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Research Article

On Energy Conditions for Electromagnetic Diffraction by Apertures

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The diffraction of light is considered for a plane screen with an open bounded aperture. The corresponding solution behind the screen is given explicitly in terms of the Fourier transforms of the tangential components of the electric boundary field in the aperture. All components of the electric as well as the magnetic field vector are considered. We introduce solutions with global finite energy behind the screen and describe them in terms of two boundary potential functions. This new approach leads to a decoupling of the vectorial boundary equations in the aperture in the case of global finite energy. For the physically admissible solutions, that is, the solutions with local finite energy, we derive a characterisation in terms of the electric boundary fields.

1. Introduction

This paper deals with the classical diffraction problem for electromagnetic waves passing a bounded aperture in an ideally conducting plane screen. We treat the problem within the exact theory, that is, we consider the corresponding solutions of the time harmonic Maxwell equations that fulfil the correct boundary conditions on the screen.

The problem of diffraction of electromagnetic waves by an infinite slit has been treated by the Fourier method in the papers [1, 2]. In [1] especially representations of the solutions that fulfil a certain energy condition have been given in terms of distributional electric boundary fields satisfying special regularity properties. In [2] mapping properties of the corresponding boundary operators between Sobolev spaces have been studied. These Sobolev spaces have been chosen such that the corresponding diffraction solutions satisfy the correct physical energy condition.

While the slit problem treated in [1, 2] can be decoupled into two scalar problems by considering two kinds of polarisations of the electromagnetic field, in the case of a bounded aperture such a decoupling is not possible in general. However, for the latter case we derive a new kind of decoupling of the vectorial system which can be performed if and only if the condition of global finite energy in part (b) of Definition 2.5 is fulfilled, see Theorem 3.2.

Finally we study the condition of local finite energy which covers all physically admissible solutions. Here we give a characterisation of solutions with local finite energy in terms of a regularity property of the electric boundary fields, see Theorem 4.1. This is done in a self-contained way by using the Paley-Wiener theorem for distributions defined on the bounded aperture as well as a special contour integration method in the Fourier domain.

2. Electromagnetic Diffraction by an Aperture in a Plane Screen

We start with an informal physical description of the electromagnetic diffraction problem and fix some notations which will be used in the sequel. Then we will develop a more general mathematical frame with boundary distributions in Sobolev spaces in order to obtain diffraction solutions satisfying physical energy conditions.

Monochromatic light waves with a fixed wavenumber $k > 0$ satisfy the first-order system of Maxwell-Helmholtz equations

$$\begin{aligned} ik\underline{E}_*(\underline{x}) + \nabla \times \underline{B}_*(\underline{x}) &= \underline{0}, \\ -ik\underline{B}_*(\underline{x}) + \nabla \times \underline{E}_*(\underline{x}) &= \underline{0}. \end{aligned} \tag{2.1}$$

In the whole paper we consider a real wavenumber $k > 0$, although the results can be generalised to the case of a complex wavenumber $k \neq 0$ with $\operatorname{Re} k \geq 0$ and $\operatorname{Im} k \geq 0$. We assume that the electromagnetic field with components e_j, b_j, $j = 1, 2, 3$,

$$\begin{pmatrix} e_1(x, x_3) \\ e_2(x, x_3) \\ e_3(x, x_3) \end{pmatrix} = \underline{E}_*(x, x_3), \qquad \begin{pmatrix} b_1(x, x_3) \\ b_2(x, x_3) \\ b_3(x, x_3) \end{pmatrix} = \underline{B}_*(x, x_3), \tag{2.2}$$

consists of functions defined in the upper half-space

$$\mathscr{H} := \left\{ \underline{x} = (x, x_3) \in \mathbb{R}^3 \mid x \in \mathbb{R}^2,\ x_3 > 0 \right\}. \tag{2.3}$$

The diffraction problem is considered for an open bounded aperture

$$\Omega \subset \left\{ (x, 0) \mid x \in \mathbb{R}^2 \right\} \tag{2.4}$$

in the screen plane $x_3 = 0$. In the sequel we will suppress the notation of the third component 0 for the points in the screen plane, and interpret Ω as well as the *screen* $\Omega^c := \mathbb{R}^2 \setminus \Omega$ as subsets

of $\mathbb{R}^2$. For describing the whole electromagnetic field in terms of its boundary values, for the moment we assume that these are functions, given for $x \in \mathbb{R}^2$ by

$$\begin{pmatrix} e_{1,0}(x) \\ e_{2,0}(x) \\ e_{3,0}(x) \end{pmatrix} = \lim_{x_3 \downarrow 0} \underline{E}_*(x, x_3), \qquad \begin{pmatrix} b_{1,0}(x) \\ b_{2,0}(x) \\ b_{3,0}(x) \end{pmatrix} = \lim_{x_3 \downarrow 0} \underline{B}_*(x, x_3). \tag{2.5}$$

The screen Ω^c is assumed to be an ideal conducting wall. This implies the physical boundary conditions

$$e_{1,0}(x) = e_{2,0}(x) = 0 \quad \forall x \in \Omega^c. \tag{2.6}$$

In the general case the boundary fields (2.5) have to be replaced by appropriate distributions, and the limit $x_3 \downarrow 0$ will be performed in the Fourier domain instead of the half-space $\mathscr{H}$.

For this purpose we write again $\underline{x} = (x, x_3)$, with $x = (x_1, x_2) \in \mathbb{R}^2$ and fixed $x_3 > 0$, and assume that each field component $u(\cdot, x_3)$ represents a tempered distribution in $\mathcal{S}'(\mathbb{R}^2)$ with Fourier transform

$$\widehat{u}(\xi_1, \xi_2, x_3) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} u(x_1, x_2, x_3) e^{-i(\xi_1 x_1 + \xi_2 x_2)} dx_1 dx_2, \quad \xi_1, \xi_2 \in \mathbb{R}. \tag{2.7}$$

Then we obtain from the first-order Maxwell-Helmholtz equations the following Fourier transformed Maxwell-Helmholtz system: for all $x_3 \geq 0$ and fixed $\xi_1, \xi_2 \in \mathbb{R}$ we have

$$\begin{aligned} ik \begin{pmatrix} \widehat{e}_1 \\ \widehat{e}_2 \\ \widehat{e}_3 \end{pmatrix} + \begin{pmatrix} i\xi_2 \widehat{b}_3 - \dfrac{d}{dx_3} \widehat{b}_2 \\ \dfrac{d}{dx_3} \widehat{b}_1 - i\xi_1 \widehat{b}_3 \\ i\xi_1 \widehat{b}_2 - i\xi_2 \widehat{b}_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ -ik \begin{pmatrix} \widehat{b}_1 \\ \widehat{b}_2 \\ \widehat{b}_3 \end{pmatrix} + \begin{pmatrix} i\xi_2 \widehat{e}_3 - \dfrac{d}{dx_3} \widehat{e}_2 \\ \dfrac{d}{dx_3} \widehat{e}_1 - i\xi_1 \widehat{e}_3 \\ i\xi_1 \widehat{e}_2 - i\xi_2 \widehat{e}_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{2.8}$$

Here we have replaced the partial derivative with respect to x_3 by the ordinary derivative d/dx_3.

We define

$$C := \frac{1}{k} \begin{pmatrix} \dfrac{\partial^2}{\partial x_1 \partial x_2} & -\left(k^2 + \dfrac{\partial^2}{\partial x_1^2}\right) \\ k^2 + \dfrac{\partial^2}{\partial x_2^2} & -\dfrac{\partial^2}{\partial x_1 \partial x_2} \end{pmatrix}. \tag{2.9}$$

With the two-dimensional Laplace operator Δ we have

$$C^2 = -\left(k^2 + \Delta\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.10}$$

For the action of C in the Fourier domain we obtain multiplication by the matrix

$$\widehat{C}(\xi) := \frac{1}{k}\begin{pmatrix} -\xi_1\xi_2 & -(k^2 - \xi_1^2) \\ k^2 - \xi_2^2 & \xi_1\xi_2 \end{pmatrix}, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{2.11}$$

We replace (2.8) with the ordinary differential equations

$$\frac{d}{dx_3}\begin{pmatrix} \widehat{e}_1 \\ \widehat{e}_2 \end{pmatrix} = -i\widehat{C}\begin{pmatrix} \widehat{b}_1 \\ \widehat{b}_2 \end{pmatrix}, \qquad \frac{d}{dx_3}\begin{pmatrix} \widehat{b}_1 \\ \widehat{b}_2 \end{pmatrix} = i\widehat{C}\begin{pmatrix} \widehat{e}_1 \\ \widehat{e}_2 \end{pmatrix}, \tag{2.12}$$

and the two algebraic conditions

$$\widehat{e}_3 = \frac{1}{k}\left(\xi_2\widehat{b}_1 - \xi_1\widehat{b}_2\right), \qquad \widehat{b}_3 = \frac{1}{k}(\xi_1\widehat{e}_2 - \xi_2\widehat{e}_1). \tag{2.13}$$

For all fixed $\xi \in \mathbb{R}^2$ we supplement the system of differential equations (2.12) by the initial conditions

$$\widehat{e}_{j,0}(\xi) := \widehat{e}_j(\xi, 0), \quad \widehat{b}_{j,0}(\xi) := \widehat{b}_j(\xi, 0), \quad j = 1, 2, \tag{2.14}$$

and put

$$m(\xi) := \begin{cases} \sqrt{k^2 - |\xi|^2}, & |\xi| \le k, \\ i\sqrt{|\xi|^2 - k^2}, & |\xi| > k. \end{cases} \tag{2.15}$$

Then the general solution of the homogeneous linear system (2.12) is

$$\begin{pmatrix} \widehat{e}_1(\xi, x_3) \\ \widehat{e}_2(\xi, x_3) \\ \widehat{b}_1(\xi, x_3) \\ \widehat{b}_2(\xi, x_3) \end{pmatrix} = \begin{pmatrix} \cos(m(\xi)x_3)E & -i\,\dfrac{\sin(m(\xi)x_3)}{m(\xi)}\widehat{C}(\xi) \\ i\,\dfrac{\sin(m(\xi)x_3)}{m(\xi)}\widehat{C}(\xi) & \cos(m(\xi)x_3)E \end{pmatrix}\begin{pmatrix} \widehat{e}_{1,0}(\xi) \\ \widehat{e}_{2,0}(\xi) \\ \widehat{b}_{1,0}(\xi) \\ \widehat{b}_{2,0}(\xi) \end{pmatrix} \tag{2.16}$$

with the 2×2 unit matrix E.

