Research Article

Bound on the Minimum Eigenvalue of H -Matrices Involving Hadamard Products

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We present a new lower bound on the minimum eigenvalue of H -matrices involving Hadamard products $\tau(A_1^{(\alpha_1)} \circ \dots \circ A_m^{(\alpha_m)})$, and we show that our lower bound is larger than the lower bound $\prod_{k=1}^m [\tau(A_k)]^{\alpha_k}$. Three examples verify our result.

1. Introduction

In [1], it is shown by Theorem 5.7.15 that if A_k are $n \times n$ H -matrices for all $k \in \{1, \dots, m\}$, and $\alpha_k \geq 0$ satisfy $\sum_{k=1}^m \alpha_k \geq 1$, then

$$\tau(A_1^{(\alpha_1)} \circ \dots \circ A_m^{(\alpha_m)}) \geq \prod_{k=1}^m [\tau(A_k)]^{\alpha_k}, \quad (1)$$

where $A^{(\alpha)}$ is defined as entrywise and any scalar definition of a^α such that $|a^\alpha| = |a|^\alpha$ is allowed. This theorem provided a beautiful result about the minimum eigenvalue of H -matrices involving Hadamard products, but sometimes this inequality could be very weak. For example,

$$A = \begin{pmatrix} 100 & -21 & -30 \\ 0 & 1 & -12 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ -19 & 100 & 0 \\ -81 & -17 & 1 \end{pmatrix}, \quad (2)$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 100 \end{pmatrix}$$

are H -matrices, and

$$\tau(A^{(2)} \circ B^{(2)} \circ C^{(2)}) = 10000 \gg \tau(A)^2 \tau(B)^2 \tau(C)^2 = 1, \quad (3)$$

see the details in Section 3.

A lot of works have been done on the minimum eigenvalue of M -matrices and H -matrices involving Hadamard products, see the results in [1–6].

In this paper, we present a new lower bound by including diagonal entries and prove that our bound is larger than the bound in (1).

We now introduce some notations, see [1]. The Hadamard product of $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ is defined by $A \circ B \equiv [a_{ij}b_{ij}]_{m \times n}$. We define $|a^\alpha| = |a|^\alpha$ and $0^0 = 1$. Let $A = [a_{ij}]_{n \times n}$ and denote a_{ij} by $A(i, j)$ for every $i, j \in \{1, \dots, n\}$. We denote $A^{(r)} = [a_{ij}^r]$, where $r \geq 0$, and denote by Z_n the class of all $n \times n$ real matrices all of whose off-diagonal entries are nonpositive. Let $A \in Z_n$, then the *minimum eigenvalue* of A is defined by $\tau(A) = \min\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\}$; $\tau(A)$ is a eigenvalue of A .

For two real matrices $A, B \in Z_n$, the Fan product of A and B , denoted by $A \star B = C = [c_{ij}]$, is defined by

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & \text{if } i \neq j, \\ a_{ii}b_{ii}, & \text{if } i = j, \end{cases} \quad (4)$$

and then $A \star B \in Z_n$. The comparison matrix $M(A) = [m_{ij}]$ of a given matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ is defined by

$$m_{ij} = \begin{cases} -|a_{ij}|, & \text{if } i \neq j, \\ |a_{ii}|, & \text{if } i = j. \end{cases} \quad (5)$$

A matrix A is an H -matrix if its comparison matrix $M(A)$ is an M -matrix. For $A \in \mathbb{C}^{n \times n}$, we define $\tau(A) \equiv \tau(M(A))$.

Last, for $A \in Z_n$, we introduce a new definition $A^{[r]} = [d_{ij}]$, where

$$d_{ij} = \begin{cases} -|a_{ij}|^r, & \text{if } i \neq j, \\ |a_{ii}|^r, & \text{if } i = j. \end{cases} \quad (6)$$

2. Main Results

To prove the main theorem, we need several lemmas.

