Hindawi Publishing Corporation
ISRN Combinatorics
Volume 2013, Article ID 107454, 6 pages
http://dx.doi.org/10.1155/2013/107454

Research Article

Permutations and Pairs of Dyck Paths

Marilena Barnabei,[1] Flavio Bonetti,[1] and Matteo Silimbani[2]

[1] *Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy*
[2] *LaBRI, Université de Bordeaux, 351 Cours de la Libération, 33405 Talence, France*

Correspondence should be addressed to Matteo Silimbani; matteo.silimbani4@unibo.it

Received 7 November 2012; Accepted 26 November 2012

Academic Editors: E. Manstavicius, S. Natanzon, and Y. Shi

We define a map v between the symmetric group S_n and the set of pairs of Dyck paths of semilength n. We show that the map v is injective when restricted to the set of 1234-avoiding permutations and characterize the image of this map.

1. Introduction

We say that a permutation $\sigma \in S_n$ contains a pattern $\tau \in S_k$ if σ contains a subsequence that is order-isomorphic to τ. Otherwise, we say that σ avoids τ. Given a pattern τ, denote by $S_n(\tau)$ the set of permutations in S_n avoiding τ.

The sets of permutations that avoid a single pattern $\tau \in S_3$ have been completely determined in last decades. More precisely, it has been shown [1] that, for every $\tau \in S_3$, the cardinality of the set $S_n(\tau)$ equals the nth Catalan number, which is also the number of Dyck paths of semilength n (see [2] for an exhaustive survey). Many bijections between $S_n(\tau)$, $\tau \in S_3$, and the set of Dyck paths of semilength n have been described (see [3] for a fully detailed overview).

The case of patterns of length 4 appears much more complicated, due both to the fact that the patterns $\tau \in S_4$ are not equidistributed on S_n, and the difficulty of describing bijections between $S_n(\tau)$, $\tau \in S_4$, and some set of combinatorial objects.

In this paper we study the case $\tau = 1234$. An explicit formula for the cardinality of $S_n(1234)$ has been computed by I. Gessel (see [2, 4]).

We present a bijection between $S_n(1234)$ and a set of pairs of Dyck paths of semilength n. More specifically, we define a map v from S_n to the set of pairs of Dyck paths, prove that every element in the image of v corresponds to a single element in $S_n(1234)$, and characterize the set of all pairs that belong to the image of the map v.

2. Dyck Paths

A Dyck path of semilength n is a lattice path starting at $(0, 0)$, ending at $(2n, 0)$, and never going below the x-axis, consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$. A return of a Dyck path is a down step ending on the x-axis. A Dyck path is irreducible if it has only one return. An irreducible component of a Dyck path P is a maximal irreducible Dyck subpath of P.

A Dyck path P is specified by the lengths $a_1, \dots, a_k$ of its ascents (viz., maximal sequences of consecutive up steps) and by the lengths $d_1, \dots, d_k$ of its descents (maximal sequences of consecutive down steps), read from left to right. Set $A_i = \sum_{j=1}^{i} a_j$ and $B_i = \sum_{j=1}^{i} d_j$. If n is the semilength of P, we have of course $A_k = B_k = n$, hence the Dyck path P is uniquely determined by the two sequences $A = A_1, \dots, A_{k-1}$ and $B = B_1, \dots, B_{k-1}$. The pair (A, B) is called the ascent-descent code of the Dyck path P.

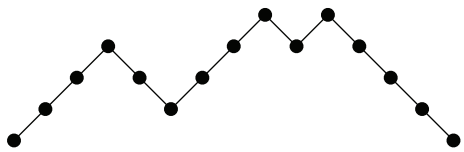
Obviously, a pair (A, B), where $A = A_1, \dots, A_{k-1}$ and $B = B_1, \dots, B_{k-1}$, is the ascent-descent code of some Dyck path of semilength n if and only if

(i) $0 < k \le n$;

(ii) $1 \le A_1 < A_2 < \cdots < A_{k-1} \le n - 1$;

(iii) $1 \le B_1 < B_2 < \cdots < B_{k-1} \le n - 1$;

(iv) $A_i \ge B_i$ for every $1 \le i \le k - 1$.