But the terms $\cos(m(\xi)x_3)$ and $\sin(m(\xi)x_3)$ are exponentially increasing for fixed $x_3 > 0$ and $|\xi| \to \infty$. For avoiding that the Fourier transformed fields are also exponentially increasing we have to require the following algebraic conditions for $|\xi| > k$:

$$m \cdot \begin{pmatrix} \widehat{b}_{1,0} \\ \widehat{b}_{2,0} \end{pmatrix} = \widehat{C}\begin{pmatrix} \widehat{e}_{1,0} \\ \widehat{e}_{2,0} \end{pmatrix}. \tag{2.17}$$

By using (2.17), for $x_3 \downarrow 0$ and $|\xi| > k$ we can replace (2.13) with

$$\widehat{e}_{3,0} = -\frac{1}{m}(\xi_1\widehat{e}_{1,0} + \xi_2\widehat{e}_{2,0}), \qquad \widehat{b}_{3,0} = -\frac{1}{k}(\xi_2\widehat{e}_{1,0} - \xi_1\widehat{e}_{2,0}). \tag{2.18}$$

From the general solution and (2.17), (2.18) we obtain the following decay conditions for $x_3 > 0$:

$$\widehat{e}_j(\xi, x_3) = e^{ix_3 m(\xi)}\widehat{e}_{j,0}(\xi), \quad \widehat{b}_j(\xi, x_3) = e^{ix_3 m(\xi)}\widehat{b}_{j,0}(\xi), \quad j = 1, 2, 3. \tag{2.19}$$

With $x = (x_1, x_2) \in \mathbb{R}^2$, $\underline{x} = (x, x_3) \in \mathbb{R}^3$ and $x_3 > 0$ there holds the important and well-known Sommerfeld-Weyl integral representation

$$F_k(x, x_3) := \frac{e^{ik|\underline{x}|}}{|\underline{x}|} = \frac{i}{2\pi}\iint_{\mathbb{R}^2} \frac{e^{ix_3 m(\xi) + i\langle x, \xi\rangle}}{m(\xi)} d\xi. \tag{2.20}$$

The left-hand side in (2.20) is the singular solution of the three-dimensional Helmholtz equation $(\Delta + k^2)F_k = -4\pi\delta$. For this reason it is natural to require the algebraic conditions (2.17), (2.18) also in the case $|\xi| < k$, such that (2.19) is generally valid for $x_3 > 0$ and $\xi \in \mathbb{R}^2$, $|\xi| \neq k$.

Distributions $u \in \mathcal{D}'(\mathbb{R}^2)$ with compact support in the screen plane are tempered, and it follows from the Paley-Wiener theorem that $\widehat{u}$ is a smooth function which has polynomial growth on $\mathbb{R}^2$. This is used in the following theorem, which results if we regard (2.17), (2.18), and (2.19) and apply the Fourier inversion formula for $x_3 > 0$ to each component $\widehat{e}_j(\cdot, x_3)$ and $\widehat{b}_j(\cdot, x_3)$.

Theorem 2.1. *Let there be given $e_{1,0}, e_{2,0} \in \mathcal{S}'(\mathbb{R}^2)$ with support in the bounded region $\overline{\Omega}$. Then the following functions $e_j, b_j : \mathcal{H} \to \mathbb{C}$ constitute a C^∞-solution of the Maxwell-Helmholtz system* (2.1) *in the upper half-space $\mathcal{H}$, $j = 1, 2, 3$;*

$$\begin{aligned}
e_1(x, x_3) &= \frac{1}{2\pi}\iint_{\mathbb{R}^2} \widehat{e}_{1,0}(\xi) e^{ix_3 m(\xi) + i\langle x, \xi\rangle} d\xi, \\
e_2(x, x_3) &= \frac{1}{2\pi}\iint_{\mathbb{R}^2} \widehat{e}_{2,0}(\xi) e^{ix_3 m(\xi) + i\langle x, \xi\rangle} d\xi, \\
b_1(x, x_3) &= -\frac{1}{2\pi k}\iint_{\mathbb{R}^2} \frac{\xi_1\xi_2\widehat{e}_{1,0}(\xi) + (k^2 - \xi_1^2)\widehat{e}_{2,0}(\xi)}{m(\xi)} e^{ix_3 m(\xi) + i\langle x, \xi\rangle} d\xi, \\
b_2(x, x_3) &= \frac{1}{2\pi k}\iint_{\mathbb{R}^2} \frac{(k^2 - \xi_2^2)\widehat{e}_{1,0}(\xi) + \xi_1\xi_2\widehat{e}_{2,0}(\xi)}{m(\xi)} e^{ix_3 m(\xi) + i\langle x, \xi\rangle} d\xi, \\
e_3(x, x_3) &= -\frac{1}{2\pi}\iint_{\mathbb{R}^2} \frac{\xi_1\widehat{e}_{1,0}(\xi) + \xi_2\widehat{e}_{2,0}(\xi)}{m(\xi)} e^{ix_3 m(\xi) + i\langle x, \xi\rangle} d\xi, \\
b_3(x, x_3) &= -\frac{1}{2\pi k}\iint_{\mathbb{R}^2} [\xi_2\widehat{e}_{1,0}(\xi) - \xi_1\widehat{e}_{2,0}(\xi)] e^{ix_3 m(\xi) + i\langle x, \xi\rangle} d\xi.
\end{aligned} \tag{2.21}$$

Proof. The calculation of the partial derivatives of e_j and b_j can be interchanged with integration. This can be used to check the Maxwell-Helmholtz equations independent from the previous representations of the Fourier-transforms $\widehat{e}_j$ and $\widehat{b}_j$ in terms of $\widehat{e}_{1,0}$ and $\widehat{e}_{2,0}$. □

Definition 2.2. The electromagnetic field in the half-space $x_3 > 0$ behind the screen is completely determined by the electric boundary components $e_{1,0}, e_{2,0}$. We call $e_j, b_j : \mathscr{H} \to \mathbb{C}$, $j = 1, 2, 3$, the half-space solution determined by the boundary distributions $e_{1,0}, e_{2,0} \in \mathcal{S}'(\mathbb{R}^2)$ with compact support in $\overline{\Omega}$.

Remark 2.3. Assume that $e_{1,0}, e_{2,0}$ are smooth functions with compact support in $\overline{\Omega}$ and that $\underline{E}_*, \underline{B}_*$ is the corresponding electromagnetic field with the components given in Theorem 2.1. Then we obtain from Fourier's inversion formula that for all $x \in \mathbb{R}^2$

$$\lim_{x_3 \downarrow 0} e_j(x, x_3) = e_{j,0}(x), \quad j = 1, 2. \tag{2.22}$$

Remark 2.4. Assume again that $e_{1,0}, e_{2,0}$ are smooth functions with compact support in $\overline{\Omega}$. Then we obtain from Theorem 2.1 and the Sommerfeld-Weyl integral (2.20) for all $x \in \mathbb{R}^2$ and all $x_3 > 0$ that

$$\begin{aligned} b_1(x, x_3) &= -\frac{i}{2\pi k} \frac{\partial^2}{\partial x_1 \partial x_2} \iint_\Omega e_{1,0}(y) F_k(x - y, x_3) dy \\ &\quad + \frac{i}{2\pi k} \left(k^2 + \frac{\partial^2}{\partial x_1^2} \right) \iint_\Omega e_{2,0}(y) F_k(x - y, x_3) dy, \\ b_2(x, x_3) &= -\frac{i}{2\pi k} \left(k^2 + \frac{\partial^2}{\partial x_2^2} \right) \iint_\Omega e_{1,0}(y) F_k(x - y, x_3) dy \\ &\quad + \frac{i}{2\pi k} \frac{\partial^2}{\partial x_1 \partial x_2} \iint_\Omega e_{2,0}(y) F_k(x - y, x_3) dy. \end{aligned} \tag{2.23}$$

Equations (2.23) involve the boundary conditions (2.6) for the electric field components on the ideal conducting plane screen Ω^c. In order to obtain a coupled system of boundary integro-differential equations we pass for $x \in \mathbb{R}^2$ to the limit $x_3 \downarrow 0$ and obtain

$$\begin{aligned} b_{1,0}(x) &= -\frac{i}{2\pi k} \frac{\partial^2}{\partial x_1 \partial x_2} \iint_\Omega e_{1,0}(y) \frac{e^{ik|x-y|}}{|x - y|} dy \\ &\quad + \frac{i}{2\pi k} \left(k^2 + \frac{\partial^2}{\partial x_1^2} \right) \iint_\Omega e_{2,0}(y) \frac{e^{ik|x-y|}}{|x - y|} dy, \\ b_{2,0}(x) &= -\frac{i}{2\pi k} \left(k^2 + \frac{\partial^2}{\partial x_2^2} \right) \iint_\Omega e_{1,0}(y) \frac{e^{ik|x-y|}}{|x - y|} dy \\ &\quad + \frac{i}{2\pi k} \frac{\partial^2}{\partial x_1 \partial x_2} \iint_\Omega e_{2,0}(y) \frac{e^{ik|x-y|}}{|x - y|} dy. \end{aligned} \tag{2.24}$$

In general the electric boundary fields $e_{1,0}$ and $e_{2,0}$ are unknown distributions with compact support in $\overline{\Omega}$, whereas $b_{1,0}$ and $b_{2,0}$ are given distributions in the aperture Ω. In order to

select physical admissible solutions of the diffraction problem we need some conditions for its electromagnetic energy content, especially in local volume elements $G \subset \mathscr{H}$. Recall that $\mathscr{H} := \mathbb{R}^2 \times \mathbb{R}^+$.

Definition 2.5. Let $e_j, b_j : \mathscr{H} \to \mathbb{C}$, $j = 1,2,3$, be the half-space solution determined by the boundary distributions $e_{1,0}, e_{2,0} \in \mathcal{S}'(\mathbb{R}^2)$ with compact support in $\overline{\Omega}$.

