1 - u^*), j(u_{n+1} - u^*) \rangle.\end{aligned}\quad (3.6)$$

Using Schwartz's inequality and the fact that F is Lipschitz, it follows that:

$$\begin{aligned}\|u_{n+1} - u^*\|_E^2 &\leq \|u_n - u^*\|_E^2 - 2\lambda_n \theta_n \|u_{n+1} - u^*\|_E^2 \\ &\quad + 2\lambda_n [\alpha_n L_1 \|u_{n+1} - u_n\|_E + \alpha_n \|v_n - v^*\|_E] \|u_{n+1} - u^*\|_E \\ &\quad + 2\lambda_n [\theta_n \|u_{n+1} - u_n\|_E + \theta_n \|u_1 - u^*\|_E] \|u_{n+1} - u^*\|_E.\end{aligned}\quad (3.7)$$

Thus, we obtain

$$\begin{aligned}\|u_{n+1} - u^*\|_E^2 &\leq \|u_n - u^*\|_E^2 - 2\lambda_n \theta_n \|u_{n+1} - u^*\|_E^2 \\ &\quad + 2\lambda_n [(\alpha_n L_1 + \theta_n) \|u_{n+1} - u_n\|_E] \|u_{n+1} - u^*\|_E \\ &\quad + 2\lambda_n [\alpha_n \|v_n - v^*\|_E + \theta_n \|u_1 - u^*\|_E] \|u_{n+1} - u^*\|_E.\end{aligned}\quad (3.8)$$

Using the fact that

$$\begin{aligned}\|u_{n+1} - u_n\|_E &= \|\lambda_n \alpha_n (Fu_n - v_n) + \lambda_n \theta_n (u_n - u_1)\|_E \\ &\leq \lambda_n \alpha_n L_1 \|u_n - u^*\|_E + \lambda_n \alpha_n \|v_n - v^*\|_E + \lambda_n \theta_n \|u_n - u_1\|_E \\ &= \lambda_n (\alpha_n L_1 \|u_n - u^*\|_E + \alpha_n \|v_n - v^*\|_E + \theta_n \|u_n - u_1\|_E),\end{aligned}\quad (3.9)$$

we have

$$\begin{aligned}
\|u_{n+1} - u^*\|_E^2 &\leq \|u_n - u^*\|_E^2 - 2\lambda_n \theta_n \|u_{n+1} - u^*\|_E^2 \\
&\quad + 2\lambda_n [(\alpha_n L_1 + \theta_n)(\lambda_n \alpha_n L_1 \|u_n - u^*\|_E)] \|u_{n+1} - u^*\|_E \\
&\quad + 2\lambda_n [(\alpha_n L_1 + \theta_n)(\lambda_n \alpha_n \|v_n - v^*\|_E)] \|u_{n+1} - u^*\|_E \\
&\quad + 2\lambda_n [(\alpha_n L_1 + \theta_n)(\lambda_n \theta_n \|u_n - u_1\|_E)] \|u_{n+1} - u^*\|_E \\
&\quad + 2\lambda_n [\alpha_n \|v_n - v^*\|_E + \theta_n \|u_1 - u^*\|_E] \|u_{n+1} - u^*\|_E.
\end{aligned} \tag{3.10}$$

Following the same argument, we also obtain

$$\begin{aligned}
\|v_{n+1} - v^*\|_E^2 &\leq \|v_n - v^*\|_E^2 - 2\lambda_n \theta_n \|v_{n+1} - v^*\|_E^2 \\
&\quad + 2\lambda_n [(\alpha_n L_2 + \theta_n)(\lambda_n \alpha_n L_2 \|v_n - v^*\|_E)] \|v_{n+1} - v^*\|_E \\
&\quad + 2\lambda_n [(\alpha_n L_2 + \theta_n)(\lambda_n \alpha_n \|u_n - u^*\|_E)] \|v_{n+1} - v^*\|_E \\
&\quad + 2\lambda_n [(\alpha_n L_2 + \theta_n)(\lambda_n \theta_n \|v_n - v_1\|_E)] \|v_{n+1} - v^*\|_E \\
&\quad + 2\lambda_n [\alpha_n \|u_n - u^*\|_E + \theta_n \|v_1 - v^*\|_E] \|v_{n+1} - v^*\|_E.
\end{aligned} \tag{3.11}$$

