Research Article

Lagrange Spaces with (γ, β) -Metric

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We study Lagrange spaces with (γ, β) -metric, where γ is a cubic metric and β is a 1-form. We obtain fundamental metric tensor, its inverse, Euler-Lagrange equations, semispray coefficients, and canonical nonlinear connection for a Lagrange space endowed with a (γ, β) -metric. Several other properties of such space are also discussed.

1. Introduction

Finsler spaces endowed with (α, β) -metric were studied by several geometers such as Matsumoto [1, 2] and Kitayama et al. [3], and various important applications of such spaces have been observed in physics and relativity theory (cf. [4, 5]). The notion of (α, β) -metric was taken to a more general space called Lagrange space and the study was performed by the authors such as Miron [6], Nicolaescu [7, 8], and the present authors [9]. An n -dimensional Lagrange space $L^n = (M, L(x, y))$ is said to be endowed with (α, β) -metric if Lagrangian $L(x, y)$ is function of $\alpha(x, y)$ and $\beta(x, y)$ only, that is, $L(x, y) = \bar{L}(\alpha, \beta)$, $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ being a Riemannian metric and $\beta(x, y) = b_i(x)y^i$ a 1-form. Recently, Pandey and Chaubey [10] discussed Lagrange spaces with (γ, β) -metric and obtained several results. They called a Lagrange space $L^n = (M, L(x, y))$ to be endowed with (γ, β) -metric if Lagrangian $L(x, y)$ is a function of $\gamma(x, y)$ and $\beta(x, y)$ only, that is, $L(x, y) = \bar{L}(\gamma, \beta)$ where $\gamma(x, y)$ is a cubic metric and $\beta(x, y)$ is a 1-form, that is, $\gamma^3 = a_{ijk}(x)y^i y^j y^k$ and $\beta(x, y) = b_i(x)y^i$. The paper [10] by Pandey and Chaubey is full of flaws and needs to be rectified. The aim of the present paper is to develop a revised and modified theory of Lagrange spaces with (γ, β) -metric.

The paper is organized as follows. In Section 2, we define a Lagrange space and discuss some preliminary results required for the discussion of the following sections. It includes the notion of a Lagrange space with (γ, β) -metric.

In Section 3, we discuss some properties of a Lagrange space with (γ, β) -metric and obtain the expression for the fundamental metric tensor g_{ij} and its inverse g^{ij} . In Section 4, we consider the variational problem in Lagrange spaces with (γ, β) -metric and obtain various forms of Euler-Lagrange equations. Section 5 deals with the semispray of a Lagrange space with (γ, β) -metric. In Section 6, we obtain the coefficients of nonlinear connection in a Lagrange space endowed with (γ, β) -metric. Section 7 consists of concluding remarks on the results obtained in the paper.

2. Preliminaries

Let M be an n -dimensional smooth manifold and let TM be its tangent bundle. Let (x^i) and (x^i, y^i) be local coordinates on M and TM , respectively. A Lagrangian is a function $L : TM \rightarrow \mathbb{R}$ which is a smooth function on $\widetilde{TM} = TM \setminus \{0\}$ and continuous on the null section. The Lagrangian $L(x, y)$ is said to be regular if $\text{rank}(g_{ij}(x, y)) = n$, where

$$g_{ij}(x, y) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L, \quad \dot{\partial}_i \equiv \frac{\partial}{\partial y^i}, \quad (1)$$

is a covariant symmetric tensor called the fundamental metric tensor of the Lagrangian $L(x, y)$. A Lagrange space is a pair $L^n = (M, L(x, y))$, $L(x, y)$ being a regular Lagrangian whose metric tensor g_{ij} has constant signature on $\widetilde{TM}$.

The integral of action of the Lagrangian $L(x, y)$ along a smooth curve $c : [0, 1] \rightarrow M$ leads to the following Euler-Lagrange equations:

$$E_i L \equiv \frac{\partial L}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial y^i} \right) = 0, \quad y^i = \frac{dx^i}{dt}. \quad (2)$$

The coefficients of the semispray S of a Lagrange space $L^n = (M, L(x, y))$ are given by

$$G^i(x, y) = \frac{1}{4} g^{ih} (y^k \dot{\partial}_h \partial_k L - \partial_h L), \quad \partial_k \equiv \frac{\partial}{\partial x^k}. \quad (3)$$

The semispray S is called a canonical semispray as its coefficients depend on $L(x, y)$ only.