(a) The solution is called *physical admissible* if and only if it satisfies the *local energy condition*

$$\frac{1}{2}\sum_{j=1}^{3}\iiint_G \left(\left|e_j(\underline{x})\right|^2 + \left|b_j(\underline{x})\right|^2\right) d\underline{x} < \infty \tag{2.25}$$

for every bounded domain $G \subset \mathscr{H}$.

(b) The solution satisfies the stronger *global energy condition* if and only if

$$\frac{1}{2}\sum_{j=1}^{3}\iiint_{\mathcal{L}_h} \left(\left|e_j(\underline{x})\right|^2 + \left|b_j(\underline{x})\right|^2\right) d\underline{x} < \infty \tag{2.26}$$

for some $h > 0$, and therewith for all $h > 0$, with the layer

$\mathcal{L}_h := \{(x_1, x_2, x_3) \in \mathscr{H} \mid x_3 < h\}$.

In the following two sections we determine the solutions with global as well as those with local finite energy in terms of an appropriate functional analytical setting for the electric boundary fields $e_{1,0}$ and $e_{2,0}$.

3. The Global Energy Condition

Throughout the rest of this paper let the open aperture $\Omega := \bigcup_{j=1}^{\kappa} \Omega_j \subset \mathbb{R}^2$ be a finite union of nonempty bounded Lipschitz domains Ω_j in the screen plane, such that the compact sets $\overline{\Omega}_j$ are pairwise disjoint.

By $H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, we denote the Sobolev space of tempered distributions h, for which the Fourier transform $\widehat{h}$ is locally integrable with

$$\|h\|_{H^s(\mathbb{R}^2)} := \left(\iint_{\mathbb{R}^2} \left|\widehat{h}(\xi)\right|^2 \left(1 + |\xi|^2\right)^s d\xi\right)^{1/2} < \infty \tag{3.1}$$

(cf. [3], Chapter 8.8). $\|\cdot\|_{H^s(\mathbb{R}^2)}$ is the norm on the Banach space $H^s(\mathbb{R}^2)$.

For $s \in \mathbb{R}$ the Sobolev space $\widetilde{H}^s(\Omega)$ is given by

$$\widetilde{H}^s(\Omega) = \left\{h \in H^s\left(\mathbb{R}^2\right) \mid \operatorname{supp} h \subset \overline{\Omega}\right\}. \tag{3.2}$$

Here supp h denotes the support of the distribution h in the compact set $\overline{\Omega}$. The space $\widetilde{H}^s(\Omega)$ is equipped with the norm of $H^s(\mathbb{R}^2)$, which makes it into a Banach space (cf. [4, Chapter 3]). The Lipschitz property of $\partial\Omega$ guarantees that $\widetilde{H}^s(\Omega)$ is the closure of $\mathfrak{D}(\Omega)$ in $H^s(\mathbb{R}^2)$. For a more general result see [4, Theorem 3.29].

Theorem 3.1. *Let $e_j, b_j : \mathscr{H} \to \mathbb{C}$, $j = 1, 2, 3$, be the half-space solution determined by the boundary distributions $e_{1,0}, e_{2,0} \in \mathcal{S}'(\mathbb{R}^2)$ with compact support in $\overline{\Omega}$. Then the diffraction solution has global finite energy if and only if*

$$e_{1,0}, e_{2,0}, \quad \nabla \cdot \begin{pmatrix} -e_{2,0} \\ e_{1,0} \end{pmatrix} \in \widetilde{H}^{-1/2}(\Omega) \tag{3.3}$$

and $\widehat{e}_{1,0}(\xi)\xi_1 + \widehat{e}_{2,0}(\xi)\xi_2 = 0$ for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ with $|\xi| = k$.

Proof. In order to perform the energy evaluation with Parseval's theorem we define with the Fourier transformed electromagnetic boundary fields $\widehat{e}_{j,0}, \widehat{b}_{j,0}$, $j = 1, 2, 3$, the quantity

$$W_0(\xi) := \frac{1}{2}\sum_{j=1}^{3}\left(\left|\widehat{e}_{j,0}(\xi)\right|^2 + \left|\widehat{b}_{j,0}(\xi)\right|^2\right). \tag{3.4}$$

It follows from a lengthy calculation with $|z|^2 = z\overline{z}$ for the values $z \in \mathbb{C}$ of the Fourier transformed boundary fields as well as from the algebraic relations (2.17), (2.18), which are valid for $|\xi| \neq k$, that

$$W_0(\xi) = \begin{cases} |\widehat{e}_{1,0}(\xi)|^2 + |\widehat{e}_{2,0}(\xi)|^2 + |\widehat{e}_{3,0}(\xi)|^2, & |\xi| < k, \\ |\widehat{e}_{3,0}(\xi)|^2 + \left|\widehat{b}_{3,0}(\xi)\right|^2, & |\xi| > k. \end{cases} \tag{3.5}$$

With $V_1 := \{\xi \in \mathbb{R}^2 \mid |\xi| < k\}$, $V_2 := \{\xi \in \mathbb{R}^2 \mid |\xi| > k\}$ we conclude from Parseval's theorem, (2.19) and the definition (2.15) of $m(\xi)$ for $h > 0$

$$\begin{aligned} \mathcal{E}(h) &:= \frac{1}{2}\sum_{j=1}^{3}\int_0^h \iint_{\mathbb{R}^2}\left(\left|e_j(x, x_3)\right|^2 + \left|b_j(x, x_3)\right|^2\right)dx\,dx_3 \\ &= h\iint_{V_1} W_0(\xi)d\xi + \iint_{V_2} \frac{1 - e^{-2h\sqrt{|\xi|^2 - k^2}}}{2\sqrt{|\xi|^2 - k^2}}\, W_0(\xi)d\xi. \end{aligned} \tag{3.6}$$

Using for all $\xi \in \mathbb{R}^2$ with $|\xi| \neq k$ the estimate

$$\begin{aligned} &\frac{1}{2}\left(|\widehat{e}_{1,0}(\xi)|^2 + |\widehat{e}_{2,0}(\xi)|^2 + |\widehat{e}_{3,0}(\xi)|^2 + \left|\widehat{b}_{3,0}(\xi)\right|^2\right) \\ &\quad \leq W_0(\xi) \leq |\widehat{e}_{1,0}(\xi)|^2 + |\widehat{e}_{2,0}(\xi)|^2 + |\widehat{e}_{3,0}(\xi)|^2 + \left|\widehat{b}_{3,0}(\xi)\right|^2, \end{aligned} \tag{3.7}$$

and observing that we have uniformly in $r > k$ for appropriate constants $\alpha, \beta > 0$

$$\frac{\alpha}{\sqrt{1+r^2}} \leq \frac{1-e^{-2h\sqrt{r^2-k^2}}}{2\sqrt{r^2-k^2}} \leq \frac{\beta}{\sqrt{1+r^2}}, \tag{3.8}$$

we conclude from (3.6) that $\mathcal{E}(h)$ is finite if and only if

$$e_{1,0}, e_{2,0}, e_{3,0}, b_{3,0} = \frac{i}{k}\nabla \cdot \begin{pmatrix} -e_{2,0} \\ e_{1,0} \end{pmatrix} \in H^{-1/2}\left(\mathbb{R}^2\right). \tag{3.9}$$

Note that with $e_{1,0}$ and $e_{2,0}$ also $\nabla \cdot (-e_{2,0}, e_{1,0})^T$ has support in $\overline{\Omega}$.

Assume first that $\mathcal{E}(h) < \infty$ and that $\widehat{e}_{1,0}(\xi)\xi_1 + \widehat{e}_{2,0}(\xi)\xi_2 \neq 0$ for a certain $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ with $|\xi| = k$. Since $e_{1,0}, e_{2,0}$ have compact support, we conclude from the Paley-Wiener theorem that $\widehat{e}_{1,0}, \widehat{e}_{2,0} : \mathbb{R}^2 \to \mathbb{C}$ are smooth and especially continuous. Thus $|\widehat{e}_{1,0}(\xi)\xi_1 + \widehat{e}_{2,0}(\xi)\xi_2|^2 \geq \delta > 0$ in a bounded domain

$$(\xi_1, \xi_2) = r(\cos\varphi, \sin\varphi) \quad \text{with } \varphi_1 < \varphi < \varphi_2,\ k < r < k + \varepsilon, \tag{3.10}$$

and hence $|\widehat{e}_{3,0}|^2$ because of (2.18) is not integrable. Due to (3.1) this violates the necessary condition $e_{3,0} \in H^{-1/2}(\mathbb{R}^2)$ in (3.9).

For showing the other direction of the equivalence stated in the theorem we assume that

$$e_{1,0}, e_{2,0}, \quad \nabla \cdot \begin{pmatrix} -e_{2,0} \\ e_{1,0} \end{pmatrix} \in \widetilde{H}^{-1/2}(\Omega), \tag{3.11}$$

$$\widehat{e}_{1,0}(\xi_1, \xi_2)\xi_1 + \widehat{e}_{2,0}(\xi_1, \xi_2)\xi_2 = 0, \tag{3.12}$$

for all $\xi_1, \xi_2 \in \mathbb{R}$ with $\xi_1^2 + \xi_2^2 = k^2$. In order to prove (3.9) it remains to show that $e_{3,0} \in H^{-1/2}(\mathbb{R}^2)$. Since $e_{1,0}$, $e_{2,0}$ have compact support in $\overline{\Omega}$, we obtain from the Paley-Wiener theorem that the Fourier transforms $\widehat{e}_{1,0}$, $\widehat{e}_{2,0}$ can be continuated to entire functions. We also denote these entire functions by $\widehat{e}_{1,0}, \widehat{e}_{2,0}$. We define the entire function f by

$$f(z_1, z_2) := \widehat{e}_{1,0}(z_1, z_2)z_1 + \widehat{e}_{2,0}(z_1, z_2)z_2. \tag{3.13}$$

Using (3.12) and (2.18), it follows for example from Theorem A.1 in the appendix that $|\widehat{e}_{3,0}|^2$ is a locally integrable function.

