Research Article

Green's Second Identity for Vector Fields

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The second derivative of two vector functions is related to the divergence of the vector functions with first order operators. Namely, $\mathbf{P} \cdot \nabla^2 \mathbf{Q} - \mathbf{Q} \cdot \nabla^2 \mathbf{P} = \nabla \cdot [\mathbf{P}(\nabla \cdot \mathbf{Q}) - \mathbf{Q}(\nabla \cdot \mathbf{P}) + \mathbf{P} \times \nabla \times \mathbf{Q} - \mathbf{Q} \times \nabla \times \mathbf{P}]$.

1. Introduction

Green's second identity establishes a relationship between second and (the divergence of) first order derivatives of two scalar functions

$$p_m \nabla^2 q_m - q_m \nabla^2 p_m = \nabla \cdot (p_m \nabla q_m - q_m \nabla p_m), \quad (1.1)$$

where p_m and q_m are two arbitrary scalar fields. This identity is of great importance in physics because continuity equations can thus be established for scalar fields such as mass or energy. It has been called forth to obtain a scalar wave energy density [1]. It is also invoked in the classical [2, 3] as well as the quantum [4, 5] time-dependent harmonic oscillator in order to obtain an exact invariant [6]. In optics, it is also used to derive the integral theorem of Kirchhoff in scalar diffraction theory.

Although the second Green's identity is always presented in vector analysis, only a scalar version is found on textbooks. Even in the specialized literature, a vector version is not easily found. In vector diffraction theory, two versions of Green's second identity are introduced. One variant invokes the divergence of a cross product [7–9] and states a relationship in terms of the curl-curl of the field $\mathbf{P} \cdot (\nabla \times \nabla \times \mathbf{Q}) - \mathbf{Q} \cdot (\nabla \times \nabla \times \mathbf{P}) = \nabla \cdot (\mathbf{Q} \times \nabla \times \mathbf{P} - \mathbf{P} \times \nabla \times \mathbf{Q})$. This equation can be written in terms of the Laplacians using the well-known identity $\nabla \times \nabla \times \mathbf{Q} = \nabla(\nabla \cdot \mathbf{Q}) - \nabla^2 \mathbf{Q}$,

$$\mathbf{P} \cdot \nabla^2 \mathbf{Q} - \mathbf{Q} \cdot \nabla^2 \mathbf{P} + \mathbf{Q} \cdot [\nabla(\nabla \cdot \mathbf{P})] - \mathbf{P} \cdot [\nabla(\nabla \cdot \mathbf{Q})] = \nabla \cdot (\mathbf{P} \times \nabla \times \mathbf{Q} - \mathbf{Q} \times \nabla \times \mathbf{P}). \quad (1.2)$$

However, the terms $\mathbf{Q} \cdot [\nabla(\nabla \cdot \mathbf{P})] - \mathbf{P} \cdot [\nabla(\nabla \cdot \mathbf{Q})]$ could not be readily written in terms of a divergence. The other approach introduces bivectors; this formulation requires a dyadic Green function [10, 11]. It is the purpose of this communication to establish an equivalent Green's identity for vector fields involving the Laplacians of vector functions written out in terms of the divergence operator.

2. Divergence of Two Vector Fields

Consider that the scalar fields in (1.1) are the Cartesian components of vector fields, that is, $\mathbf{P} = \sum_m p_m \hat{\mathbf{e}}_m$ and $\mathbf{Q} = \sum_m q_m \hat{\mathbf{e}}_m$. Each component m obeys an equation of the form of (1.1). Summing up these equations, we obtain

$$\sum_m [p_m \nabla^2 q_m - q_m \nabla^2 p_m] = \sum_m [\nabla \cdot (p_m \nabla q_m - q_m \nabla p_m)]. \quad (2.1)$$

The LHS according to the definition of the dot product may be written in vector form as

$$\sum_m [p_m \nabla^2 q_m - q_m \nabla^2 p_m] = \mathbf{P} \cdot \nabla^2 \mathbf{Q} - \mathbf{Q} \cdot \nabla^2 \mathbf{P}. \quad (2.2)$$