The coefficients of canonical nonlinear connection $N(N_j^i(x, y))$ of a Lagrange space $L^n = (M, L(x, y))$ are given by

$$N_j^i = \dot{\partial}_j G^i. \quad (4)$$

In the present paper, we study a Lagrange space whose Lagrangian L is a function of $\gamma(x, y)$ and $\beta(x, y)$ only, where

$$\gamma^3 = a_{ijk}(x) y^i y^j y^k, \quad (5)$$

$$\beta(x, y) = b_i(x) y^i. \quad (6)$$

Let us denote this Lagrangian by $\bar{L}$. Thus

$$L(x, y) = \bar{L}(\gamma, \beta). \quad (7)$$

The space $L^n = (M, L(x, y))$ is called a *Lagrange space with (γ, β) -metric* (cf. [10]). Following are some examples of regular Lagrangians with (γ, β) -metric:

- (i) $\bar{L}(\gamma, \beta) = \gamma^3 + \beta$,
- (ii) $\bar{L}(\gamma, \beta) = \gamma^3 + a\gamma + b\beta + c\beta^2$, $a, b, c \in \mathbb{R}$, (8)
- (iii) $\bar{L}(\gamma, \beta) = (\gamma + \beta)^2$.

The Lagrange space determined by the Lagrangian in (8)(iii) is reducible to a Finsler space whereas those determined by the Lagrangians in (8)(i) and (8)(ii) are not so.

For basic notations and terminology related to a Lagrange space, we refer the reader to [11, 12].

3. Fundamental Metric Tensor of $L^n = (M, L(x, y))$

If we differentiate (5) partially with respect to y^j and use the symmetry of a_{rst} in its indices, we obtain

$$\dot{\partial}_j \gamma = \gamma^{-2} a_j(x, y), \quad (9)$$

where $a_j(x, y) = a_{jst}(x) y^s y^t$.

Again differentiating (9) partially with respect to y^h , using symmetry of $a_{rst}(x)$ in its indices and simplifying, we find

$$\dot{\partial}_j \dot{\partial}_h \gamma = 2\gamma^{-2} a_{jh}(x, y) - 2\gamma^{-5} a_j a_h, \quad (10)$$

where $a_{jh} = a_{jhs} y^s$.

Differentiating (6) partially with respect to y^j , we have

$$\dot{\partial}_j \beta = b_j(x). \quad (11)$$

Differentiating (11) partially with respect to y^h , we get

$$\dot{\partial}_j \dot{\partial}_h \beta = 0. \quad (12)$$

Thus, we have the following.

Proposition 1. *In a Lagrange space L^n with (γ, β) -metric, the following hold good:*

$$\begin{aligned} \dot{\partial}_j \gamma &= \gamma^{-2} a_j(x, y), \\ \dot{\partial}_j \dot{\partial}_h \gamma &= 2\gamma^{-2} a_{jh}(x, y) - 2\gamma^{-5} a_j a_h, \\ \dot{\partial}_j \beta &= b_j(x), \quad \dot{\partial}_j \dot{\partial}_h \beta = 0, \end{aligned} \quad (13)$$

where

$$a_j(x, y) = a_{jst}(x) y^s y^t, \quad a_{jh}(x, y) = a_{jhs} y^s. \quad (14)$$

The moments of Lagrangian $L(x, y)$ are given by

$$p_i := \frac{1}{2} \dot{\partial}_i L. \quad (15)$$

In our case, the Lagrangian $L(x, y)$ is a function of γ and β only (vide (7)). Therefore, we have

$$p_i = \frac{1}{2} (\bar{L}_\gamma \dot{\partial}_i \gamma + \bar{L}_\beta \dot{\partial}_i \beta), \quad (16)$$

where $\bar{L}_\gamma = \partial \bar{L} / \partial \gamma$, $\bar{L}_\beta = \partial \bar{L} / \partial \beta$.