From the first equation in (2.18) and the Cauchy-Schwarz inequality we obtain the estimate

$$|\widehat{e}_{3,0}(\xi)|^2 \leq \frac{|\xi|^2}{|\xi|^2 - k^2} \cdot \left(|\widehat{e}_{1,0}(\xi)|^2 + |\widehat{e}_{2,0}(\xi)|^2\right), \quad |\xi| > k. \tag{3.14}$$

Now $e_{3,0} \in H^{-1/2}(\mathbb{R}^2)$ follows from the assumption $e_{1,0}, e_{2,0} \in \widetilde{H}^{-1/2}(\Omega)$, because $|e_{3,0}|^2$ is locally integrable and $|\xi|^2/(|\xi|^2 - k^2)$ is bounded for sufficiently large $|\xi|$. □

Next we present a characterisation of the global finite energy solutions in terms of two boundary potential functions. In the following we use the vectorial differential operator (2.9) satisfying (2.10).

Theorem 3.2. *Boundary potential functions*

(a) *Given are two functions* $u_1, u_2 \in \widetilde{H}^{1/2}(\Omega)$, *in the following called boundary potential functions, satisfying the regularity condition*

$$\nabla \cdot \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} \in \widetilde{H}^{1/2}(\Omega). \tag{3.15}$$

Define

$$\begin{pmatrix} e_{1,0} \\ e_{2,0} \end{pmatrix} := C \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{3.16}$$

Then the corresponding electromagnetic field e_j, b_j, $j = 1, 2, 3$, *determined by* $e_{1,0}$ *and* $e_{2,0}$ *according to Theorem 2.1, has global finite energy.*

(b) *Let* $e_{1,0}$, $e_{2,0}$ *be given such that the half-space solution in Theorem 2.1 has global finite energy. If the set* $\mathbb{R}^2 \setminus \overline{\Omega}$ *is connected, then the boundary fields* $e_{1,0}$ *and* $e_{2,0}$ *can be represented by two boundary potential functions* $u_1, u_2 \in \widetilde{H}^{1/2}(\Omega)$ *satisfying* (3.15) *and* (3.16) *in part (a).*

(c) *From the assumptions of part (a), or alternatively part (b), one obtains in the distributional sense*

$$\begin{aligned} b_{1,0}(x) &= \frac{i}{2\pi}\left(k^2 + \Delta\right) \iint_\Omega u_1(y) \frac{e^{ik|x-y|}}{|x-y|} dy, \\ b_{2,0}(x) &= \frac{i}{2\pi}\left(k^2 + \Delta\right) \iint_\Omega u_2(y) \frac{e^{ik|x-y|}}{|x-y|} dy \end{aligned} \tag{3.17}$$

on the whole screen plane $\mathbb{R}^2$ *with*

$$b_{1,0}, b_{2,0}, \quad \nabla \cdot \begin{pmatrix} -b_{2,0} \\ b_{1,0} \end{pmatrix} \in H^{-1/2}\left(\mathbb{R}^2\right). \tag{3.18}$$

Proof. Part (a) follows in the Fourier domain by representing $\widehat{e}_{1,0}$, $\widehat{e}_{2,0}$, $\widehat{e}_{3,0}$ and $\widehat{b}_{3,0}$ in terms of $\widehat{u}_1$ and $\widehat{u}_2$ as

$$\begin{aligned} \widehat{e}_{1,0} &= -k\widehat{u}_2 - \frac{\xi_1}{k}(\xi_2 \widehat{u}_1 - \xi_1 \widehat{u}_2), \\ \widehat{e}_{2,0} &= +k\widehat{u}_1 - \frac{\xi_2}{k}(\xi_2 \widehat{u}_1 - \xi_1 \widehat{u}_2), \end{aligned}$$

$$\widehat{e}_{3,0} = -\frac{m}{k}(\xi_2\widehat{u}_1 - \xi_1\widehat{u}_2),$$
$$\widehat{b}_{3,0} = \xi_1\widehat{u}_1 + \xi_2\widehat{u}_2. \tag{3.19}$$

These equations and the assumptions imply that $e_{1,0}, e_{2,0}, b_{3,0} \in H^{-1/2}(\mathbb{R}^2)$, and regarding $|m(\xi)| \leq \alpha\sqrt{1+|\xi|^2}$ uniformly in $\xi \in \mathbb{R}^2$ for a certain constant $\alpha > 0$ also $e_{3,0} \in H^{-1/2}(\mathbb{R}^2)$. Since $e_{1,0}$, $e_{2,0}$ have compact support in $\overline{\Omega}$ like u_1, u_2, the proof of part (a) follows from the fact that condition (3.9) is equivalent to $\mathcal{E}(h) < \infty$.

For proving part (b) we assume that the diffraction solution corresponding to $e_{1,0}, e_{2,0} \in \widetilde{H}^{-1/2}(\Omega)$ has global finite energy. We conclude from Theorem 3.1 and Theorem A.1 in the appendix that we obtain entire functions $v_1, v_2 : \mathbb{C}^2 \to \mathbb{C}$ by

$$v_1(\xi_1,\xi_2) = +\frac{1}{k}\widehat{e}_{2,0}(\xi_1,\xi_2) - \frac{\xi_2}{k}\cdot\frac{\widehat{e}_{1,0}(\xi)\xi_1 + \widehat{e}_{2,0}(\xi)\xi_2}{\xi_1^2+\xi_2^2-k^2},$$
$$v_2(\xi_1,\xi_2) = -\frac{1}{k}\widehat{e}_{1,0}(\xi_1,\xi_2) + \frac{\xi_1}{k}\cdot\frac{\widehat{e}_{1,0}(\xi)\xi_1 + \widehat{e}_{2,0}(\xi)\xi_2}{\xi_1^2+\xi_2^2-k^2}. \tag{3.20}$$

From (3.20) we get for $j = 1,2$ that

$$\iint_{\mathbb{R}^2} \left|v_j(\xi)\right|^2\left(1+|\xi|^2\right)^{1/2} d\xi < \infty. \tag{3.21}$$

Thus for $j = 1,2$ the inverse Fourier transform u_j of v_j lies in $H^{1/2}(\mathbb{R}^2)$.

Now we show that u_1 and u_2 have their support in $\overline{\Omega}$ and hence, by reason of (3.21), $u_1, u_2 \in \widetilde{H}^{1/2}(\Omega)$. To this aim we choose some $R > 0$ such that

$$\Omega \subset B_R := \left\{x \in \mathbb{R}^2 \mid |x| < R\right\}. \tag{3.22}$$

Since $e_{1,0}, e_{2,0}$ are supported in $\overline{\Omega}$, we have supp $e_{1,0}$, supp $e_{2,0} \subset \overline{B}_R$. Therefore we obtain from the Paley-Wiener theorem and (3.20) that u_1 and u_2 are also supported in $\overline{B}_R$, because v_1, v_2, as $\widehat{e}_{1,0}$, $\widehat{e}_{2,0}$, are of exponential type not larger than R.

Equations (3.20) can be resolved with respect to $\widehat{e}_{1,0}$, $\widehat{e}_{2,0}$. This gives (3.16). Together with (2.10) it follows that in the complement of $\overline{\Omega}$ there holds

$$-\left(k^2+\Delta\right)\begin{pmatrix}u_1\\u_2\end{pmatrix} = C^2\begin{pmatrix}u_1\\u_2\end{pmatrix} = C\begin{pmatrix}e_{1,0}\\e_{2,0}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{3.23}$$

Now we make use of the fact that the complement of $\overline{\Omega}$ is connected. Since supp $u_j \subset \overline{B}_R$ for $j = 1,2$, Holmgren's unique continuation principle, applied to the two scalar Helmholtz equations in (3.23), implies that u_1, u_2 have their support in $\overline{\Omega}$.

Finally, since $v_j = \widehat{u}_j$ for $j = 1, 2$, from (3.20) we obtain for $\xi \in \mathbb{R}^2$

$$-\xi_1 \widehat{u}_2 + \xi_2 \widehat{u}_1 = -k \frac{\xi_1 \widehat{e}_{1,0} + \xi_2 \widehat{e}_{2,0}}{\xi_1^2 + \xi_2^2 - k^2}. \tag{3.24}$$

The asymptotic behaviour of this expression for $|\xi| \to \infty$ shows the validity of the regularity condition (3.15).

The convolution integrals in part (c) can be rewritten in the Fourier domain by using the Sommerfeld-Weyl integral representation (2.20) in the limit $x_3 \downarrow 0$. The resulting relations in the Fourier domain follow from the representations of the components b_1 and b_2 in Theorem 2.1.

The validity of (3.18) is a consequence of the algebraic relations between $\widehat{u}_1$, $\widehat{u}_2$, $\widehat{b}_{1,0}$, and $\widehat{b}_{2,0}$. □

Remark 3.3. Consider the Sobolev spaces $H^s(\Omega)$ given for $s \in \mathbb{R}$ by

$$H^s(\Omega) = \left\{ F|_\Omega \mid F \in H^s\left(\mathbb{R}^2\right) \right\}, \tag{3.25}$$

with the restriction $F|_\Omega$ of the tempered distribution $F : \mathcal{S}(\mathbb{R}^2) \to \mathbb{C}$ to the subspace $\mathcal{S}(\Omega)$ of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$, where $\mathcal{S}(\Omega)$ is the closure of the set $\mathfrak{D}(\Omega)$ in $\mathcal{S}(\mathbb{R}^2)$ with respect to the topology of $\mathcal{S}(\mathbb{R}^2)$ [5], §1 in Section 5.

$H^s(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{H^s(\Omega)}$ given by

$$\|f\|_{H^s(\Omega)} = \inf\left\{ \|F\|_{H^s(\mathbb{R}^2)} \mid F \in H^s\left(\mathbb{R}^2\right) \text{ is a continuation of } f \right\}. \tag{3.26}$$

The magnetic boundary fields in part (c) of Theorem 3.2 corresponding to global finite energy solutions may also be reinterpreted as distributions restricted to Ω. In this case we obtain

$$b_{1,0}, b_{2,0}, \quad \nabla \cdot \begin{pmatrix} -b_{2,0} \\ b_{1,0} \end{pmatrix} \in H^{-1/2}(\Omega). \tag{3.27}$$

In general the conditions (3.27) are weaker than the conditions (3.18) in Theorem 3.2, and we assume that they are fulfilled for diffraction solutions with *local* finite energy.