The RHS is a bit more awkward to express in terms of vector operators. Due to the distributivity of the divergence operator over addition, the sum of the divergence is equal to the divergence of the sum, that is, $\sum_m [\nabla \cdot (p_m \nabla q_m - q_m \nabla p_m)] = \nabla \cdot (\sum_m p_m \nabla q_m - \sum_m q_m \nabla p_m)$. Recall the vector identity for the gradient of a dot product [12]

$$\nabla(\mathbf{P} \cdot \mathbf{Q}) = (\mathbf{P} \cdot \nabla) \mathbf{Q} + (\mathbf{Q} \cdot \nabla) \mathbf{P} + \mathbf{P} \times \nabla \times \mathbf{Q} + \mathbf{Q} \times \nabla \times \mathbf{P} \quad (2.3)$$

which, written out in vector components, is given by

$$\nabla(\mathbf{P} \cdot \mathbf{Q}) = \nabla \sum_m p_m q_m = \sum_m p_m \nabla q_m + \sum_m q_m \nabla p_m. \quad (2.4)$$

This result is similar to what we wish to evince in vector terms "except" for the minus sign. Since the differential operators in each term of (2.3) act either over one vector (say p_m 's) or the other (q_m 's), the contribution to each term must be

$$\begin{aligned} \sum_m p_m \nabla q_m &= (\mathbf{P} \cdot \nabla) \mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q}, \\ \sum_m q_m \nabla p_m &= (\mathbf{Q} \cdot \nabla) \mathbf{P} + \mathbf{Q} \times \nabla \times \mathbf{P}. \end{aligned} \quad (2.5)$$

These results are rigorously proven to be correct in Appendix A through evaluation of the vector components. Therefore, the RHS of (2.1) can be written in vector form as

$$\sum_m p_m \nabla q_m - \sum_m q_m \nabla p_m = (\mathbf{P} \cdot \nabla) \mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q} - (\mathbf{Q} \cdot \nabla) \mathbf{P} - \mathbf{Q} \times \nabla \times \mathbf{P}. \quad (2.6)$$

Putting together these two results, a theorem for vector fields analogous to Green's theorem for scalar fields is obtained

$$\mathbf{P} \cdot \nabla^2 \mathbf{Q} - \mathbf{Q} \cdot \nabla^2 \mathbf{P} = \nabla \cdot [(\mathbf{P} \cdot \nabla) \mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q} - (\mathbf{Q} \cdot \nabla) \mathbf{P} - \mathbf{Q} \times \nabla \times \mathbf{P}]. \quad (2.7)$$

Reassuringly, from the vector relationship (2.7), we can go back to the scalar case as shown in Appendix B. The curl of a cross product can be written as $\nabla \times (\mathbf{P} \times \mathbf{Q}) = (\mathbf{Q} \cdot \nabla) \mathbf{P} - (\mathbf{P} \cdot \nabla) \mathbf{Q} + \mathbf{P}(\nabla \cdot \mathbf{Q}) - \mathbf{Q}(\nabla \cdot \mathbf{P})$; Green's vector identity can then be rewritten as

$$\mathbf{P} \cdot \nabla^2 \mathbf{Q} - \mathbf{Q} \cdot \nabla^2 \mathbf{P} = \nabla \cdot [\mathbf{P}(\nabla \cdot \mathbf{Q}) - \mathbf{Q}(\nabla \cdot \mathbf{P}) - \nabla \times (\mathbf{P} \times \mathbf{Q}) + \mathbf{P} \times \nabla \times \mathbf{Q} - \mathbf{Q} \times \nabla \times \mathbf{P}]. \quad (2.8)$$

Since the divergence of a curl is zero, the third term vanishes and the identity can be written as

$$\boxed{\mathbf{P} \cdot \nabla^2 \mathbf{Q} - \mathbf{Q} \cdot \nabla^2 \mathbf{P} = \nabla \cdot [\mathbf{P}(\nabla \cdot \mathbf{Q}) - \mathbf{Q}(\nabla \cdot \mathbf{P}) + \mathbf{P} \times \nabla \times \mathbf{Q} - \mathbf{Q} \times \nabla \times \mathbf{P}].} \quad (2.9)$$

This result should prove useful when the divergence and curl of the fields can be established in terms of other quantities, as is the case in electromagnetism. There are several particular cases of interest of this expression: if the fields satisfy Helmholtz equation, the LHS of (2.9) is zero. Thus, a conserved quantity with zero divergence is obtained; if the fields are curl-free so that they can be written in terms of the gradients of scalar functions α and β , expression (2.9) becomes

$$\nabla \alpha \cdot \nabla^2 (\nabla \beta) - \nabla \beta \cdot \nabla^2 (\nabla \alpha) = \nabla \cdot [\nabla \alpha (\nabla^2 \beta) - \nabla \beta (\nabla^2 \alpha)]. \quad (2.10)$$