Using (9) and (11) in (16), we obtain

$$p_i = \frac{1}{2} (\gamma^{-2} \bar{L}_\gamma a_i + \bar{L}_\beta b_i). \quad (17)$$

Thus, we have the following.

Theorem 2. *In a Lagrange space L^n with (γ, β) -metric, the moments of Lagrangian $L(x, y)$ are given by*

$$p_i = \rho a_i + \rho_1 b_i, \quad (18)$$

where

$$\rho = \frac{1}{2} \gamma^{-2} \bar{L}_\gamma, \quad (19)$$

$$\rho_1 = \frac{1}{2} \bar{L}_\beta. \quad (20)$$

Remarks 3. The scalars ρ and ρ_1 appearing in Theorem 2 are called the principal invariants of the space L^n .

Differentiating (19) and (20), partially with respect to y^j and simplifying, we, respectively, have

$$\dot{\partial}_j \rho = \frac{1}{2} \gamma^{-4} (\bar{L}_{\gamma\gamma} - 2\gamma^{-1} \bar{L}_\gamma) a_j + \frac{1}{2} \gamma^{-2} \bar{L}_{\gamma\beta} b_j, \quad (21)$$

$$\dot{\partial}_j \rho_1 = \frac{1}{2} \gamma^{-2} \bar{L}_{\beta\gamma} a_j + \frac{1}{2} \bar{L}_{\beta\beta} b_j, \quad (22)$$

where

$$\begin{aligned} \bar{L}_{\gamma\gamma} &= \frac{\partial^2 \bar{L}}{\partial \gamma^2}, & \bar{L}_{\gamma\beta} &= \frac{\partial^2 \bar{L}}{\partial \gamma \partial \beta} = \frac{\partial^2 \bar{L}}{\partial \beta \partial \gamma} = \bar{L}_{\beta\gamma}, \\ \bar{L}_{\beta\beta} &= \frac{\partial^2 \bar{L}}{\partial \beta^2}. \end{aligned} \quad (23)$$

Thus, we have the following.

Proposition 4. The derivatives of the principal invariants of a Lagrange space L^n with (γ, β) -metric are given by

$$\dot{\partial}_j \rho = \rho_{-2} a_j + \rho_{-1} b_j, \quad \dot{\partial}_j \rho_1 = \rho_{-1} a_j + \rho_0 b_j, \quad (24)$$

with

$$\begin{aligned} \rho_{-2} &= \frac{1}{2} \gamma^{-4} (\bar{L}_{\gamma\gamma} - 2\gamma^{-1} \bar{L}_\gamma), \\ \rho_{-1} &= \frac{1}{2} \gamma^{-2} \bar{L}_{\gamma\beta}, \quad \rho_0 = \frac{1}{2} \bar{L}_{\beta\beta}. \end{aligned} \quad (25)$$

The energy of Lagrangian $L(x, y)$ is defined as

$$E_L := y^i \dot{\partial}_i L - L. \quad (26)$$

Using (7) in (26), we have

$$E_{\bar{L}} = y^i (\bar{L}_\gamma \dot{\partial}_i \gamma + \bar{L}_\beta \dot{\partial}_i \beta) - \bar{L}. \quad (27)$$

Since γ and β are positively homogeneous of degree one in y^i , by virtue of Euler's theorem on homogeneous functions, we have

$$y^i \dot{\partial}_i \gamma = \gamma, \quad y^i \dot{\partial}_i \beta = \beta. \quad (28)$$

In view of (28), (27) takes the form

$$E_{\bar{L}} = \gamma \bar{L}_\gamma + \beta \bar{L}_\beta - \bar{L}. \quad (29)$$

Thus, we have the following.

Theorem 5. In a Lagrange space with (γ, β) -metric, the energy of the Lagrangian $L(x, y)$ is given by (29).