The conditions (3.27), where $b_{1,0}$ and $b_{2,0}$ are considered only in the aperture Ω, reflects the physical fact that $b_{1,0}$ and $b_{2,0}$ are only prescribed in Ω. Namely, $b_{1,0}$ and $b_{2,0}$ are the tangential magnetic components of the incoming electromagnetic wave in the aperture Ω.

4. The Local Energy Condition

In this section we derive the following characterisation for the diffraction solutions of local finite energy.

Theorem 4.1. *Let be* $e_{1,0}, e_{2,0} \in \mathcal{S}'(\mathbb{R}^2)$ *and* $\operatorname{supp} e_{1,0}, \operatorname{supp} e_{2,0} \subset \overline{\Omega}$. *Let* $e_j, b_j : \mathscr{H} \to \mathbb{C}$, $j = 1,2,3$, *be defined as in Theorem 2.1. Then the diffraction solution* e_j, b_j, $j = 1,2,3$, *has local finite energy if and only if*

$$e_{1,0}, e_{2,0}, \quad \nabla \cdot \begin{pmatrix} -e_{2,0} \\ e_{1,0} \end{pmatrix} \in \widetilde{H}^{-1/2}(\Omega). \tag{4.1}$$

For proving Theorem 4.1 firstly we formulate the subsequent Lemma 4.2. Then, using this lemma, we give the proof of the theorem. Afterwards we prove the lemma.

In the sequel we will make use of the following notations. For $r > 0$ we define the open ball B_r by

$$B_r := \left\{ x \in \mathbb{R}^2 \mid |x| < r \right\}. \tag{4.2}$$

For $r > 0$ and $h > 0$ the open cylinder $Z_{r,h}$ is defined by

$$Z_{r,h} := \left\{ \underline{x} = (x, x_3) \in \mathbb{R}^3 \mid x \in B_r,\ 0 < x_3 < h \right\}. \tag{4.3}$$

Lemma 4.2. *Let be* $e_{1,0}, e_{2,0} \in \mathcal{S}'(\mathbb{R}^2)$ *and* $\operatorname{supp} e_{1,0}, \operatorname{supp} e_{2,0} \subset \overline{\Omega}$. *Let* $e_j, b_j : \mathscr{H} \to \mathbb{C}$, $j = 1,2,3$, *be the diffraction solution given in Theorem 2.1. Let be* $R' > 0$ *with* $\Omega \subset B_{R'}$ *and let be* $R > R'$ *and* $H > 0$. *Then one has the following equivalences and implications.*

(a) $e_j \in L^2(Z_{R,H}) \Leftrightarrow e_j \in L^2(\mathbb{R}^2 \times (0,H))$ *for* $j \in \{1,2\}$.

(b) $b_3 \in L^2(Z_{R,H}) \Leftrightarrow b_3 \in L^2(\mathbb{R}^2 \times (0,H))$.

(c) $e_j \in L^2(\mathbb{R}^2 \times (0,H)) \Leftrightarrow e_{j,0} \in \widetilde{H}^{-1/2}(\Omega)$ *for* $j \in \{1,2\}$.

(d) $b_3 \in L^2(\mathbb{R}^2 \times (0,H)) \Leftrightarrow b_{3,0} \in \widetilde{H}^{-1/2}(\Omega)$.

(e) $e_{1,0}, e_{2,0} \in \widetilde{H}^{-1/2}(\Omega) \Rightarrow e_3 \in L^2(Z_{R,H})$.

(f) $e_{1,0}, e_{2,0}, \nabla \cdot \left(\begin{smallmatrix} -e_{2,0} \\ e_{1,0} \end{smallmatrix}\right) \in \widetilde{H}^{-1/2}(\Omega) \Rightarrow b_1, b_2 \in L^2(Z_{R,H})$.

Proof of Theorem 4.1. Let there be given $e_{1,0}, e_{2,0} \in \mathcal{S}'(\mathbb{R}^2)$ with support in $\overline{\Omega}$ and let e_j, b_j, $j = 1,2,3$, be defined as in Theorem 2.1.

Firstly, we assume that the diffraction solution e_j, b_j, $j = 1,2,3$, has local finite energy and show the validity of (4.1). Since the diffraction solution has local finite energy especially we have $e_1, e_2, b_3 \in L^2(Z_{R,H})$. From the parts (a) and (b) of Lemma 4.2 we find that $e_1, e_2, b_3 \in L^2(\mathbb{R}^2 \times (0,H))$. From the parts (c) and (d) of the lemma we thus obtain $e_{1,0}, e_{2,0}, b_{3,0} \in \widetilde{H}^{-1/2}(\Omega)$. Now the validity of (4.1) follows from

$$b_{3,0} = \frac{i}{k} \nabla \cdot \begin{pmatrix} -e_{2,0} \\ e_{1,0} \end{pmatrix} \tag{4.4}$$

(cf. (3.9)).

For proving the other direction, we assume that relation (4.1) is valid. Because of (4.4) from the parts (c) and (d) of Lemma 4.2 it follows that $e_1, e_2, b_3 \in L^2(\mathbb{R}^2 \times (0,H))$. Regarding

the other three field components, from the parts (e) and (f) of the lemma we obtain $e_3, b_1, b_2 \in L^2(Z_{R,H})$. Since $R > R'$ and $H > 0$ can be chosen arbitrarily large, the diffraction solution e_j, b_j, $j = 1, 2, 3$, has local finite energy. □

We have yet to prove Lemma 4.2.

Proof of Lemma 4.2. Let the assumptions of Lemma 4.2 be fulfilled.

Proof of Part (a)

Let be $j \in \{1, 2\}$. Obviously we have only to show the validity of the implication

$$e_j \in L^2(Z_{R,H}) \Longrightarrow e_j \in L^2\Big(\mathbb{R}^2 \times (0, H)\Big). \tag{4.5}$$

We set $\varepsilon = (R - R')/2$ and $R'' = R' + \varepsilon$. For $\varphi \in [0, 2\pi)$ we consider the rotation matrix

$$A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \tag{4.6}$$

and define the function $e_j^{(\varphi)} : \mathscr{H} \to \mathbb{C}$ by $e_j^{(\varphi)}(x, x_3) = e_j(A_\varphi x, x_3)$ for $x \in \mathbb{R}^2$ and $x_3 > 0$. We show that

$$e_j^{(\varphi)} \in L^2\big((R'', \infty) \times \mathbb{R} \times (0, H)\big) \quad \text{for } \varphi \in [0, 2\pi), \tag{4.7}$$

regardless of whether $e_j \in L^2(Z_{R,H})$ or not.

From (4.7) we have

$$e_j \in L^2\big(A_\varphi\big((R'', \infty) \times \mathbb{R}\big) \times (0, H)\big) \quad \text{for } \varphi \in [0, 2\pi), \tag{4.8}$$

where $A_\varphi((R'', \infty) \times \mathbb{R})$ is the image of the set $(R'', \infty) \times \mathbb{R}$ under the linear mapping A_φ, that is, the set $(R'', \infty) \times \mathbb{R}$ rotated by the angle φ.

Since $R'' < R$, by considering a sufficient number of angles φ, from (4.8) we find that there is a polygon $P \subset B_R$ with $e_j \in L^2((\mathbb{R}^2 \setminus P) \times (0, H))$. Thus

$$e_j \in L^2\Big(\Big(\mathbb{R}^2 \times (0, H)\Big) \setminus Z_{R,H}\Big), \tag{4.9}$$

and therefore (4.5) holds true.

It remains to prove (4.7). To this end we use a contour integration technique which one of the authors had already used earlier in the treatment of diffraction problems [6, 7].

From the representation of e_j given in Theorem 2.1 we conclude that

$$e_j^{(\varphi)}(x_1, x_2, x_3) = \frac{1}{2\pi} \iint_{\mathbb{R}} \gamma_\varphi(\xi) e^{i x_3 m(\xi) + i\langle x, \xi\rangle} d\xi_1 d\xi_2, \quad (x_1, x_2, x_3) \in \mathscr{H}, \tag{4.10}$$

where the function γ_φ is given by

$$\gamma_\varphi(\xi) = \widehat{e}_{j,0}(A_\varphi \xi). \tag{4.11}$$

By assumption, it holds that $\operatorname{supp} e_{j,0} \subset \overline{\Omega} \subset \overline{B}_{R'}$. Thus, from the Paley-Wiener theorem and the fact that A_φ is an orthogonal matrix it follows that

$$\left|\gamma_\varphi(\xi)\right| \le c\left(1 + |\xi|^2\right)^{N/2} e^{R'|\operatorname{Im}\xi|}, \quad \xi \in \mathbb{C}^2, \tag{4.12}$$

for some constants $c > 0$ and $N \in \mathbb{N}_0$.

We split the inner integral in (4.10) into one integral over the interval $(-\infty, 0)$ and one over $(0, \infty)$. Firstly we consider the integral over $(-\infty, 0)$.

For $r > 0$ we define the curve η_r by

$$\eta_r = \left\{ re^{i\alpha} \mid \frac{\pi}{2} \le \alpha \le \pi \right\}. \tag{4.13}$$

Let Γ_r be the closed contour consisting of the parts $\{it \mid 0 \le t < r\}$, η_r and $(-r, 0)$. By Cauchy's theorem we have

$$\int_{\Gamma_r} \gamma_\varphi(\xi) e^{ix_3 m(\xi) + i\langle x, \xi\rangle} d\xi_1 = 0, \tag{4.14}$$

here for nonreal values of ξ_1 the function m is defined by analytic continuation of the function $\xi_1 \mapsto m(\xi_1, \xi_2)$, where $\xi_1 < 0$ and $|\xi| \ne k$, and $\langle x, \xi\rangle = x_1\xi_1 + x_2\xi_2$.