Another identity that may prove useful is obtained from the divergence of (2.3)

$$\nabla \cdot \nabla (\mathbf{P} \cdot \mathbf{Q}) = \nabla \cdot [(\mathbf{P} \cdot \nabla) \mathbf{Q} + (\mathbf{Q} \cdot \nabla) \mathbf{P} + \mathbf{P} \times \nabla \times \mathbf{Q} + \mathbf{Q} \times \nabla \times \mathbf{P}], \quad (2.11)$$

invoking the Green's vector identity (2.7) derived above; the Laplacian of the dot product can be expressed in terms of the Laplacians of the factors

$$\nabla^2 (\mathbf{P} \cdot \mathbf{Q}) = \mathbf{P} \cdot \nabla^2 \mathbf{Q} - \mathbf{Q} \cdot \nabla^2 \mathbf{P} + 2 \nabla \cdot [(\mathbf{Q} \cdot \nabla) \mathbf{P} + \mathbf{Q} \times \nabla \times \mathbf{P}]. \quad (2.12)$$

If the substitution of the vector identity (2.7) is performed eliminating the terms $(\mathbf{Q} \cdot \nabla) \mathbf{P} + \mathbf{Q} \times \nabla \times \mathbf{P}$, the Laplacian of the dot product is

$$\nabla^2 (\mathbf{P} \cdot \mathbf{Q}) = -(\mathbf{P} \cdot \nabla^2 \mathbf{Q} - \mathbf{Q} \cdot \nabla^2 \mathbf{P}) + 2 \nabla \cdot [(\mathbf{P} \cdot \nabla) \mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q}]. \quad (2.13)$$

3. Conclusions

Green's second identity relating the Laplacians with the divergence has been derived for vector fields. No use of bivectors or dyadics has been made as in some previous approaches. In diffraction theory, the vector identity was stated before in terms of the curl. However, this earlier formulation had the drawback that the Laplacian could not be invoked without involving extra terms. As a corollary, the awkward terms in (1.2) can now be written in terms of a divergence by comparison with (2.9)

$$\mathbf{P} \cdot [\nabla(\nabla \cdot \mathbf{Q})] - \mathbf{Q} \cdot [\nabla(\nabla \cdot \mathbf{P})] = \nabla \cdot [\mathbf{P}(\nabla \cdot \mathbf{Q}) - \mathbf{Q}(\nabla \cdot \mathbf{P})]. \quad (3.1)$$

This result can be verified by expanding the divergence of a vector times a scalar for the two addends on the RHS.

The condition imposed by Helmholtz equation $\nabla^2 \mathbf{P} = -k^2 \mathbf{P}$ can be readily incorporated in the present formulation of Green's second identity. This result is particularly useful if the vector fields satisfy the wave equation.

Appendices

A. Derivation by Components

In order to evaluate

$$(\mathbf{P} \cdot \nabla) \mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q}, \quad (A.1)$$

consider the first term in three-dimensional Cartesian components

$$(\mathbf{P} \cdot \nabla) \mathbf{Q} = \left(p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z} \right) (q_x \hat{\mathbf{e}}_x + q_y \hat{\mathbf{e}}_y + q_z \hat{\mathbf{e}}_z) \quad (A.2)$$

that may be written as

$$\begin{aligned} (\mathbf{P} \cdot \nabla) \mathbf{Q} = & \left(p_x \frac{\partial q_x}{\partial x} + p_y \frac{\partial q_x}{\partial y} + p_z \frac{\partial q_x}{\partial z} \right) \hat{\mathbf{e}}_x \\ & + \left(p_x \frac{\partial q_y}{\partial x} + p_y \frac{\partial q_y}{\partial y} + p_z \frac{\partial q_y}{\partial z} \right) \hat{\mathbf{e}}_y \\ & + \left(p_x \frac{\partial q_z}{\partial x} + p_y \frac{\partial q_z}{\partial y} + p_z \frac{\partial q_z}{\partial z} \right) \hat{\mathbf{e}}_z. \end{aligned} \quad (A.3)$$