Now, we find out expression for the fundamental metric tensor $g_{ij}(x, y)$ of a Lagrange space with (γ, β) -metric. Using (7) in (1), we have

$$\begin{aligned} g_{ij} &= \frac{1}{2} [(\bar{L}_{\gamma\gamma} \dot{\partial}_i \gamma + \bar{L}_{\gamma\beta} \dot{\partial}_i \beta) \dot{\partial}_j \gamma + \bar{L}_\gamma \dot{\partial}_i \dot{\partial}_j \gamma \\ &\quad + (\bar{L}_{\beta\gamma} \dot{\partial}_i \gamma + \bar{L}_{\beta\beta} \dot{\partial}_i \beta) \dot{\partial}_j \beta + \bar{L}_\beta \dot{\partial}_i \dot{\partial}_j \beta]. \end{aligned} \quad (30)$$

In view of Proposition 1, (30) takes the form

$$g_{ij}(x, y) = 2\rho a_{ij} + \rho_{-2} a_i a_j + \rho_{-1} (a_i b_j + a_j b_i) + \rho_0 b_i b_j. \quad (31)$$

Equation (31) can be written as

$$g_{ij}(x, y) = 2\rho a_{ij} + c_i c_j, \quad (32)$$

where

$$c_i = q_{-1} a_i + q_0 b_i \quad (33)$$

and q_{-1}, q_0 satisfy

$$\begin{aligned} (a) \quad q_0 q_{-1} &= \rho_{-1}, & (b) \quad (q_{-1})^2 &= \rho_{-2}, \\ (c) \quad q_0^2 &= \rho_0. \end{aligned} \quad (34)$$

Thus, we have the following.

Theorem 6. The fundamental metric tensor of a Lagrange space with (γ, β) -metric is given by (32).

The following result gives the expression for the inverse of g_{ij} .

Theorem 7. The inverse g^{ij} of the fundamental metric tensor g_{ij} of a Lagrange space with (γ, β) -metric is given by

$$g^{ij} = \frac{1}{2\rho} \left(a^{ij} - \frac{1}{2\rho + c^2} c^i c^j \right), \quad (35)$$

where

$$(a) \quad c^i = a^{ir} c_r, \quad (b) \quad c^2 = a^{ij} c_i c_j. \quad (36)$$

Proof. Utilizing Lemma 6.1.2.1 of [11] for the nonsingular matrix g_{ij} given by (32) we have the result. $\square$

Remarks 8. In [10], Pandey and Chaubey obtained the following:

$$\dot{\partial}_i \gamma = \gamma^{-1} y_i, \quad \dot{\partial}_i \dot{\partial}_j \gamma = 2\gamma^{-1} a_{ij}(x, y) - \gamma^{-3} y_i y_j, \quad (37)$$

$$y_i = a_{ij}(x, y) y^j,$$

$$p_i = \rho y_i + \rho_1 b_i, \quad (38)$$

where ρ and ρ_1 are, respectively, given by (19) and (20),

$$\dot{\partial}_i \rho = \rho_{-2} y_i + \rho_{-1} b_i, \quad \dot{\partial}_i \rho_1 = \rho_{-1} y_i + \rho_0 b_i, \quad (39)$$

where $\rho_{-2} = (1/2)\gamma^{-2}(\bar{L}_{\gamma\gamma} - \gamma^{-1}\bar{L}_\gamma)$, $\rho_{-1} = (1/2)\gamma^{-1}\bar{L}_{\gamma\beta}$ and ρ_0 is the same as given in (25),

$$E_{\bar{L}} = \gamma^{-1} \bar{L}_\gamma + \beta \bar{L}_\beta - \bar{L}, \quad (40)$$

$$g_{ij}(x, y) = 2\rho a_{ij} + c_i c_j, \quad (41)$$

where $c_i = q_{-1}y_i + q_0b_i$ and q_{-1}, q_0 satisfy $q_0q_{-1} = \rho_{-1}$, $(q_{-1})^2 = \rho_{-2}$, $q_0^2 = \rho_0$, and

$$g^{ij} = \frac{1}{2\rho}a^{ij} - \frac{1}{1+c^2}c^ic^j, \quad (42)$$

where $c^i = (1/2)\rho^{-1}a^{ir}c_r$ and $c^2 = c^ic_i$.

In view of corresponding results obtained by us, these results are erroneous.