Lemma 1. Let $x_j = (x_j(1), \dots, x_j(n))^T \geq 0$ for all $j \in \{1, \dots, m\}$. If $p_j > 0$ and $\sum_{j=1}^m (1/p_j) = r \geq 1$, then

$$\prod_{i=1}^n \prod_{j=1}^m x_j(i) \leq \prod_{j=1}^m \left\{ \sum_{i=1}^n [x_j(i)]^{p_j} \right\}^{1/p_j}. \quad (7)$$

Proof. When $r = 1$, we can prove this inequality by the Hölder inequality and by induction. When $r > 1$, apparently, $\sum_{j=1}^m (1/r p_j) = 1$, similarly, we can prove this inequality. $\square$

If $A \in Z_n$, then $A = \alpha I - B$ for some $B \geq 0$ and some $\alpha \in \mathbb{R}$. Let $\sigma(B) = \{\rho(B), \lambda_2, \dots, \lambda_n\}$. Then $\sigma(A) = \{\alpha - \rho(B), \alpha - \lambda_2, \dots, \alpha - \lambda_n\}$. If there is a λ_i such that $\alpha - \rho(B) = \text{Re}(\alpha - \rho(B)) > \text{Re}(\alpha - \lambda_i) = \alpha - \text{Re}(\lambda_i)$, then $\rho(B) < \text{Re}(\lambda_i)$, but this is a contradiction. So, $\tau(A) = \alpha - \rho(B)$. If $x \geq 0$ is the right Perron eigenvector of B , then $Ax = (\alpha I - B)x = \alpha x - Bx = \alpha x - \rho(B)x = (\alpha - \rho(B))x = \tau(A)x$. If $y \geq 0$ is the left Perron eigenvector of B , then $y^T A = y^T (\alpha I - B) = \alpha y^T - y^T B = \alpha y^T - \rho(B)y^T = (\alpha - \rho(B))y^T = \tau(A)y^T$. So, similar to the Perron-Frobenius theorem, we have the following: if $A \in Z_n$ became irreducible, then there exist positive vectors u and v such that $Au = \tau(A)u$, and $v^T A = \tau(A)v^T$, u and v being called right and left Perron eigenvectors of A , respectively.

Lemma 2. If $A \in Z_n$ is irreducible, and $Az \geq kz$ for a nonnegative nonzero vector z , then $k \leq \tau(A)$.

Proof. $A = \alpha I - P$, where $P \geq 0$ is irreducible and $\alpha \in \mathbb{R}$. By the Perron-Frobenius theorem, P has a positive left Perron vector y , that is, $y^T P = \rho(P)y^T$. Note that $Az - kz = (\alpha - k)z - Pz \geq 0$. Hence, $y^T (Az - kz) = (\alpha - k)y^T z - \rho(P)y^T z = (\alpha - k - \rho(P))y^T z \geq 0$. Since $y^T z > 0$, we have $k \leq \alpha - \rho(P) = \tau(A)$. $\square$

Lemma 3. Let $A_k \in Z_n$ for all $k \in \{1, \dots, m\}$. If $p_k > 0$ and $\sum_{k=1}^m (1/p_k) \geq 1$, then

$$\begin{aligned} & \tau(A_1 \star \dots \star A_m) \\ & \geq \min_{1 \leq i \leq n} \left\{ \prod_{k=1}^m A_k(i, i) \right. \\ & \quad \left. - \prod_{k=1}^m [|A_k(i, i)|^{p_k} - \tau(A_k^{[p_k]})]^{1/p_k} \right\}. \end{aligned} \quad (8)$$

Proof. It is quite evident that the conclusion holds with equality for $n = 1$.

For $n \geq 2$, we have two cases.

Case 1. $A_1 \star \dots \star A_m \in Z_n$ is irreducible. Then $A_k \in Z_n$ are irreducible for all $k \in \{1, \dots, m\}$. Thus, $A_k^{[p_k]} \in Z_n$ are irreducible for all $k \in \{1, \dots, m\}$. Let $u_k^{(p_k)} = (u_k(1)^{p_k}, \dots, u_k(n)^{p_k})^T > 0$ be the right Perron eigenvectors of $A_k^{[p_k]}$ for all $k \in \{1, \dots, m\}$. If $u_k^{(1)} = u_k = (u_k(1), \dots, u_k(n))^T > 0$ for all $k \in \{1, \dots, m\}$, then for every $i \in \{1, \dots, n\}$, we have

$$\begin{aligned} & A_k^{[p_k]} u_k^{(p_k)} = \tau(A_k^{[p_k]}) u_k^{(p_k)}, \\ & |A_k(i, i)|^{p_k} u_k(i)^{p_k} - \sum_{j \neq i} |A_k(i, j)|^{p_k} u_k(j)^{p_k} \\ & = \tau(A_k^{[p_k]}) u_k(i)^{p_k}, \\ & \sum_{j \neq i} |A_k(i, j)|^{p_k} u_k(j)^{p_k} \\ & = [|A_k(i, i)|^{p_k} - \tau(A_k^{[p_k]})] u_k(i)^{p_k} \geq 0. \end{aligned} \quad (9)$$