The curl in the second term is

$${}^R \mathbf{Q} = \nabla \times \mathbf{Q} = \left(\frac{\partial q_z}{\partial y} - \frac{\partial q_y}{\partial z} \right) \hat{\mathbf{e}}_x + \left(\frac{\partial q_x}{\partial z} - \frac{\partial q_z}{\partial x} \right) \hat{\mathbf{e}}_y + \left(\frac{\partial q_y}{\partial x} - \frac{\partial q_x}{\partial y} \right) \hat{\mathbf{e}}_z. \quad (A.4)$$

The cross product is

$$\mathbf{P} \times {}^R\mathbf{Q} = (p_y^R q_z - p_z^R q_y) \hat{\mathbf{e}}_x + (p_z^R q_x - p_x^R q_z) \hat{\mathbf{e}}_y + (p_x^R q_y - p_y^R q_x) \hat{\mathbf{e}}_z. \quad (\text{A.5})$$

The second term is then

$$\begin{aligned} \mathbf{P} \times \nabla \times \mathbf{Q} &= \left(p_y \left(\frac{\partial q_y}{\partial x} - \frac{\partial q_x}{\partial y} \right) - p_z \left(\frac{\partial q_x}{\partial z} - \frac{\partial q_z}{\partial x} \right) \right) \hat{\mathbf{e}}_x \\ &\quad + \left(p_z \left(\frac{\partial q_z}{\partial y} - \frac{\partial q_y}{\partial z} \right) - p_x \left(\frac{\partial q_y}{\partial x} - \frac{\partial q_x}{\partial y} \right) \right) \hat{\mathbf{e}}_y \\ &\quad + \left(p_x \left(\frac{\partial q_x}{\partial z} - \frac{\partial q_z}{\partial x} \right) - p_y \left(\frac{\partial q_z}{\partial y} - \frac{\partial q_y}{\partial z} \right) \right) \hat{\mathbf{e}}_z \end{aligned} \quad (\text{A.6})$$

that expands to

$$\begin{aligned} \mathbf{P} \times \nabla \times \mathbf{Q} &= \left(p_y \frac{\partial q_y}{\partial x} - p_y \frac{\partial q_x}{\partial y} - p_z \frac{\partial q_x}{\partial z} + p_z \frac{\partial q_z}{\partial x} \right) \hat{\mathbf{e}}_x \\ &\quad + \left(p_z \frac{\partial q_z}{\partial y} - p_z \frac{\partial q_y}{\partial z} - p_x \frac{\partial q_y}{\partial x} + p_x \frac{\partial q_x}{\partial y} \right) \hat{\mathbf{e}}_y \\ &\quad + \left(p_x \frac{\partial q_x}{\partial z} - p_x \frac{\partial q_z}{\partial x} - p_y \frac{\partial q_z}{\partial y} + p_y \frac{\partial q_y}{\partial z} \right) \hat{\mathbf{e}}_z. \end{aligned} \quad (\text{A.7})$$

Evaluate $(\mathbf{P} \cdot \nabla)\mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q}$ in the x direction

$$[(\mathbf{P} \cdot \nabla)\mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q}]_x = \left(p_x \frac{\partial q_x}{\partial x} + p_y \frac{\partial q_x}{\partial y} + p_z \frac{\partial q_x}{\partial z} + p_y \frac{\partial q_y}{\partial x} - p_y \frac{\partial q_x}{\partial y} - p_z \frac{\partial q_x}{\partial z} + p_z \frac{\partial q_z}{\partial x} \right) \hat{\mathbf{e}}_x, \quad (\text{A.8})$$

canceling out terms

$$[(\mathbf{P} \cdot \nabla)\mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q}]_x = \left(p_x \frac{\partial q_x}{\partial x} + p_y \frac{\partial q_y}{\partial x} + p_z \frac{\partial q_z}{\partial x} \right) \hat{\mathbf{e}}_x. \quad (\text{A.9})$$