Using the fact that

$$\|u\|_E + \|v\|_E \leq \sqrt{2} \|w\|_{X'}, \quad \forall w = (u, v) \in X, \tag{3.12}$$

we have

$$\begin{aligned}
\|w_{n+1} - w^*\|_X^2 &\leq \|w_n - w^*\|_X^2 - 2\lambda_n \theta_n \|w_{n+1} - w^*\|_X^2 \\
&\quad + 2\lambda_n \left[\sqrt{2}(\alpha_n L + \theta_n)(\lambda_n \alpha_n L \|w_n - w^*\|_X) \right] \|w_{n+1} - w^*\|_X \\
&\quad + 2\lambda_n \left[\sqrt{2}(\alpha_n L + \theta_n)(\lambda_n \alpha_n \|w_n - w^*\|_X) \right] \|w_{n+1} - w^*\|_X \\
&\quad + 2\lambda_n \left[\sqrt{2}(\alpha_n L + \theta_n)(\lambda_n \theta_n \|w_n - w_1\|_X) \right] \|w_{n+1} - w^*\|_X \\
&\quad + 2\lambda_n \left[\sqrt{2}\alpha_n \|w_n - w^*\|_X + \sqrt{2}\theta_n \|w_1 - w^*\|_X \right] \|w_{n+1} - w^*\|_X,
\end{aligned} \tag{3.13}$$

which implies that

$$\begin{aligned}
\|w_{n+1} - w^*\|_X^2 &\leq \|w_n - w^*\|_X^2 - 2\lambda_n \theta_n \|w_{n+1} - w^*\|_X^2 \\
&\quad + 2\lambda_n \left[2\sqrt{2}(\alpha_n L + \theta_n)(\lambda_n \alpha_n L \|w_n - w^*\|_X) \right] \|w_{n+1} - w^*\|_X \\
&\quad + 2\lambda_n \left[\sqrt{2}(\alpha_n L + \theta_n)(\lambda_n \theta_n \|w_n - w_1\|_X) \right] \|w_{n+1} - w^*\|_X \\
&\quad + 2\lambda_n \left[\sqrt{2}\alpha_n \|w_n - w^*\|_X + \sqrt{2}\theta_n \|w_1 - w^*\|_X \right] \|w_{n+1} - w^*\|_X.
\end{aligned} \tag{3.14}$$

Since $w_n \in B$ and $w_1 \in B(w^*, r/2)$, we have

$$\begin{aligned} \|w_{n+1} - w^*\|_X^2 &\leq \|w_n - w^*\|_X^2 - 2\lambda_n\theta_n\|w_{n+1} - w^*\|_X^2 \\ &\quad + 2\lambda_n\left[2\sqrt{2}(\alpha_n L + \theta_n)\lambda_n(\alpha_n L r + 2\theta_n r)\right]\|w_{n+1} - w^*\|_X \\ &\quad + 2\lambda_n\left[2\sqrt{2}\alpha_n r + \frac{\sqrt{2}}{2}\theta_n r\right]\|w_{n+1} - w^*\|_X. \end{aligned} \quad (3.15)$$

By assumption, $\|w_{n+1} - w^*\|_X^2 > \|w_n - w^*\|_X^2$. So we have

$$\theta_n\|w_{n+1} - w^*\|_X \leq 2\sqrt{2}(\alpha_n L + \theta_n)\lambda_n(\alpha_n L r + 2\theta_n r) + 2\sqrt{2}\alpha_n r + \frac{\sqrt{2}}{2}\theta_n r. \quad (3.16)$$

Hence

$$\|w_{n+1} - w^*\|_X \leq 2\sqrt{2}\frac{\lambda_n}{\theta_n}(L+1)(L+2)r + 2\sqrt{2}\frac{\alpha_n}{\theta_n}r + \frac{\sqrt{2}}{2}r. \quad (3.17)$$

Therefore, using the fact that $\lambda_n/\theta_n < \gamma_0$ and $\alpha_n/\theta_n < \gamma_0$, it follows that $\|w_{n+1} - w^*\|_X < r$. A contradiction. So $w_{n+1} \in B$. This prove the boundedness of the sequences $\{u_n\}$ and $\{v_n\}$. $\square$

Theorem 3.2. Let E be reflexive real Banach space with uniformly Gâteaux differentiable norm and $F, K : E \rightarrow E$ be Lipschitz accretive mappings. Let $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ be sequences in E defined iteratively from arbitrary $u_1, v_1 \in E$ by