4. Euler-Lagrange Equations

Using (7) in (2), we obtain

$$E_i(\bar{L}) \equiv \frac{\partial \bar{L}}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial y^i} \right) = 0, \quad y^i = \frac{dx^i}{dt}. \quad (43)$$

For the Lagrangian $\bar{L}$ given by (7), we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial y^i} \right) &= \left(\bar{L}_{\gamma\gamma} \frac{d\gamma}{dt} + \bar{L}_{\gamma\beta} \frac{d\beta}{dt} \right) \frac{\partial \gamma}{\partial y^i} \\ &\quad + \left(\bar{L}_{\beta\gamma} \frac{d\gamma}{dt} + \bar{L}_{\beta\beta} \frac{d\beta}{dt} \right) \frac{\partial \beta}{\partial y^i} \\ &\quad + \bar{L}_\gamma \frac{d}{dt} \left(\frac{\partial \gamma}{\partial y^i} \right) + \bar{L}_\beta \frac{d}{dt} \left(\frac{\partial \beta}{\partial y^i} \right). \end{aligned} \quad (44)$$

In view of $\partial_i \bar{L} = \bar{L}_\gamma \partial_i \gamma + \bar{L}_\beta \partial_i \beta$ and (44), (43) takes the form

$$\begin{aligned} E_i(\bar{L}) &\equiv \bar{L}_\gamma E_i(\gamma) + \bar{L}_\beta E_i(\beta) - \left(\bar{L}_{\gamma\gamma} \frac{d\gamma}{dt} + \bar{L}_{\gamma\beta} \frac{d\beta}{dt} \right) \frac{\partial \gamma}{\partial y^i} \\ &\quad - \left(\bar{L}_{\beta\gamma} \frac{d\gamma}{dt} + \bar{L}_{\beta\beta} \frac{d\beta}{dt} \right) \frac{\partial \beta}{\partial y^i}. \end{aligned} \quad (45)$$

Since

$$E_i(\gamma^3) = 3\gamma^2 E_i(\gamma) - 3 \frac{\partial \gamma}{\partial y^i} \frac{d\gamma^2}{dt}, \quad (46)$$

we get

$$E_i(\gamma) = \frac{1}{3} \gamma^{-2} E_i(\gamma^3) + \gamma^{-2} \frac{\partial \gamma}{\partial y^i} \frac{d\gamma^2}{dt}. \quad (47)$$

From $E_i(\beta) = (\partial \beta / \partial x^i) - (d/dt)(\partial \beta / \partial y^i)$, we have

$$E_i(\beta) = 2F_{ir}y^r, \quad y^r = \frac{dx^r}{dt}, \quad (48)$$

where

$$F_{ir} = \frac{1}{2} \left(\frac{\partial b_r}{\partial x^i} - \frac{\partial b_i}{\partial x^r} \right) \quad (49)$$

is the electromagnetic tensor field of the potentials b_i .

Using (47) and (48) in (45), we obtain

$$\begin{aligned} E_i(\bar{L}) &= \frac{2}{3} \left(\frac{1}{2} \gamma^{-2} \bar{L}_\gamma \right) E_i(\gamma^3) + 2 \left(\frac{1}{2} \gamma^{-2} \bar{L}_\gamma \right) \frac{\partial \gamma}{\partial y^i} \frac{d\gamma^2}{dt} \\ &\quad + 4 \left(\frac{1}{2} \bar{L}_\beta \right) F_{ir}y^r - \frac{\partial \gamma}{\partial y^i} \left(\bar{L}_{\gamma\gamma} \frac{d\gamma}{dt} + \bar{L}_{\gamma\beta} \frac{d\beta}{dt} \right) \\ &\quad - \frac{\partial \beta}{\partial y^i} \left(\bar{L}_{\beta\gamma} \frac{d\gamma}{dt} + \bar{L}_{\beta\beta} \frac{d\beta}{dt} \right). \end{aligned} \quad (50)$$

Thus, we have the following.