Because of (4.12), for $x_1 > R'$ it holds that

$$\lim_{r\to\infty} \int_{\eta_r} \gamma_\varphi(\xi) e^{ix_3 m(\xi) + i\langle x, \xi\rangle} d\xi_1 = 0. \tag{4.15}$$

From the last two equations we obtain

$$\int_{-\infty}^{0} \gamma_\varphi(\xi) e^{ix_3 m(\xi) + i\langle x, \xi\rangle} d\xi_1 = -\int_{0}^{i\infty} \gamma_\varphi(\xi) e^{ix_3 m(\xi) + i\langle x, \xi\rangle} d\xi_1, \quad x_1 > R'. \tag{4.16}$$

Now we treat the integral over $(0, \infty)$. Let $\sqrt{\ }^*$ be the principal branch of the square root function (branch cut $(-\infty, 0)$). For $z < 0$ we set $\sqrt{z}^* = -i\sqrt{|z|}$, that is, $\sqrt{\ }^*$ is thought to be defined on the negative real half-axis by continuation from the *lower* complex half-plane. In what follows, the function m^* is defined by $m^*(\xi) = \sqrt{k^2 - \xi_1^2 - \xi_2^2}^*$. Regarding the integration variable ξ_2 we distinguish the two cases $|\xi_2| < k$ and $|\xi_2| > k$.

Firstly we assume that $|\xi_2| < k$. We have

$$\begin{aligned}
&\int_0^\infty \gamma_\varphi(\xi) e^{ix_3 m(\xi)+i\langle x,\xi\rangle} d\xi_1 \\
&\quad = \int_0^{\sqrt{k^2-\xi_2^2}} \gamma_\varphi(\xi) e^{ix_3 m^*(\xi)+i\langle x,\xi\rangle} d\xi_1 + \int_{\sqrt{k^2-\xi_2^2}}^\infty \gamma_\varphi(\xi) e^{-ix_3 m^*(\xi)+i\langle x,\xi\rangle} d\xi_1 \\
&\quad = \int_0^\infty \gamma_\varphi(\xi) e^{-ix_3 m^*(\xi)+i\langle x,\xi\rangle} d\xi_1 + 2i \int_0^{\sqrt{k^2-\xi_2^2}} \gamma_\varphi(\xi) \sin(x_3 m^*(\xi)) e^{i\langle x,\xi\rangle} d\xi_1.
\end{aligned} \tag{4.17}$$

By contour integration analogous as in the derivation of (4.16), applied to the next-to-last integral, eventually we get

$$\begin{aligned}
&\int_0^\infty \gamma_\varphi(\xi) e^{ix_3 m(\xi)+i\langle x,\xi\rangle} d\xi_1 \\
&\quad = \int_0^{i\infty} \gamma_\varphi(\xi) e^{-ix_3 m^*(\xi)+i\langle x,\xi\rangle} d\xi_1 \\
&\qquad + 2i \int_0^{\sqrt{k^2-\xi_2^2}} \gamma_\varphi(\xi) \sin(x_3 m^*(\xi)) e^{i\langle x,\xi\rangle} d\xi_1 \quad \text{for } |\xi_2| < k,\ x_1 > R'.
\end{aligned} \tag{4.18}$$

In the case $|\xi_2| > k$ it holds that $m(\xi) = -m^*(\xi)$ for $\xi_1 > 0$, and contour integration yields

$$\int_0^\infty \gamma_\varphi(\xi) e^{ix_3 m(\xi)+i\langle x,\xi\rangle} d\xi_1 = \int_0^{i\infty} \gamma_\varphi(\xi) e^{-ix_3 m^*(\xi)+i\langle x,\xi\rangle} d\xi_1 \quad \text{for } |\xi_2| > k,\ x_1 > R'. \tag{4.19}$$

In the following, for $\xi_1 \in \mathbb{R}$ or $\xi_1 \in \{it \mid 0 < t < \infty\}$ we set $m(\xi) = \sqrt{k^2 - \xi_1^2 - \xi_2^2}$, where the square root is chosen in the way that $\operatorname{Re} m(\xi) \geq 0$ and $\operatorname{Im} m(\xi) \geq 0$. This definition is in accordance with (2.15). In this notation, from (4.18) and (4.19) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^\infty \gamma_\varphi(\xi) e^{ix_3 m(\xi)+i\langle x,\xi\rangle} d\xi_1 d\xi_2 \\
&\quad = \int_{-k}^{k} \int_0^{i\infty} \gamma_\varphi(\xi) e^{-ix_3 m(\xi)+i\langle x,\xi\rangle} d\xi_1 d\xi_2 \\
&\qquad + 2i \int_{-k}^{k} \int_0^{\sqrt{k^2-\xi_2^2}} \gamma_\varphi(\xi) \sin(x_3 m(\xi)) e^{i\langle x,\xi\rangle} d\xi_1 d\xi_2 \\
&\qquad + \int_{\mathbb{R}\setminus[-k,k]} \int_0^{i\sqrt{\xi_2^2-k^2}} \gamma_\varphi(\xi) e^{ix_3 m(\xi)+i\langle x,\xi\rangle} d\xi_1 d\xi_2 \\
&\qquad + \int_{\mathbb{R}\setminus[-k,k]} \int_{i\sqrt{\xi_2^2-k^2}}^{i\infty} \gamma_\varphi(\xi) e^{-ix_3 m(\xi)+i\langle x,\xi\rangle} d\xi_1 d\xi_2 \quad \text{for } x_1 > R'.
\end{aligned} \tag{4.20}$$

Together with (4.16) we thus find

$$\begin{aligned}
&\iint_{\mathbb{R}} \gamma_\varphi(\xi) e^{ix_3 m(\xi)+i\langle x,\xi\rangle} d\xi_1 d\xi_2 \\
&\quad = 2i \int_{-k}^{k} \int_{0}^{\sqrt{k^2-\xi_2^2}} \gamma_\varphi(\xi) \sin(x_3 m(\xi)) e^{i\langle x,\xi\rangle} d\xi_1 d\xi_2 \\
&\qquad - 2i \int_{-k}^{k} \int_{0}^{i\infty} \gamma_\varphi(\xi) \sin(x_3 m(\xi)) e^{i\langle x,\xi\rangle} d\xi_1 d\xi_2 \\
&\qquad - 2i \int_{\mathbb{R}\setminus[-k,k]} \int_{i\sqrt{\xi_2^2-k^2}}^{i\infty} \gamma_\varphi(\xi) \sin(x_3 m(\xi)) e^{i\langle x,\xi\rangle} d\xi_1 d\xi_2 \quad \text{for } x_1 > R'.
\end{aligned} \tag{4.21}$$

Now we show that each of the three addends in the right-hand side of (4.21), considered as a function of $\underline{x} = (x, x_3)$, lies in $L^2((R'', \infty) \times \mathbb{R} \times (0, H))$. Then, because of the representation (4.10), the stated relation (4.7) is proved.

We begin with the first addend. Let $A := \{\xi \in \mathbb{R}^2 \mid |\xi| \le k,\ \xi_1 \ge 0\}$, let χ_A be the characteristic function of the set A and let $\check{}$ denote the inverse Fourier transform. There is a constant $c > 0$, which does not depend on $x_3 \in (0, H)$, such that

$$\left\| \left[\chi_A \gamma_\varphi \sin(x_3 m(\cdot))\right]^{\check{}} \right\|_{L^2(\mathbb{R}^2)} = \left\| \chi_A \gamma_\varphi \sin(x_3 m(\cdot)) \right\|_{L^2(\mathbb{R}^2)} \le c. \tag{4.22}$$

From this it follows that the first addend in the right-hand side of (4.21) is quadratically integrable over the set $\mathbb{R}^2 \times (0, H)$ and thus especially quadratically integrable over $(R'', \infty) \times \mathbb{R} \times (0, H)$.

Now we come to the second addend. We define $\tilde{R}$ by $\tilde{R} = (R' + R'')/2 = R' + (\varepsilon/2)$. From (4.12) we find that there is a constant $c_1 > 0$ with

$$\left|\gamma_\varphi(it, \xi_2)\right| \le c_1 e^{\tilde{R}t} \quad \text{for } \xi_2 \in [-k, k],\ t \ge 0. \tag{4.23}$$

Since for $\xi_2 \in [-k, k]$ and $t \ge 0$ the quantity $m(it, \xi_2)$ is a real number and therefore $|\sin(x_3 m(it, \xi_2))| \le 1$ for $x_3 \in (0, H)$, we conclude that

$$\begin{aligned}
&\left| \int_0^\infty \gamma_\varphi(it, \xi_2) \sin(x_3 m(it, \xi_2)) e^{-tx_1} dt \right| \\
&\quad \le c_1 \int_0^\infty e^{(\tilde{R}-x_1)t} dt = \frac{c_1}{x_1 - \tilde{R}} \quad \text{for } \xi_2 \in [-k, k],\ x_3 \in (0, H),\ x_1 > \tilde{R}.
\end{aligned} \tag{4.24}$$

Thus we have

$$\int_{-k}^{k} \left| \int_0^\infty \gamma_\varphi(it, \xi_2) \sin(x_3 m(it, \xi_2)) e^{-tx_1} dt \right|^2 d\xi_2 \le \frac{2kc_1^2}{\left(x_1 - \tilde{R}\right)^2} \quad \text{for } x_3 \in (0, H),\ x_1 > \tilde{R}. \tag{4.25}$$

Because the inverse Fourier transform (here with regard to the variable ξ_2) is isometric on L^2, we see that

$$\begin{aligned}\int_{\mathbb{R}} \left| \int_{-k}^{k} \int_{0}^{\infty} \gamma_\varphi(it,\xi_2) \sin(x_3 m(it,\xi_2)) e^{-tx_1} dt e^{i\xi_2 x_2} d\xi_2 \right|^2 dx_2 \\ \leq \frac{4\pi k c_1^2}{\left(x_1 - \widetilde{R}\right)^2} \quad \text{for } x_3 \in (0,H),\ x_1 > \widetilde{R}.\end{aligned} \tag{4.26}$$

It follows that

$$\int_0^H \int_{R''}^{\infty} \int_{\mathbb{R}} \left| \int_{-k}^{k} \int_{0}^{\infty} \gamma_\varphi(it,\xi_2) \sin(x_3 m(it,\xi_2)) e^{-tx_1} dt e^{i\xi_2 x_2} d\xi_2 \right|^2 dx_2 dx_1 dx_3 < \infty. \tag{4.27}$$

The substitution $t = -i\xi_1$ in the innermost integral now shows that the second addend in the right-hand side of (4.21) indeed lies in $L^2((R'',\infty) \times \mathbb{R} \times (0,H))$.