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n \alpha_n (F u_n - v_n) - \lambda_n \theta_n (u_n - u_1), \quad n \geq 1, \\ v_{n+1} &= v_n - \lambda_n \alpha_n (K v_n + u_n) - \lambda_n \theta_n (v_n - v_1), \quad n \geq 1, \end{aligned} \quad (3.18)$$

where $\{\lambda_n\}_{n \geq 1}$, $\{\alpha_n\}_{n \geq 1}$, and $\{\theta_n\}_{n \geq 1}$ are real sequences in $(0, 1)$ such that $\lambda_n = o(\theta_n)$, $\alpha_n = o(\theta_n)$, and $\sum_{n=1}^{\infty} \lambda_n \theta_n = +\infty$. Suppose that the equation $u + KFu = 0$ has a solution u^* . Then there exists a subset $K_{\min}$ of $E \times E$ such that if $(u^*, v^*) \in K_{\min}$ with $v^* = Fu^*$, then the sequence $\{u_n\}_{n \geq 1}$ converges strongly to u^* .

Proof. Since, by Theorem 3.1, the sequences $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ are bounded and F is bounded, there exists an $R > 0$ such that $u_n \in B_1 := \overline{B(u^*, R)}$ and $v_n \in B_2 := \overline{B(v^*, R)}$ for all $n \geq 1$. Furthermore, the sets B_1 and B_2 are nonempty closed convex and bounded subsets of E . Let μ_n be a Banach limit. Define the maps $\varphi_1 : E \rightarrow \mathbb{R}$ and $\varphi_2 : E \rightarrow \mathbb{R}$ by

$$\varphi_1(u) := \mu_n \|u_n - u\|^2, \quad \varphi_2(v) := \mu_n \|v_n - v\|^2. \quad (3.19)$$

Then for $i = 1, 2$, φ_i is weakly lower semicontinuous. Since E reflexive, then

$$\text{Arg min}_{u \in B_1} \varphi_1(u) \neq \emptyset, \quad \text{Arg min}_{v \in B_2} \varphi_2(v) \neq \emptyset. \quad (3.20)$$

Set

$$K_{\min} := \operatorname{Arg} \min_{u \in B_1} \varphi_1(u) \times \operatorname{Arg} \min_{v \in B_2} \varphi_2(v). \quad (3.21)$$

Now assume that $w^* = (u^*, v^*) \in K_{\min}$ and let $t \in (0, 1)$. Then, by convexity of B_1 , we have that $(1-t)u^* + tu_1 \in B_1$. Thus, $\varphi_1(u^*) \leq \varphi_1((1-t)u^* + tu_1)$. It follows from Lemma 2.5 that

$$\|u_n - u^* - t(u_1 - u^*)\|^2 \leq \|u_n - u^*\|^2 - 2t \langle u_1 - u^*, j(u_n - u^* - t(u_1 - u^*)) \rangle \quad \forall n \in \mathbb{N}. \quad (3.22)$$

Thus, taking Banach limits gives

$$\mu_n(\|u_n - u^* - t(u_1 - u^*)\|^2) \leq \mu_n(\|u_n - u^*\|^2) - 2t\mu_n(\langle u_1 - u^*, j(u_n - u^* - t(u_1 - u^*)) \rangle). \quad (3.23)$$

That is,

$$\varphi_1((1-t)u^* + tu_1) \leq \varphi_1(u^*) - 2t\mu_n(\langle u_1 - u^*, j(u_n - u^* - t(u_1 - u^*)) \rangle). \quad (3.24)$$

This implies that $\mu_n(\langle u_1 - u^*, j(u_n - u^* - t(u_1 - u^*)) \rangle) \leq 0$ for all $n \in \mathbb{N}$. Furthermore, the fact that E has a uniformly Gâteaux differentiable norm gives, as $t \rightarrow 0$, that

$$\langle u_1 - u^*, j(u_n - u^*) \rangle - \langle u_1 - u^*, j(u_n - u^* - t(u_1 - u^*)) \rangle \rightarrow 0. \quad (3.25)$$

Thus, for all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $t \in]0, \delta_\epsilon[$ and for all $n \in \mathbb{N}$,

$$\langle u_1 - u^*, j(u_n - u^*) \rangle - \langle u_1 - u^*, j(u_n - u^* - t(u_1 - u^*)) \rangle < \epsilon. \quad (3.26)$$

Thus,

$$\mu_n(\langle u_1 - u^*, j(u_n - u^*) \rangle) - \mu_n(\langle u_1 - u^*, j(u_n - u^* - t(u_1 - u^*)) \rangle) \leq \epsilon, \quad (3.27)$$

which implies that

$$\mu_n(\langle u_1 - u^*, j(u_n - u^*) \rangle) \leq 0. \quad (3.28)$$

Using the same argument, we have

$$\mu_n(\langle v_1 - v^*, j(v_n - v^*) \rangle) \leq 0. \quad (3.29)$$

Moreover, since $\{u_n\}_{n \geq 1}$, $\{v_n\}_{n \geq 1}$, $\{Fu_n\}_{n \geq 1}$, and $\{Kv_n\}_{n \geq 1}$ are all bounded, then, from (3.18) and that there exist positive constants M_1 and M_2 such that