Theorem 9. *The Euler-Lagrange equations of a Lagrange space with (γ, β) -metric are of the following form:*

$$\begin{aligned} E_i(\bar{L}) &\equiv \frac{2}{3} \rho E_i(\gamma^3) + 2\rho \frac{\partial \gamma}{\partial y^i} \frac{d\gamma^2}{dt} + 4\rho_1 F_{ir}y^r \\ &\quad - \frac{\partial \gamma}{\partial y^i} \left(\bar{L}_{\gamma\gamma} \frac{d\gamma}{dt} + \bar{L}_{\gamma\beta} \frac{d\beta}{dt} \right) \\ &\quad - \frac{\partial \beta}{\partial y^i} \left(\bar{L}_{\beta\gamma} \frac{d\gamma}{dt} + \bar{L}_{\beta\beta} \frac{d\beta}{dt} \right) = 0, \quad y^i = \frac{dx^i}{dt}. \end{aligned} \quad (51)$$

For the natural parametrization of the curve $c : t \in [0, 1] \mapsto x^i(t) \in M$ with respect to the cubic metric $a_{ijk}(x)$, $\gamma(x, dx/dt) = 1$. Thus, we have the following.

Theorem 10. *In the natural parametrization, the Euler-Lagrange equations of a Lagrange space with (γ, β) -metric are*

$$\begin{aligned} E_i(\bar{L}) &\equiv \frac{2}{3} \rho E_i(\gamma^3) + 4\rho_1 F_{ir}y^r - \frac{\partial \gamma}{\partial y^i} \bar{L}_{\gamma\beta} \frac{d\beta}{ds} \\ &\quad - \frac{\partial \beta}{\partial y^i} \bar{L}_{\beta\beta} \frac{d\beta}{ds} = 0. \end{aligned} \quad (52)$$

If β is constant on the integral curve c of the Euler-Lagrange equations with natural parametrization, then (52) takes the form

$$E_i(\bar{L}) \equiv \frac{2}{3} \rho E_i(\gamma^3) + 4\rho_1 F_{ir}y^r = 0. \quad (53)$$

Thus, we have the following.

Theorem 11. *If β is constant along the integral curve of the Euler-Lagrange equations with natural parametrization, then the Euler-Lagrange equations of the Lagrange space with (γ, β) -metric are given by (53).*

Remarks 12. Pandey and Chaubey [10] obtained the following form of Euler-Lagrange equations:

$$\begin{aligned} E_i(\bar{L}) &\equiv \frac{2}{3\gamma} \rho E_i(\gamma^3) + \frac{2}{3\gamma} \rho \frac{\partial \gamma}{\partial y^i} \frac{d\gamma^2}{dt} + 2\rho_1 F_{ir}y^r \\ &\quad - \frac{\partial \gamma}{\partial y^i} \left(\bar{L}_{\gamma\gamma} \frac{d\gamma}{dt} + \bar{L}_{\gamma\beta} \frac{d\beta}{dt} \right) \\ &\quad - \frac{\partial \beta}{\partial y^i} \left(\bar{L}_{\beta\gamma} \frac{d\gamma}{dt} + \bar{L}_{\beta\beta} \frac{d\beta}{dt} \right) = 0. \end{aligned} \quad (54)$$

In case of natural parametrization, their result is

$$\begin{aligned} \frac{2}{3\gamma} \rho E_i (\gamma^3) + \frac{2}{3\gamma} \rho \frac{\partial \gamma}{\partial y^i} \frac{d\gamma^2}{dt} + 2\rho_1 F_{ir} y^r \\ - \frac{\partial \beta}{\partial y^i} \left(\bar{L}_{\beta\gamma} \frac{d\gamma}{dt} + \bar{L}_{\beta\beta} \frac{d\beta}{dt} \right) = 0. \end{aligned} \quad (55)$$

If β is constant along the integral curve c of the Euler-Lagrange equations (with natural parametrization), the Euler-Lagrange equations obtained by Pandey and Chaubey [10] are given by

$$\frac{2}{3\gamma} \rho E_i (\gamma^3) + \frac{2}{3\gamma} \rho \frac{\partial \gamma}{\partial y^i} \frac{d\gamma^2}{dt} + 2\rho_1 F_{ir} y^r = 0. \quad (56)$$

In view of Theorem 9, Theorem 10, and Theorem 11, all the above discussed results of Pandey and Chaubey [10] are erroneous.

5. Canonical Semispray

In this section, we obtain the coefficients of the canonical semispray of a Lagrange space with (γ, β) -metric.

