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*Research Article*

# Approximate Complete Solutions of DKP Equation under a Vector Exponential Interaction via a Pekeris-Type Approximation

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The DKP equation for an exponential potential is exactly solved via an appropriate approximation using the methodology of supersymmetric quantum mechanics. We see that the solutions are already known without any cumbersome algebra we face in any numerical or analytical approach. Closed forms of eigenfunctions and eigenvalues are reported.

## 1. Introduction

Working on the basis of Dirac and Klein-Gordon equations, spin-0 and spin-1/2 particles have been extensively discussed via many analytical and numerical techniques. For the spin-1 particles, however, there are only few investigations. The main reason for this lack of literature is probably the mathematical complexity of Proca equation describing the spin-1 particles. The Duffin-Kemmer-Petiau (DKP) [1–4] equation, however, provides us with a theoretical basis for describing both spin-0 and spin-1 particles on a relatively easier background. For many years, the DKP equation was thought to be exactly equivalent to Klein-Gordon (KG) and Proca equations and consequently was in the shadow of them. Now, we know that the equations are not exactly the same and the equivalence does not hold generally [5–14]. In addition, the DKP equation is richer in the investigation of interactions and is even closer to some experimental data in comparison with KG or Proca equations [15–20]. Moreover, besides cosmology and gravity, this equation has been tested in many branches of physics including particle and nuclear physics [21–25].

As usual, the most appealing case studies are Coulomb and quadratic terms [26–28], and other ones including the woods-Saxon and Hulthen are investigated by different

approaches as well [29–32]. Within the present work, we first review the DKP equation. Next, an introductory section is included on supersymmetry (SUSY) quantum mechanics. In the last part, we obtain the approximate analytical solutions of the problem.

## 2. DKP Equation

For the sake of briefness, our starting square is

$$\left(\beta \cdot \vec{p}c + mc^2 + U_s + \beta^0 U_v^o\right)\psi(\vec{r}) = \beta^0 E\psi(\vec{r}), \tag{2.1}$$

where

$$\psi(\vec{r}) = \begin{pmatrix} \psi_{\text{upper}} \\ i\psi_{\text{lower}} \end{pmatrix}, \tag{2.2}$$

with the upper and lower components, respectively, being

$$\begin{aligned} \psi_{\text{upper}} &\equiv \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \\ \psi_{\text{lower}} &\equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \end{aligned} \tag{2.3}$$

$\beta^0$ is the usual $5\times5$ matrix, and $U_s, U_v^o$ denote the scalar and vector interactions. The equation, in $(3+0)$-dimensions, is therefore written as

$$\begin{aligned} \left(mc^2 + U_s\right)\phi &= (E - U_v^o)\varphi + \hbar c\vec{\nabla}\cdot\vec{A}, \\ \vec{\nabla}\phi &= \left(mc^2 + U_s\right)\vec{A}, \\ \left(mc^2 + U_s\right)\varphi &= (E - U_v^o)\phi, \end{aligned} \tag{2.4}$$

where $\vec{A} = (A_1, A_2, A_3)$. It should be noted that $\psi$ is a simultaneous eigenfunction of $J^2$ and $J_3$, that is

$$\begin{aligned} J^2\begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix} &= \begin{pmatrix} L^2\psi_{\text{upper}} \\ (L+S)^2\psi_{\text{lower}} \end{pmatrix} = J(J+1)\begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix}, \\ J_3\begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix} &= \begin{pmatrix} L_3\psi_{\text{upper}} \\ (L_3+s_3)\psi_{\text{lower}} \end{pmatrix} = M\begin{pmatrix} \psi_{\text{upper}} \\ \psi_{\text{lower}} \end{pmatrix}, \end{aligned} \tag{2.5}$$

and the general solution is

$$\psi_{JM}(r) = \begin{pmatrix} f_{nJ}(r)Y_{JM}(\Omega) \\ g_{nJ}(r)Y_{JM}(\Omega) \\ i\sum_{L} h_{nJL}(r)Y^{M}_{JL1}(\Omega) \end{pmatrix}, \tag{2.6}$$