Analogous results are obtained in the other directions so that

$$\begin{aligned}
 (\mathbf{P} \cdot \nabla) \mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q} = & \left(p_x \frac{\partial q_x}{\partial x} + p_y \frac{\partial q_y}{\partial x} + p_z \frac{\partial q_z}{\partial x} \right) \hat{\mathbf{e}}_x \\
 & + \left(p_x \frac{\partial q_x}{\partial y} + p_y \frac{\partial q_y}{\partial y} + p_z \frac{\partial q_z}{\partial y} \right) \hat{\mathbf{e}}_y \\
 & + \left(p_x \frac{\partial q_x}{\partial z} + p_y \frac{\partial q_y}{\partial z} + p_z \frac{\partial q_z}{\partial z} \right) \hat{\mathbf{e}}_z
 \end{aligned} \tag{A.10}$$

that may be written out in vector form as

$$(\mathbf{P} \cdot \nabla) \mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q} = \mathbf{P} \cdot \frac{\partial \mathbf{Q}}{\partial x} + \mathbf{P} \cdot \frac{\partial \mathbf{Q}}{\partial y} + \mathbf{P} \cdot \frac{\partial \mathbf{Q}}{\partial z}. \tag{A.11}$$

However, the terms can be rearranged as

$$\begin{aligned}
 (\mathbf{P} \cdot \nabla) \mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q} = & \left(p_x \frac{\partial q_x}{\partial x} \hat{\mathbf{e}}_x + p_x \frac{\partial q_x}{\partial y} \hat{\mathbf{e}}_y + p_x \frac{\partial q_x}{\partial z} \hat{\mathbf{e}}_z \right) \\
 & + \left(p_y \frac{\partial q_y}{\partial x} \hat{\mathbf{e}}_x + p_y \frac{\partial q_y}{\partial y} \hat{\mathbf{e}}_y + p_y \frac{\partial q_y}{\partial z} \hat{\mathbf{e}}_z \right) \\
 & + \left(p_z \frac{\partial q_z}{\partial x} \hat{\mathbf{e}}_x + p_z \frac{\partial q_z}{\partial y} \hat{\mathbf{e}}_y + p_z \frac{\partial q_z}{\partial z} \hat{\mathbf{e}}_z \right),
 \end{aligned} \tag{A.12}$$

and thus

$$(\mathbf{P} \cdot \nabla) \mathbf{Q} + \mathbf{P} \times \nabla \times \mathbf{Q} = (p_x \nabla q_x + p_y \nabla q_y + p_z \nabla q_z). \tag{A.13}$$

An equivalent procedure for $(\mathbf{Q} \cdot \nabla) \mathbf{P} + \mathbf{Q} \times \nabla \times \mathbf{P}$ gives

$$(\mathbf{Q} \cdot \nabla) \mathbf{P} + \mathbf{Q} \times \nabla \times \mathbf{P} = q_x \nabla p_x + q_y \nabla p_y + q_z \nabla p_z. \tag{A.14}$$

B. Scalar Case

If we take one component vectors, for example, $\mathbf{P} \rightarrow p_x \hat{\mathbf{e}}_x$, $\mathbf{Q} \rightarrow q_x \hat{\mathbf{e}}_x$, the vector relationship (2.7) becomes

$$p_x \nabla^2 q_x - q_x \nabla^2 p_x = \nabla \cdot \left[p_x \frac{\partial q_x}{\partial x} \hat{\mathbf{e}}_x - q_x \frac{\partial p_x}{\partial x} \hat{\mathbf{e}}_x + \mathbf{P} \times \nabla \times \mathbf{Q} - \mathbf{Q} \times \nabla \times \mathbf{P} \right]. \tag{B.1}$$

Since $\nabla \times \mathbf{Q} = (\partial q_x / \partial z) \hat{\mathbf{e}}_y - (\partial q_x / \partial y) \hat{\mathbf{e}}_z$,

$$\mathbf{P} \times \nabla \times \mathbf{Q} = \left(p_x \frac{\partial q_x}{\partial y} \right) \hat{\mathbf{e}}_y + \left(p_x \frac{\partial q_x}{\partial z} \right) \hat{\mathbf{e}}_z \quad (\text{B.2})$$

and $\mathbf{Q} \times \nabla \times \mathbf{P} = (q_x (\partial p_x / \partial y)) \hat{\mathbf{e}}_y + (q_x (\partial p_x / \partial z)) \hat{\mathbf{e}}_z$. Therefore,

$$p_x \nabla^2 q_x - q_x \nabla^2 p_x = \nabla \cdot [p_x \nabla q_x - q_x \nabla p_x], \quad (\text{B.3})$$

and we recover Green's second identity for the functions p_x, q_x .

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