Using (7) in (3), we obtain

$$G^i(x, y) = \frac{1}{4} g^{ih} (y^k \dot{\partial}_h \partial_k \bar{L} - \partial_h \bar{L}). \quad (57)$$

Since $\gamma^3 = a_{ijk}(x) y^i y^j y^k$ and $\beta = b_i(x) y^i$, we have

$$\partial_h \gamma = A_h \gamma^{-2}, \quad \partial_h \beta = B_h, \quad (58)$$

where

$$A_h = \frac{1}{3} (\partial_h a_{ijk}) y^i y^j y^k, \quad B_h = (\partial_h b_i) y^i. \quad (59)$$

Using (58), (19), and (20) in $\partial_k \bar{L} = \bar{L}_\gamma \partial_k \gamma + \bar{L}_\beta \partial_k \beta$, we get

$$\partial_k \bar{L} = 2\rho A_k + 2\rho_1 B_k. \quad (60)$$

Differentiating (60) partially with respect to y^h and simplifying, we have

$$\begin{aligned} \dot{\partial}_h \partial_k \bar{L} = 2(\rho_{-2} a_h + \rho_{-1} b_h) A_k + 2\rho A_{kh} \\ + 2(\rho_{-1} a_h + \rho_0 b_h) B_k + 2\rho_1 b_{kh}, \end{aligned} \quad (61)$$

where

$$(a) \ A_{kh} = \dot{\partial}_h A_k, \quad (b) \ b_{kh} = \dot{\partial}_h B_k. \quad (62)$$

Using (60) and (61) in (57), we obtain

$$\begin{aligned} G^i = \frac{1}{2} g^{ih} [(\rho_{-2} A_0 + \rho_{-1} B_0) a_h + (\rho_{-1} A_0 + \rho_0 B_0) b_h \\ + \rho A_{0h} + \rho_1 b_{0h} - (\rho A_h + \rho_1 B_h)], \end{aligned} \quad (63)$$

where

$$\begin{aligned} (i) \ A_0 = A_k(x, y) y^k, \quad (ii) \ B_0 = B_k(x, y) y^k, \\ (iii) \ A_{0h} = A_{kh}(x, y) y^k, \quad (iv) \ b_{0h} = b_{kh}(x, y) y^k. \end{aligned} \quad (64)$$

Thus, we have the following.

Theorem 13. *The local coefficients of canonical semispray of a Lagrange space with (γ, β) -metric are given by (63).*

6. Canonical Nonlinear Connection

In this section, we obtain the local coefficients of the canonical nonlinear connection of a Lagrange space with (γ, β) -metric.

Partial differentiation of $g^{ih} g_{is} = \delta_s^h$, with respect to y^j , yields

$$\dot{\partial}_j g^{ih} = -2g^{rh} C_{rj}^i. \quad (65)$$

If we partially differentiate the quantities appearing in (25) and (64) with respect to y^j , we find the following quantities:

$$\begin{aligned} \dot{\partial}_j \rho_{-2} = \mu_{-3} a_j + \mu_{-2} b_j, \quad \dot{\partial}_j \rho_{-1} = \mu_{-2} a_j + \mu_{-1} b_j, \\ \dot{\partial}_j \rho_0 = \mu_{-1} a_j + \mu_0 b_j, \\ \dot{\partial}_j A_0 = A_j + A_{0j}, \quad \dot{\partial}_j B_0 = \mathfrak{S}_{(sj)} \{ \partial_s b_j \} y^s, \\ \dot{\partial}_j A_{0h} = 2A_{0hj} + A_{jh}, \quad \dot{\partial}_j b_{0h} = b_{jh}, \end{aligned} \quad (66)$$

where

$$\begin{aligned} \mu_{-3} = \frac{1}{2} \gamma^{-6} [\bar{L}_{\gamma\gamma\gamma} - 6\gamma^{-1} \bar{L}_{\gamma\gamma} + 10\gamma^{-2} \bar{L}_\gamma], \\ \mu_{-2} = \frac{1}{2} \gamma^{-4} [\bar{L}_{\gamma\gamma\beta} - 2\gamma^{-1} \bar{L}_{\gamma\beta}], \\ \mu_{-1} = \frac{1}{2} \gamma^{-2} \bar{L}_{\gamma\gamma\beta}, \quad \mu_0 = \frac{1}{2} \bar{L}_{\beta\beta\beta}, \\ A_{0hj} = A_{rhj} y^r, \quad A_{rhj} = \partial_r a_{hj} \end{aligned} \quad (67)$$