where the spherical harmonics $Y_{JM}(\Omega)$ are of order $J$, $Y^{M}_{JL1}(\Omega)$ are the normalized vector spherical harmonics, and $f_{nJ}$, $g_{nJ}$, and $h_{nJL}$ denote the radial wave functions. It is shown that the above equations result in the coupled differential equations [26, 33]:

$$\begin{gathered}
\left(E - U_v^0\right)F(r) = \left(mc^2 + U_s\right)G(r), \\
\left(\frac{dF(r)}{dr} - \frac{J+1}{r}F(r)\right) = -\frac{1}{\alpha_J}\left(mc^2 + U_s\right)H_1(r), \\
\left(\frac{dF(r)}{dr} + \frac{J}{r}F(r)\right) = \frac{1}{\zeta_J}\left(mc^2 + U_s\right)H_{-1}(r), \\
-\alpha_J\left(\frac{dH_1(r)}{dr} + \frac{J+1}{r}H_1(r)\right) + \zeta\left(\frac{dH_{-1}(r)}{dr} - \frac{J}{r}H_{-1}(r)\right) \\
= \frac{1}{\hbar c}\left(\left(mc^2 + U_s\right)F(r) - \left(E - U_v^0\right)G(r)\right),
\end{gathered} \tag{2.7}$$

which give

$$\begin{aligned}
&\frac{d^2F(r)}{dr^2}\left[1 + \frac{\zeta_J^2}{\alpha_J^2}\right] - \frac{dF(r)}{dr}\left[\frac{U_s'}{(m+U_s)}\left(1 + \frac{\zeta_J^2}{\alpha_J^2}\right)\right] \\
&\quad + F(r)\left[-\frac{J(J+1)}{r^2}\left(1 + \frac{\zeta_J^2}{\alpha_J^2}\right) + \frac{U_s'}{(m+U_s)}\left(\frac{J+1}{r} - \frac{\zeta_J^2}{\alpha_J^2}\frac{J}{r}\right)\right] \\
&\quad - \left[\frac{1}{\alpha_J^2}\left((m+U_s)^2 - \left(E - U_v^0\right)^2\right)\right] = 0,
\end{aligned} \tag{2.8}$$

where $\alpha_J = \sqrt{(J+1)/(2J+1)}$ and $\zeta_J = \sqrt{J/(2J+1)}$. In the case of $U_s = 0$, we recover the well-known formula

$$\left(\frac{d^2}{dr^2} - \frac{J(J+1)}{r^2} + \left(E - U_v^0\right)^2 - m^2\right)F(r) = 0. \tag{2.9}$$

Within the next section, we give a brief introduction to the SUSY method.

### 2.1. SUSY and Shape Invariance

The basic idea of the SUSY approach is based on finding the solutions of the Riccati equation

$$V_{\mp} = \Phi^2 \mp \Phi', \tag{2.10}$$

with $V$ being the potential of Schrödinger equation. If the condition

$$V_+(a_0, x) = V_-(a_1, x) + R(a_1) \tag{2.11}$$

is satisfied, we call the partner Hamiltonians shape invariant. In the latter relation, $a_1$ is a new set of parameters uniquely determined from the old set $a_0$ via the mapping $F : a_0 \mapsto a_1 = F(a_0)$ and the residual term $R(a_1)$ does not include $x$. Actually, the shape invariance implies that the partner potential, apart from some constant terms, is interpreted as a new partner potential $V_-(a_1, x)$ associated with a new SUSY potential $\Phi(a_1, x)$ [34]. In such a case, the problem is simplified to a high degree and everything of interest is calculated via the elegant idea of [34–36]:

$$E_n = \sum_{s=1}^{n} R(a_s), \tag{2.12}$$

$$\phi_n^-(a_0, x) = \prod_{s=0}^{n-1} \left( \frac{A^\dagger(a_s)}{[E_n - E_s]^{1/2}} \right) \phi_0^-(a_n, x), \tag{2.13}$$

$$\phi_0^-(a_n, x) = C \exp\left\{ -\int_0^x dz \Phi(a_n, z) \right\}. \tag{2.14}$$