Regarding the third addend, from (4.12) we find

$$\begin{aligned}\left| \int_{\sqrt{\xi_2^2-k^2}}^{\infty} \gamma_\varphi(it,\xi_2) \sin(x_3 m(it,\xi_2)) e^{-tx_1} dt \right| \\ \leq c \int_{\sqrt{\xi_2^2-k^2}}^{\infty} \left(1 + t^2 + \xi_2^2\right)^{N/2} e^{(R'-x_1)t} dt \quad \text{for } \xi_2 \in \mathbb{R} \setminus [-k,k],\ x_3 \in (0,H),\ x_1 > R'.\end{aligned} \tag{4.28}$$

Now let be $\widetilde{R} = R' + (\varepsilon/2)$ as above. By reason of

$$\left(1 + t^2 + \xi_2^2\right)^{N/2} \leq \left(1 + t^2\right)^{N/2} \left(1 + \xi_2^2\right)^{N/2}, \tag{4.29}$$

we conclude that there are constants $c_2, c_3 > 0$ such that

$$\begin{aligned}&\left| \int_{\sqrt{\xi_2^2-k^2}}^{\infty} \gamma_\varphi(it,\xi_2) \sin(x_3 m(it,\xi_2)) e^{-tx_1} dt \right| \\ &\quad \leq c_2 \left(1 + \xi_2^2\right)^{N/2} \int_{\sqrt{\xi_2^2-k^2}}^{\infty} e^{(\widetilde{R}-x_1)t} dt \\ &\quad = \frac{c_2}{x_1 - \widetilde{R}} \left(1 + \xi_2^2\right)^{N/2} e^{(\widetilde{R}-x_1)\sqrt{\xi_2^2-k^2}} \\ &\quad \leq \frac{c_3}{x_1 - \widetilde{R}} e^{-(\varepsilon/4)\sqrt{\xi_2^2-k^2}} \quad \text{for } \xi_2 \in \mathbb{R} \setminus [-k,k],\ x_3 \in (0,H),\ x_1 > R''.\end{aligned} \tag{4.30}$$

This estimate corresponds to formula (4.24), used in the case of the second addend. Continuing as in this latter case, it is seen that the third addend in the right-hand side of (4.21) lies in $L^2((R'',\infty) \times \mathbb{R} \times (0,H))$ too.

Proof of Part (b)

The preceding proof of part (a) is based on the fact that for $j \in \{1,2\}$ it holds that

$$\widehat{e}_j(\xi, x_3) = e^{ix_3 m(\xi)} \widehat{e}_{j,0}(\xi) \quad \text{for } \xi \in \mathbb{R}^2,\ x_3 > 0, \tag{4.31}$$

$$\operatorname{supp} e_{j,0} \subset \overline{\Omega}. \tag{4.32}$$

The relation (4.31), which is equivalent to the representation of e_j given in Theorem 2.1, has led to (4.10).

Since the third magnetic component b_3 fulfills conditions which are analogous to (4.31) and (4.32), the proof of part (b) of the lemma is completely analogous to the proof of part (a). The condition for b_3 that is analogous to (4.31) is given in (2.19); note that (2.19) also holds for $|\xi| < k$. The condition $\operatorname{supp} b_{3,0} \subset \overline{\Omega}$ is a direct consequence of

$$b_{3,0} = \frac{i}{k} \nabla \cdot \begin{pmatrix} -e_{2,0} \\ e_{1,0} \end{pmatrix}, \tag{4.33}$$

following from the second equation in (2.18), and $\operatorname{supp} e_{1,0}, \operatorname{supp} e_{2,0} \subset \overline{\Omega}$.

Proof of Parts (c) and (d)

These parts of the lemma can be proved along the lines of the proof of Theorem 3.1. For example, with respect to e_j, $j \in \{1,2\}$, one has

$$\begin{aligned}
&\int_0^H \iint_{\mathbb{R}^2} \left|e_j(x, x_3)\right|^2 dx\, dx_3 \\
&\quad = \int_0^H \iint_{\mathbb{R}^2} \left|\widehat{e}_j(\xi, x_3)\right|^2 d\xi dx_3 \\
&\quad = \iint_{\mathbb{R}^2} \left|\widehat{e}_{j,0}(\xi)\right|^2 \int_0^H \left|e^{ix_3 m(\xi)}\right|^2 dx_3 d\xi \\
&\quad = H \iint_{|\xi|<k} \left|\widehat{e}_{j,0}(\xi)\right|^2 d\xi + \iint_{|\xi|>k} \left|\widehat{e}_{j,0}(\xi)\right|^2 \frac{1 - e^{-2H\sqrt{|\xi|^2 - k^2}}}{2\sqrt{|\xi|^2 - k^2}} d\xi.
\end{aligned} \tag{4.34}$$

The condition (3.12), used in the proof of Theorem 3.1 because the factor $m(\xi)^{-1}$ is not locally square integrable, is not needed for the special field components e_1, e_2, and b_3.

Proof of Part (e)

From the first equation in (2.18) and the first equation in (2.19) we obtain

$$\widehat{e}_3(\xi, x_3) = -\frac{1}{m(\xi)} \left[\xi_1 \widehat{e}_1(\xi, x_3) + \xi_2 \widehat{e}_2(\xi, x_3)\right], \qquad \xi \in \mathbb{R}^2,\ \ x_3 \geq 0. \tag{4.35}$$

Thus for $(x, x_3) \in \mathscr{H}$ we have

$$e_3(x, x_3) = -\frac{1}{2\pi}\iint_{\mathbb{R}^2} \frac{1}{m(\xi)}[\xi_1\widehat{e}_1(\xi, x_3) + \xi_2\widehat{e}_2(\xi, x_3)]e^{i\langle x,\, \xi\rangle}d\xi. \tag{4.36}$$

Now we define the function $m_0 : \mathbb{R}^2 \to \mathbb{C}$ by

$$m_0(\xi) = i\sqrt{k^2 + |\xi|^2} \tag{4.37}$$

and the function $e_3^{(0)} : \mathscr{H} \to \mathbb{C}$ by

$$e_3^{(0)}(x, x_3) = -\frac{1}{2\pi}\iint_{\mathbb{R}^2} \frac{1}{m_0(\xi)}[\xi_1\widehat{e}_1(\xi, x_3) + \xi_2\widehat{e}_2(\xi, x_3)]e^{i\langle x,\, \xi\rangle}d\xi. \tag{4.38}$$

Using (4.36), we find

$$\left|e_3^{(0)}(x, x_3) - e_3(x, x_3)\right| \le \frac{1}{2\pi}\sum_{j=1}^{2}\left(I_j^{(1)}(x_3) + I_j^{(2)}(x_3)\right), \tag{4.39}$$

with

$$\begin{aligned} I_j^{(1)}(x_3) &= \iint_{|\xi|<2k}\left|\frac{1}{m_0(\xi)} - \frac{1}{m(\xi)}\right| \left|\xi_j\right| \left|\widehat{e}_j(\xi, x_3)\right|d\xi, \qquad j = 1, 2, \\ I_j^{(2)}(x_3) &= \iint_{|\xi|>2k}\left|\frac{1}{m_0(\xi)} - \frac{1}{m(\xi)}\right|\left|\xi_j\right|\left|\widehat{e}_j(\xi, x_3)\right|d\xi, \qquad j = 1, 2. \end{aligned} \tag{4.40}$$

From (4.31) we see that there is a constant $c_1 > 0$ such that

$$\left|I_j^{(1)}(x_3)\right| \le c_1 \quad \text{for } j \in \{1, 2\},\ x_3 > 0. \tag{4.41}$$

For $|\xi| > 2k$ it holds that

$$\left|\frac{1}{m_0(\xi)} - \frac{1}{m(\xi)}\right| = O\left(\frac{1}{|\xi|^3}\right). \tag{4.42}$$

Therefore with some constant $c_2 > 0$ we have for $j \in \{1, 2\}$ and $x_3 > 0$

$$\left|I_j^{(2)}(x_3)\right| \le c_2\iint_{|\xi|>2k}\frac{1}{|\xi|^{3/2}}\left|\frac{1}{|\xi|^{1/2}}\widehat{e}_j(\xi, x_3)\right|d\xi. \tag{4.43}$$

From this we get from the Cauchy-Schwarz inequality that

$$\left|I_j^{(2)}(x_3)\right| \le c_2\left(\iint_{|\xi|>2k}\frac{1}{|\xi|^3}d\xi\right)^{1/2}\left(\iint_{|\xi|>2k}\frac{1}{|\xi|}\left|\widehat{e}_j(\xi,x_3)\right|^2 d\xi\right)^{1/2}. \tag{4.44}$$

Using the fact that the first integral in the right-hand side of the last expression is finite and using (4.31), we conclude that there is a $c_3 > 0$ with

$$\left|I_j^{(2)}(x_3)\right| \le c_3\left(\iint_{|\xi|>2k}\frac{1}{|\xi|}\left|e^{ix_3m(\xi)}\right|^2\left|\widehat{e}_{j,0}(\xi)\right|^2 d\xi\right)^{1/2}. \tag{4.45}$$

Since $|e^{ix_3m(\xi)}| \le 1$ for $|\xi| > 2k$ and $x_3 > 0$, from the assumption $e_{1,0}, e_{2,0} \in \widetilde{H}^{-1/2}(\Omega)$ of part (e) of the lemma it now follows that there is a $c_4 > 0$ with

$$\left|I_j^{(2)}(x_3)\right| \le c_4 \quad \text{for } j \in \{1,2\},\ x_3 > 0. \tag{4.46}$$

From (4.39), (4.41), and (4.46) we obtain that there is a constant $c_5 > 0$ such that

$$\left|e_3^{(0)}(x,x_3) - e_3(x,x_3)\right| \le c_5 \quad \text{for } (x,x_3) \in \mathscr{H}. \tag{4.47}$$

Thus we have

$$e_3^{(0)} - e_3 \in L^2(Z_{R,H}), \tag{4.48}$$

since $Z_{R,H}$ is a bounded set.