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \lambda_n[\alpha_n(\|Fu_n\| + \|v_n\|) + (\theta_n\|u_n - u_1\|)] \leq \lambda_n M_1, \\ \|v_{n+1} - v_n\| &\leq \lambda_n[\alpha_n(\|Kv_n\| + \|u_n\|) + (\theta_n\|v_n - v_1\|)] \leq \lambda_n M_2.\end{aligned}\tag{3.30}$$

Thus, $\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0$ and $\lim_{n \rightarrow \infty} \|v_{n+1} - v_n\| = 0$.

Again, using the fact that E has a uniformly Gâteaux differentiable norm, we obtain that

$$\begin{aligned}\lim_{n \rightarrow \infty} (\langle u_1 - u^*, j(u_{n+1} - u^*) \rangle - \langle u_1 - u^*, j(u_n - u^*) \rangle) &= 0, \\ \lim_{n \rightarrow \infty} (\langle v_1 - v^*, j(v_{n+1} - v^*) \rangle - \langle v_1 - v^*, j(v_n - v^*) \rangle) &= 0.\end{aligned}\tag{3.31}$$

Therefore, the sequences $\{\langle u_1 - u^*, j(u_n - u^*) \rangle\}_{n \geq 1}$ and $\{\langle v_1 - v^*, j(v_n - v^*) \rangle\}_{n \geq 1}$ satisfy the conditions of Lemma 2.4.

Hence,

$$\begin{aligned}\limsup_{n \rightarrow \infty} \langle u_1 - u^*, j(u_n - u^*) \rangle &\leq 0, \\ \limsup_{n \rightarrow \infty} \langle v_1 - v^*, j(v_n - v^*) \rangle &\leq 0.\end{aligned}\tag{3.32}$$

Define

$$\begin{aligned}\sigma_n &:= \max\{\langle u_1 - u^*, j(u_{n+1} - u^*) \rangle, 0\}, \\ \xi_n &:= \max\{\langle v_1 - v^*, j(v_{n+1} - v^*) \rangle, 0\},\end{aligned}\tag{3.33}$$

then $\lim_{n \rightarrow \infty} \sigma_n = 0 = \lim_{n \rightarrow \infty} \xi_n$. Moreover,

$$\begin{aligned}\langle u_1 - u^*, j(u_n - u^*) \rangle &\leq \sigma_n, \\ \langle v_1 - v^*, j(v_n - v^*) \rangle &\leq \xi_n, \quad \forall n \in \mathbb{N}.\end{aligned}\tag{3.34}$$

Using Lemma 2.5, and (3.18), we have:

$$\begin{aligned}\|u_{n+1} - u^*\|_E^2 &\leq \|u_n - u^*\|_E^2 - 2\lambda_n \theta_n \|u_{n+1} - u^*\|_E^2 \\ &\quad + 2\lambda_n \langle \alpha_n(Fu_{n+1} - Fu_n) + \alpha_n(v_n - v^*), j(u_{n+1} - u^*) \rangle \\ &\quad + 2\lambda_n \langle \theta_n(u_{n+1} - u_n) + \theta_n(u_1 - u^*), j(u_{n+1} - u^*) \rangle, \\ \|v_{n+1} - v^*\|_E^2 &\leq \|v_n - v^*\|_E^2 - 2\lambda_n \theta_n \|v_{n+1} - v^*\|_E^2 \\ &\quad + 2\lambda_n \langle \alpha_n(Kv_{n+1} - Kv_n) + \alpha_n(v^* - v_n), j(v_{n+1} - v^*) \rangle \\ &\quad + 2\lambda_n \langle \theta_n(v_{n+1} - v_n) + \theta_n(v_1 - v^*), j(v_{n+1} - v^*) \rangle.\end{aligned}\tag{3.35}$$