That is, the energy eigenvalues as well as the corresponding eigenfunctions are obtained without cumbersome algebra, the

$$A_s^\dagger = -\frac{\partial}{\partial x} + \Phi(a_s, x). \tag{2.15}$$

Hence, shape invariance gives the complete and exact information about the spectrum of the bound states of the following Hamiltonians:

$$H_s = -\frac{\partial^2}{\partial x^2} + V_-(a_s, x) + E_s. \tag{2.16}$$

It should be noted that the energy eigenfunctions

$$H_s \phi_{n-s}^-(a_s, x) = E_n \phi_{n-s}^-(a_s, x), \quad n \geq s, \tag{2.17}$$

of this family of Hamiltonians are related by [3, 4]

$$\phi^{-}_{n-s}(a_s, x) = \frac{A^{\dagger}}{[E_n - E_s]^{1/2}} \phi^{-}_{n-(s+1)}(a_{s+1}, x). \tag{2.18}$$

In other words, everything is obtained without any cumbersome algebra provided that the superpotential is simply found and the shape invariance exists.

### 2.2. A Famous SUSY Example

Within this section, we review an SUSY example. The results can be found in [34–36]. Choosing the superpotential as

$$\Phi(x) = A - B\exp(-\beta x), \tag{2.19}$$

we find

$$\begin{aligned} V_{-}(x) &= \Phi^2(x) + \Phi'(x) = A^2 + B^2\exp(-2\beta x) - 2B\left(A + \frac{\beta}{2}\right)\exp(-\beta x), \\ V_{+}(x) &= \Phi^2(x) - \Phi'(x) = A^2 + B^2\exp(-2\beta x) - 2B\left(A - \frac{\beta}{2}\right)\exp(-\beta x). \end{aligned} \tag{2.20}$$

Obviously, the partner potentials are shape invariant via a mapping of the form

$$A_1 = F(A) = A - \beta. \tag{2.21}$$

For the energy relation, as

$$\begin{aligned} A_1 &= F(A) = A - \beta, \\ A_2 &= F(A_1) = A_1 - \beta = A - 2\beta, \\ &\vdots \\ A_s &= F(A_{s-1}) = A - n\beta, \end{aligned} \tag{2.22}$$

using (2.11) and (2.12), we obtain

$$\begin{aligned} E_n &= \sum_{s=1}^{n} R(A_s) = R(A_1) + R(A_2) + \cdots + R(A_n) \\ &= \left(A^2 - A_1^2\right) + \left(A_1^2 - A_2^2\right) + \cdots + \left(A_{n-1}^2 - A_n^2\right) \\ &= A^2 - A_n^2 = A^2 - (A - n\beta)^2. \end{aligned} \tag{2.23}$$

For the wave functions, based on (2.13) to (2.18), we have

$$y^{s-n}\exp\left(\frac{-y}{2}\right)L_n^{2s-2n}(y), \tag{2.24}$$

where

$$\begin{aligned} y &= \left(\frac{2B}{\beta}\right)\exp(-\beta x), \\ s &= \frac{A}{\beta}. \end{aligned} \tag{2.25}$$

## 3. Approximate Analytical Solution of the Radial Part

We now see that the problem appears as the latter SUSY problem. Choosing the potential as $U_v = Ve^{-a(r-r_0)}$, we get

$$\left(\frac{d^2}{dr^2} - \frac{J(J+1)}{r^2} + E^2 + 2EV\exp(-a(r-r_0)) + V^2\exp(-2a(r-r_0)) - m^2\right)F(r) = 0. \tag{3.1}$$