and $\mathfrak{S}_{(sj)}$ denotes interchanging indices s and j and taking sum. Also, we have

$$\dot{\partial}_j a_h = 2a_{jh}. \quad (68)$$

Now, applying (63) in (4), we get

$$\begin{aligned}
 N_j^i = & \frac{1}{2} (\partial_j g^{ih}) ((\rho_{-2} A_0 + \rho_{-1} B_0) a_h + (\rho_{-1} A_0 + \rho_0 B_0) b_h \\
 & + \rho A_{0h} + \rho_1 b_{0h} - (\rho A_h + \rho_1 B_h)) \\
 & + \frac{1}{2} g^{ih} [((\partial_j \rho_{-2}) A_0 + \rho_{-2} (\partial_j A_0) + (\partial_j \rho_{-1}) B_0 \\
 & + \rho_{-1} (\partial_j B_0)) a_h \\
 & + (\rho_{-2} A_0 + \rho_{-1} B_0) \partial_j a_h \\
 & + ((\partial_j \rho_{-1}) A_0 + \rho_{-1} (\partial_j A_0) \\
 & + (\partial_j \rho_0) B_0 + \rho_0 (\partial_j B_0)) b_h \\
 & + (\partial_j \rho) A_{0h} + (\partial_j \rho_1) b_{0h} + \rho \partial_j A_{0h} + \rho_1 \partial_j b_{0h} \\
 & - ((\partial_j \rho) A_h + \rho (\partial_j A_h) \\
 & + (\partial_j \rho_1) B_h + \rho_1 (\partial_j B_h))] .
 \end{aligned} \tag{69}$$

Using (24), (62)–(64), (65), (66) and (68) in (69) and simplifying, we obtain

$$\begin{aligned}
 N_j^i = & -2C_{rj}^i G^r + \frac{1}{2} g^{ih} \\
 & \times [\rho_{-2} ((A_j + A_{0j}) a_h + (A_{0h} - A_h) a_j + 2A_0 a_{jh}) \\
 & + \rho_{-1} ((A_j + A_{0j}) b_h + (A_{0h} - A_h) b_j + a_j b_{0h} \\
 & + \mathfrak{S}_{(sj)} \{ \partial_s b_j \} y^s a_h + 2B_0 a_{jh} - a_j B_h) \\
 & + \rho_0 (\mathfrak{S}_{(sj)} \{ \partial_s b_j \} y^s b_h - b_j B_h + b_j b_{0h}) \\
 & + \rho_1 \mathfrak{M}_{(jh)} \{ b_{jh} \} + \rho (2A_{0hj} + A_{jh} - A_{hj}) \\
 & + \mu_{-3} A_0 a_j a_h + \mu_{-2} (A_0 b_j a_h + a_j (B_0 a_h + A_0 b_h)) \\
 & + \mu_{-1} (b_j (B_0 a_h + A_0 b_h) + a_j B_0 b_h) \\
 & + \mu_0 B_0 b_j b_h] ,
 \end{aligned} \tag{70}$$

where $\mathfrak{M}_{(jh)} \{ b_{jh} \} = b_{jh} - b_{hj}$.

Thus, we have the following.

Theorem 14. *The local coefficients of the canonical nonlinear connection of a Lagrange space with (γ, β) -metric are given by (70).*

7. Conclusions

We have developed the theory of Lagrange spaces with (γ, β) -metric. The paper presents a significant generalization of the

theory of Lagrange spaces with (α, β) -metrics (cf. [6–8]). The expressions for the geometric objects obtained in the paper may be useful in further work on the spaces under consideration. The importance of the results lies in the study of canonical metrical d -connection, curvatures, and torsions in such spaces. The expressions for canonical semispray and nonlinear connection, obtained, respectively, in Section 5 and Section 6, may be applicable in geodesic correspondences between two Lagrange spaces with different (γ, β) -metrics on the same underlying manifold. It is a matter of later investigations to look into the aforesaid applications of the results obtained in the paper.

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