Using the fact that F and K are Lipschitz and the sequences $\{u_n\}$ and $\{v_n\}$ are bounded, we obtain the following estimate:

$$\begin{aligned}\|w_{n+1} - w^*\|_E^2 &\leq \|w_n - w^*\|_E^2 - 2\lambda_n\theta_n\|w_{n+1} - w^*\|_E^2 \\ &\quad + 2(\lambda_n\alpha_n + \lambda_n^2)c + 2\lambda_n\theta_n(\sigma_n + \xi_n) \\ &= \|w_n - w^*\|_E^2 - 2\lambda_n\theta_n\|w_{n+1} - w^*\|_E^2 + \gamma_n,\end{aligned}\tag{3.36}$$

where $\gamma_n = 2(\lambda_n\alpha_n + \lambda_n^2)c + 2\lambda_n\theta_n(\sigma_n + \xi_n) = o(\lambda_n\theta_n)$ for some $c > 0$. Hence, by Lemma 2.1, $w_n \rightarrow w^*$ as $n \rightarrow \infty$. But $w_n = (u_n, v_n)$ and $w^* = (u^*, v^*)$. This implies that $u_n \rightarrow u^*$ as $n \rightarrow \infty$. This completes the proof. $\square$

Corollary 3.3. *Let E be a q -uniformly real Banach spaces, $q > 1$ and $F, K : E \rightarrow E$ be Lipschitz accretive mappings. Let $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ be sequences in E defined iteratively from arbitrary $u_1, v_1 \in E$ by*

$$\begin{aligned}u_{n+1} &= u_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1), \quad n \geq 1, \\ v_{n+1} &= v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), \quad n \geq 1,\end{aligned}\tag{3.37}$$

where $\{\lambda_n\}_{n \geq 1}$, $\{\alpha_n\}_{n \geq 1}$, and $\{\theta_n\}_{n \geq 1}$ are real sequences in $(0, 1)$ such that $\lambda_n = o(\theta_n)$, $\alpha_n = o(\theta_n)$, and $\sum_{n=1}^{\infty} \lambda_n\theta_n = +\infty$. Suppose that the equation $u + KF u = 0$ has a solution u^* . Then there exists a set $K_{\min}$ in $E \times E$ such that if $(u^*, v^*) \in K_{\min}$ with $v^* = Fu^*$, then the sequence $\{u_n\}_{n \geq 1}$ converges strongly to u^* .

Corollary 3.4. *Let $E = L_p$ (or ℓ_p) space ($1 < p < +\infty$) and $F, K : E \rightarrow E$ be Lipschitz accretive mappings. Let $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ be sequences in E defined iteratively from arbitrary $u_1, v_1 \in E$ by*

$$\begin{aligned}u_{n+1} &= u_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1), \quad n \geq 1, \\ v_{n+1} &= v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), \quad n \geq 1,\end{aligned}\tag{3.38}$$

where $\{\lambda_n\}_{n \geq 1}$, $\{\alpha_n\}_{n \geq 1}$, and $\{\theta_n\}_{n \geq 1}$ are real sequences in $(0, 1)$ such that $\lambda_n = o(\theta_n)$, $\alpha_n = o(\theta_n)$, and $\sum_{n=1}^{\infty} \lambda_n\theta_n = +\infty$. Suppose that the equation $u + KF u = 0$ has a solution u^* . Then there exists a subset $K_{\min}$ of $E \times E$ such that if $(u^*, v^*) \in K_{\min}$ with $v^* = Fu^*$, then the sequence $\{u_n\}_{n \geq 1}$ converges strongly to u^* .

Corollary 3.5. *Let H be a real Hilbert space and $F, K : H \rightarrow H$ be Lipschitz monotone mappings. Let $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ be sequences in H defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{aligned}u_{n+1} &= u_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1), \quad n \geq 1, \\ v_{n+1} &= v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), \quad n \geq 1,\end{aligned}\tag{3.39}$$

where $\{\lambda_n\}_{n \geq 1}$, $\{\alpha_n\}_{n \geq 1}$, and $\{\theta_n\}_{n \geq 1}$ are real sequences in $(0, 1)$ such that $\lambda_n = o(\theta_n)$, $\alpha_n = o(\theta_n)$, and $\sum_{n=1}^{\infty} \lambda_n\theta_n = +\infty$. Suppose that the equation $u + KF u = 0$ has a solution u^* . Then there exists

a subset $K_{\min}$ of $E \times E$ such that if $(u^*, v^*) \in K_{\min}$ with $v^* = Fu^*$, then the sequence $\{u_n\}_{n \geq 1}$ converges strongly to u^* .

Remark 3.6. Real sequences that satisfy the hypotheses of Theorems 3.1 are $\lambda_n = \alpha_n = n^{-a}$ and $\theta_n = n^{-b}$, $n \geq 1$ with $0 < b < a$ and $a + b < 1$.

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