We wish to emphasize that, on the one hand, exponential-type potentials find application in a wide class of physical sciences including cosmology [37–40], nuclear and particle physics [41–45], solid state [46–48], atomic and molecular physics [49–58], and chemical physics [59, 60]. On the other hand, the DKP equation itself, as an outstanding relativistic wave equation which well explains both microscale phenomena in particle physics and large-scale ones in cosmology, motivated us to do the present calculations. Naturally, depending on the system under consideration, the phenomenological fits are quite different. To be able to analytically solve the problem, let us use the well-known approximation [61]

$$\frac{1}{r^2} \approx \left(C_0 + C_1e^{-\alpha x} + C_2e^{-2\alpha x}\right), \tag{3.2}$$

where

$$x = \frac{r-r_0}{r_0}, \tag{3.3a}$$

$$\alpha = ar_0, \tag{3.3b}$$

$$\begin{aligned} C_0 &= \frac{1}{r_0^2}\left(1 - \frac{3}{\alpha} + \frac{3}{\alpha^2}\right), \\ C_1 &= \frac{1}{1r_0^2}\left(\frac{4}{\alpha} - \frac{6}{\alpha^2}\right), \\ C_2 &= \frac{1}{r_0^2}\left(\frac{-1}{\alpha} + \frac{3}{\alpha^2}\right). \end{aligned} \tag{3.3c}$$

We find

$$\left(\frac{d^2}{dx^2} - e^{-\alpha x}(C_1 J(J+1) - 2EV)r_0^2 + e^{-2\alpha x}\left(V^2 - C_2 J(J+1)\right)r_0^2 + \left(C_0 J(J+1) + E^2 - m^2\right)r_0^2\right) \times F(x) = 0, \tag{3.4}$$

which is a Schrödinger-like equation corresponding to the Morse potential. Before proceeding further, it should be noted that the choice of the system under consideration definitely puts limitations on the approximation (3.2) via (3.3a), (3.3b) and (3.3c) to yield acceptable results. Here, however, as we intend to give a general background for related studies, we avoid focusing on a special system since the numerical data are quite different from one system, for example, a diatomic molecule, to another such as cosmological model.

Also, there exist many other papers which use Pekeris-type approximations for Schrödinger, Dirac, and KG equations for a lengthy list of potentials including the Hulthen, Eckart, Rosen-Morse, and Pöschl-Teller (see [62–69] and references therein).

In addition, the interested reader might find attractive discussion on the SUSY structure of DKP equation in [70]. Moreover, we wish to address the interesting papers [71–76] which first discussed investigation of half-line problems on the basis of full-line SUSY examples. Let us now return to our problem. Comparison of (3.4) with (2.20) indicates the correspondence

$$\begin{aligned} B &= \pm r_0\sqrt{(V^2 - C_2 J(J+1))}, \\ A &= \pm r_0\sqrt{C_0 J(J+1)}, \\ \beta &= \frac{r_0^2(C_1 J(J+1) - 2EV)}{B} - 2A. \end{aligned} \tag{3.5}$$

Therefore, the eigenfunctions are

$$\begin{aligned} F(x) &= C_n y^{s-n}\exp\left(\frac{-y}{2}\right)L_n^{2s-2n}(y), \\ y &= \left(\frac{2B}{\beta}\right)\exp(-\beta x), \\ s &= \frac{A}{\beta}, \end{aligned} \tag{3.6}$$

and the energy eigenvalues satisfy

$$\begin{aligned} \varepsilon_n &= r_0^2\left(E^2 - m^2\right) - \lambda_0 = \cdots \\ &= C_0 J(J+1)r_0^2 - r_0\left(\pm\sqrt{C_0 J(J+1)} - n\left(\frac{(C_1 J(J+1) - 2EV)}{\pm\sqrt{(V^2 - C_2 J(J+1))}} \mp 2\sqrt{C_0 J(J+1)}\right)\right)^2. \end{aligned} \tag{3.7}$$

## 4. Conclusion

We obtained approximate analytical solutions of the DKP equation for an exponential term which in many cases provides more exact solutions than linear or quadratic terms. The results are applicable to many branches of physics from cosmology to particle physics.

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