Let $C = A_1 \star \dots \star A_m \in Z_n$. Let $z = u_1 \circ \dots \circ u_m = (z(1), \dots, z(n))^T > 0$. Then $z(i) = \prod_{k=1}^m u_k(i)$ for all $i \in \{1, \dots, n\}$. Then, for all $i \in \{1, \dots, n\}$, we have

$$\begin{aligned} (Cz)_i &= \prod_{k=1}^m A_k(i, i) z(i) - \sum_{j \neq i} \left(\prod_{k=1}^m |A_k(i, j)| \right) z(j) \\ &= \prod_{k=1}^m A_k(i, i) z(i) - \sum_{j \neq i} \prod_{k=1}^m (|A_k(i, j)| u_k(j)) \\ &\geq \prod_{k=1}^m A_k(i, i) z(i) \\ &\quad - \prod_{k=1}^m \left\{ \sum_{j \neq i} |A_k(i, j)|^{p_k} u_k(j)^{p_k} \right\}^{1/p_k} \\ &= \prod_{k=1}^m A_k(i, i) z(i) \\ &\quad - \prod_{k=1}^m [|A_k(i, i)|^{p_k} - \tau(A_k^{[p_k]})] u_k(i)^{p_k}]^{1/p_k} \\ &= \left\{ \prod_{k=1}^m A_k(i, i) \right. \\ &\quad \left. - \prod_{k=1}^m [|A_k(i, i)|^{p_k} - \tau(A_k^{[p_k]})]^{1/p_k} \right\} z(i). \end{aligned} \quad (10)$$

By Lemma 1, the “ $\geq$ ” hold. By Lemma 2, we have

$$\begin{aligned} & \tau(A_1 \star \cdots \star A_m) \\ & \geq \min_{1 \leq i \leq n} \left\{ \prod_{k=1}^m A_k(i, i) \right. \\ & \quad \left. - \prod_{k=1}^m [|A_k(i, i)|^{p_k} - \tau(A_k^{[p_k]})]^{1/p_k} \right\}. \end{aligned} \quad (11)$$

Case 2. $A_1 \star \cdots \star A_m \in Z_n$ is reducible. We denote by T the $n \times n$ permutation matrix (t_{ij}) with

$$t_{12} = t_{23} = \cdots = t_{n-1,n} = t_{n1} = 1, \quad (12)$$

the remaining t_{ij} zero, then both $A_k - \varepsilon T \in Z_n$ are irreducible for any chosen positive real number ε . Now we substitute $A_k - \varepsilon T$ for A_k in the previous case, and then by letting $\varepsilon \rightarrow 0$, the result follows by continuity. $\square$

Lemma 4 (see [7]). *Let for all $i \in \{1, \dots, n\}$, $x_i \geq y_i \geq 0$ and $p_i > 0$ satisfying $\sum_{i=1}^n (1/p_i) \geq 1$. Then*

$$\prod_{i=1}^n x_i - \prod_{i=1}^n y_i \geq \prod_{i=1}^n (x_i^{p_i} - y_i^{p_i})^{1/p_i}. \quad (13)$$

The following is our main theorem.

Theorem 5. *Let $\alpha_k \geq 0$ and $\sum_{k=1}^m \alpha_k \geq 1$. If A_k is $n \times n$ H -matrices for all $k \in \{1, \dots, m\}$, then*

$$\begin{aligned} & \tau(A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)}) \\ & \geq \min_{1 \leq i \leq n} \left\{ \prod_{k=1}^m |A_k(i, i)|^{\alpha_k} \right. \\ & \quad \left. - \prod_{k=1}^m [|A_k(i, i)| - \tau(A_k)]^{\alpha_k} \right\} \\ & \geq \prod_{k=1}^m [\tau(A_k)]^{\alpha_k}. \end{aligned} \quad (14)$$

Proof. Without loss of generality, we can assume that $\alpha_k > 0$ for all $k \in \{1, 2, \dots, m\}$. Let $p_k = 1/\alpha_k > 0$, then $\sum_{k=1}^m (1/p_k) = \sum_{k=1}^m \alpha_k \geq 1$. Now we have:

$$\begin{aligned} & \tau(A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)}) \\ & = \tau(M(A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)})) \\ & = \tau(M(A_1^{(\alpha_1)}) \star \cdots \star M(A_m^{(\alpha_m)})) \end{aligned}$$