Now we show that also $e_3^{(0)} \in L^2(Z_{R,H})$. From this, together with (4.48), we get the assertion $e_3 \in L^2(Z_{R,H})$ of part (e) of the lemma.

We show that $e_3^{(0)}$ actually is square integrable over the whole of $\mathbb{R}^2 \times (0,H)$. By definition of $e_3^{(0)}$, it suffices to prove that

$$\int_0^H \iint_{\mathbb{R}^2}\left|\iint_{\mathbb{R}^2}\frac{\xi_j}{m_0(\xi)}\widehat{e}_j(\xi,x_3)e^{i\langle x,\xi\rangle}d\xi\right|^2 dx\,dx_3 < \infty \quad \text{for } j \in \{1,2\}. \tag{4.49}$$

By applying the Parseval formula with respect to the variables x and ξ and using Formula (4.31) we find that the last integral is equal to

$$4\pi^2\int_0^H \iint_{\mathbb{R}^2}\frac{\xi_j^2}{k^2+|\xi|^2}\left|e^{ix_3m(\xi)}\right|^2\left|\widehat{e}_{j,0}(\xi)\right|^2 d\xi\,dx_3. \tag{4.50}$$

Now we split the domain of integration with respect to ξ into the parts $|\xi| \leq k$ and $|\xi| > k$. Due to the continuity of the integrand we have

$$\int_0^H \iint_{|\xi|\leq k} \frac{\xi_j^2}{k^2+|\xi|^2} \left|e^{ix_3 m(\xi)}\right|^2 \left|\widehat{e}_{j,0}(\xi)\right|^2 d\xi\, dx_3 < \infty. \tag{4.51}$$

The fact that also

$$\int_0^H \iint_{|\xi|>k} \frac{\xi_j^2}{k^2+|\xi|^2} \left|e^{ix_3 m(\xi)}\right|^2 \left|\widehat{e}_{j,0}(\xi)\right|^2 d\xi\, dx_3 < \infty \tag{4.52}$$

follows from the assumption $e_{1,0}, e_{2,0} \in \widetilde{H}^{-1/2}(\Omega)$ similar to the lines of the parts (c) and (d).

Proof of Part (f)

From the representations of b_1 and b_2 in Theorem 2.1 together with (2.19) we see that for $\xi \in \mathbb{R}^2$ and $x_3 \geq 0$ there holds

$$\begin{aligned} \widehat{b}_1(\xi, x_3) &= -\frac{k}{m(\xi)}\widehat{e}_2(\xi, x_3) + \frac{\xi_1}{km(\xi)}\left[\xi_1\widehat{e}_2(\xi, x_3) - \xi_2\widehat{e}_1(\xi, x_3)\right], \\ \widehat{b}_2(\xi, x_3) &= \frac{k}{m(\xi)}\widehat{e}_1(\xi, x_3) + \frac{\xi_2}{km(\xi)}\left[\xi_1\widehat{e}_2(\xi, x_3) - \xi_2\widehat{e}_1(\xi, x_3)\right]. \end{aligned} \tag{4.53}$$

Now the proof of part (f) proceeds along the same lines as the proof of part (e) if one takes the last two equations as starting point, as we have used (4.35) as starting point for the proof of part (e). The assumption $\nabla \cdot (-e_{2,0}, e_{1,0})^T \in \widetilde{H}^{-1/2}(\Omega)$ of part (f) is needed because in the formulas (4.53) there occur the terms

$$\frac{\xi_j}{km(\xi)}\left[\xi_1\widehat{e}_2(\xi, x_3) - \xi_2\widehat{e}_1(\xi, x_3)\right] = \frac{i\xi_j}{km(\xi)} e^{ix_3 m(\xi)} \widehat{\left[\nabla \cdot \begin{pmatrix} -e_{2,0} \\ e_{1,0} \end{pmatrix}\right]}(\xi). \tag{4.54}$$

□

5. Conclusions and Outlook

If we interpret the system (2.24) in the distributional sense, then we obtain a vectorial pseudodifferential operator A, acting on the electric boundary fields $e_{1,0}$, $e_{2,0}$ and resulting in the magnetic boundary fields $b_{1,0}$, $b_{2,0}$:

$$A\begin{pmatrix} e_{1,0} \\ e_{2,0} \end{pmatrix} = \begin{pmatrix} b_{1,0} \\ b_{2,0} \end{pmatrix}. \tag{5.1}$$

For the electromagnetic diffraction problem $b_{1,0}$ and $b_{2,0}$ are distributions which are only prescribed in the aperture Ω whereas $e_{1,0}$ and $e_{2,0}$ are unknown. Then the diffracted

electromagnetic field is completely determined for $x_3 > 0$, that is, behind the screen, by Theorem 2.1.

We define the vectorial Sobolev-Hilbert spaces V and W by

$$\begin{aligned} V &:= \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid v_1, v_2,\ \nabla \cdot \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} \in \widetilde{H}^{-1/2}(\Omega) \right\}, \\ \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_V &:= \left(\|v_1\|^2_{\widetilde{H}^{-1/2}(\Omega)} + \|v_2\|^2_{\widetilde{H}^{-1/2}(\Omega)} + \left\| \nabla \cdot \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} \right\|^2_{\widetilde{H}^{-1/2}(\Omega)} \right)^{1/2}, \\ W &:= \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mid w_1, w_2,\ \nabla \cdot \begin{pmatrix} -w_2 \\ w_1 \end{pmatrix} \in H^{-1/2}(\Omega) \right\}, \\ \left\| \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\|_W &:= \left(\|w_1\|^2_{H^{-1/2}(\Omega)} + \|w_2\|^2_{H^{-1/2}(\Omega)} + \left\| \nabla \cdot \begin{pmatrix} -w_2 \\ w_1 \end{pmatrix} \right\|^2_{H^{-1/2}(\Omega)} \right)^{1/2}. \end{aligned} \tag{5.2}$$

Now it is an interesting future task to prove that $A : V \to W$ is a homeomorphism. For this purpose we want to develop a purely "intrinsic" proof, related only to the boundary equations in the aperture and without making use of the time harmonic Maxwell equations (2.1). For the slit diffraction problem this approach has been realised by the authors in the paper [2].

Note that $(e_{1,0}, e_{2,0}) \in V$ is a necessary and sufficient condition for a diffraction solution with local finite energy, see part (a) of Definition 2.5 and Theorem 4.1.

In the case of global finite energy in Definition 2.5, part (b), we have decoupled the vectorial boundary equations (cf. Theorem 3.2). In the future it should also be examined how this result can be used to derive representations of the physical admissible solutions satisfying only the condition of local finite energy.

Appendix

Theorem A.1. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be an entire function, $k > 0$, and assume that $f(\xi_1, \xi_2) = 0$ for all $\xi_1, \xi_2 \in \mathbb{R}$ with $\xi_1^2 + \xi_2^2 = k^2$. Then one also has $f(z_1, z_2) = 0$ for all $z_1, z_2 \in \mathbb{C}$ with $z_1^2 + z_2^2 = k^2$, and there is an uniquely determined entire function $q : \mathbb{C}^2 \to \mathbb{C}$ such that for all $z_1, z_2 \in \mathbb{C}$ with $z_1^2 + z_2^2 \neq k^2$ it holds that*

$$q(z_1, z_2) = \frac{f(z_1, z_2)}{k^2 - z_1^2 - z_2^2}. \tag{A.1}$$

Proof. We only have to show the existence of an entire function q satisfying (A.1), since uniqueness results by analytic continuation. By employing Taylor expansion of f with respect to z_2 we find entire functions $\widetilde{g}, \widetilde{h} : \mathbb{C}^2 \to \mathbb{C}$ such that

$$f(z_1, z_2) = \widetilde{g}\left(z_1, z_2^2\right) + z_2 \widetilde{h}\left(z_1, z_2^2\right). \tag{A.2}$$

Now we define the entire functions $g, h : \mathbb{C}^2 \to \mathbb{C}$ by

$$g(z_1, z) = \widetilde{g}\left(z_1, z - z_1^2\right), \qquad h(z_1, z) = \widetilde{h}\left(z_1, z - z_1^2\right), \tag{A.3}$$

and conclude for $\xi_1 \in (-k, k)$, $\xi_2 = \pm\sqrt{k^2 - \xi_1^2}$ that

$$g\left(\xi_1, \xi_1^2 + \xi_2^2\right) + \xi_2 h\left(\xi_1, \xi_1^2 + \xi_2^2\right) = \widetilde{g}\left(\xi_1, \xi_2^2\right) + \xi_2 \widetilde{h}\left(\xi_1, \xi_2^2\right) = f(\xi_1, \xi_2) = 0. \tag{A.4}$$

But from the resulting two equations

$$g\left(\xi_1, k^2\right) \pm \sqrt{k^2 - \xi_1^2}\; h\left(\xi_1, k^2\right) = 0, \tag{A.5}$$

we conclude $g(\xi_1, k^2) = h(\xi_1, k^2) = 0$, and hence by analytic continuation

$$g\left(z_1, k^2\right) = h\left(z_1, k^2\right) = 0 \quad \forall z_1 \in \mathbb{C}. \tag{A.6}$$

Taylor expansion gives two entire functions $g^*, h^* : \mathbb{C}^2 \to \mathbb{C}$ with

$$g^*(z_1, z) = \frac{g(z_1, z)}{k^2 - z}, \qquad h^*(z_1, z) = \frac{h(z_1, z)}{k^2 - z}. \tag{A.7}$$

Finally we define the entire function q by

$$q(z_1, z_2) := g^*\left(z_1, z_1^2 + z_2^2\right) + z_2\; h^*\left(z_1, z_1^2 + z_2^2\right), \tag{A.8}$$

and obtain for all $z_1, z_2 \in \mathbb{C}$ with $z_1^2 + z_2^2 \neq k^2$

$$\frac{f(z_1, z_2)}{k^2 - z_1^2 - z_2^2} = \frac{g\left(z_1, z_1^2 + z_2^2\right)}{k^2 - z_1^2 - z_2^2} + z_2\; \frac{h\left(z_1, z_1^2 + z_2^2\right)}{k^2 - z_1^2 - z_2^2} = q(z_1, z_2). \tag{A.9}$$

□

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