$$\begin{aligned} & \geq \min_{1 \leq i \leq n} \left\{ \prod_{k=1}^m |A_k(i, i)|^{\alpha_k} \right. \\ & \quad \left. - \prod_{k=1}^m [|A_k(i, i)| - \tau(M(A_k^{(\alpha_k)}))^{[p_k]}]^{1/p_k} \right\} \\ & = \min_{1 \leq i \leq n} \left\{ \prod_{k=1}^m |A_k(i, i)|^{\alpha_k} \right. \\ & \quad \left. - \prod_{k=1}^m [|A_k(i, i)| - \tau(A_k)]^{\alpha_k} \right\} \\ & \geq \min_{1 \leq i \leq n} \prod_{k=1}^m [|A_k(i, i)| - (|A_k(i, i)| - \tau(A_k))]^{1/p_k} \\ & = \prod_{k=1}^m [\tau(A_k)]^{\alpha_k}. \end{aligned} \quad (15)$$

The first “ $\geq$ ” hold by Lemma 3. Because A_k is H -matrix, we have $0 \leq \tau(A_k) \leq |A_k(i, i)|$, which means that $|A_k(i, i)| \geq |A_k(i, i)| - \tau(A_k) \geq 0$. So the second “ $\geq$ ” hold by $|A_k(i, i)|^{\alpha_k} \geq [|A_k(i, i)| - \tau(A_k)]^{\alpha_k} \geq 0$ and Lemma 4. $\square$

3. Examples

In this section, we present three examples to illustrate our improved bound.

Example 6. We take the matrices A , B , and C in Section 1. It is easy to get $\tau(A) = \tau(B) = \tau(C) = 1$, so the low bound in (1) is $\tau(A)^2 \tau(B)^2 \tau(C)^2 = 1$; on the other side, we can get $\tau(A^{(2)} \circ B^{(2)} \circ C^{(2)}) = 10000$, which is greatly larger than the lower bound $\tau(A) = \tau(B) = \tau(C)$, that is,

$$\tau(A^{(2)} \circ B^{(2)} \circ C^{(2)}) = 10000 \gg \tau(A)^2 \tau(B)^2 \tau(C)^2 = 1. \quad (16)$$

However, our lower bound in Theorem 5 is

$$\begin{aligned} & \min_{1 \leq i \leq 3} \{a_{ii}^2 b_{ii}^2 c_{ii}^2 - [a_{ii} - \tau(A)]^2 \\ & \quad \times [b_{ii} - \tau(B)]^2 [c_{ii} - \tau(C)]^2\} \\ & = \min_{1 \leq i \leq 3} \{100^2 \times 1^2 \times 1^2 - (100 - 1)^2 \\ & \quad \times (1 - 1)^2 (1 - 1)^2\} = 10000, \end{aligned} \quad (17)$$

which is the exact value of $\tau(A^{(2)} \circ B^{(2)} \circ C^{(2)})$.

Example 7. Let

$$A = \begin{pmatrix} 11 & -5 & -8 \\ -2 & 14 & -7 \\ -1 & -3 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 9 & -8 & -7 \\ -2 & 13 & -4 \\ -1 & -6 & 15 \end{pmatrix}. \quad (18)$$

It is easy to see that A and B are H -matrices and $\tau(A) = 3.9152$, $\tau(B) = 4.1345$. Thus, we have

$$\tau(A^{(2)} \circ B^{(2)}) = 9799 > \tau(A)^2 \tau(B)^2 = 262.0319, \quad (19)$$

but our bound is

$$\min_{1 \leq i \leq 3} \{a_{ii}^2 b_{ii}^2 - [a_{ii} - \tau(A)]^2 [b_{ii} - \tau(B)]^2\} = 8612.8, \quad (20)$$

which is close to the exact value of 9799.

Example 8. Take the H -matrices

$$A = \begin{pmatrix} 15 & -12 & -4 \\ -3 & 14 & -2 \\ -4 & 0 & 11 \end{pmatrix}, \quad B = \begin{pmatrix} 27 & -14 & -6 \\ -4 & 20 & -8 \\ -10 & -5 & 23 \end{pmatrix}. \quad (21)$$

It is easy to see that

$$\tau(A^{(2)} \circ B^{(1)}) = 2775.5 > \tau(A)^2 \tau(B)^1 = 244.0, \quad (22)$$

but our bound is

$$\min_{1 \leq i \leq 3} \{a_{ii}^2 b_{ii}^1 - [a_{ii} - \tau(A)]^2 [b_{ii} - \tau(B)]^1\} = 2340.3. \quad (23)$$

Also, our bound is much closer to the exact value than the low bound in (1).

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