P

Figure 1

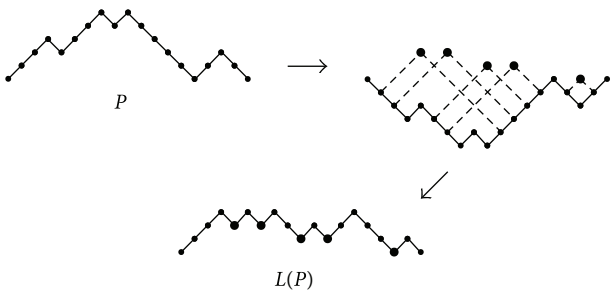


Figure 2: The map L.

It is easy to check that the returns of a Dyck path are in one-to-one correspondence with the indices $1 \leq i \leq k$ such that $A_i = B_i$. Hence, a Dyck path is irreducible whenever we have $A_i > B_i$ for every $1 \leq i \leq k-1$.

For example, the ascent-descent code of the Dyck path P in Figure 1 is (A, B), where $A = 3, 6$ and $B = 2, 3$. Note that $A_1 > B_1$ and $A_2 > B_2$. In fact, P is irreducible.

We describe an involution L due to Kreweras (a description of this bijection, originally defined in [5], can be found in [6]) and discussed by Lalanne (see [7, 8]) on the set of Dyck paths. Given a Dyck path P, the path $L(P)$ can be constructed as follows:

(i) if P is the empty path ϵ, then $L(P) = \epsilon$;

(ii) if P is nonempty:

(a) flip the Dyck path P around the x-axis, obtaining a path E;

(b) draw northwest (resp. northeast) lines starting from the midpoint of each double descent (resp. ascent);

(c) mark the intersection between the ith northwest and ith northeast line, for every i;

(d) $L(P)$ is the unique Dyck path that has valleys at the marked points (see Figure 2).

We define a further involution L' on the set of Dyck paths, which is a variation of the involution L, as follows:

(i) if P is the empty path ϵ, then $L(P) = \epsilon$;

(a) consider a Dyck path P and flip it with respect to a vertical line;

(b) decompose the obtained path into its irreducible components $U\ P_i\ D$;

(c) replace every component $U\ P_i\ D$ with $U\ L(P_i)\ D$ in order to get $L'(P)$ (see Figure 3).

We point out that the map L' appears in a slightly modified version in the paper [6].

We now give a description of the map L' in terms of ascent-descent code. Obviously, it is sufficient to consider the case of an irreducible Dyck path P.

Let (A, B) be the ascent-descent code of an irreducible path P of semilength n, with $A = A_1, \ldots, A_h$ and $B = B_1, \ldots, B_h$. Straightforward arguments show that the ascent-descent code (A', B') of $L'(P)$ can be described as follows:

(i) set $\overline{A}_i = A_i - 1$ and set $\widehat{A} = [n-2] \setminus \{\overline{A}_1, \ldots, \overline{A}_h\} = \{\widehat{A}_1, \ldots, \widehat{A}_{n-2-h}\}$, where the $\widehat{A}_i$'s are written in decreasing order. Then, $A'_i = n - \widehat{A}_i$;

(ii) consider the set $[n-2] \setminus \{B_1, \ldots, B_h\} = \{\widehat{B}_1, \ldots, \widehat{B}_{n-2-h}\}$, where the $\widehat{B}_i$'s are written in decreasing order. Then, $B'_i = n - 1 - \widehat{B}_i$.

Finally, we introduce an order relation $\leq$ on the set of Dyck paths of the same semilength. This order relation will be defined in three steps:

(i) consider two irreducible Dyck paths P and Q of semilength n. Let (A, B) be the ascent-descent code of P, with $A = A_1, \ldots, A_k$ and $B = B_1, \ldots, B_k$. We say that Q covers P in the relation $\leq$ if the ascent code of Q is obtained by removing an integer A_i from A and the descent code of Q is obtained by removing an integer B_j from B, with $j \geq i$.

Roughly speaking, Q covers P if it can be obtained from P by "closing" the rectangles corresponding to an arbitrary collection of consecutive valleys of P (see Figure 4);

(ii) the desired order relation $\leq$ on the set of irreducible Dyck paths is the transitive closure of the above covering relation;

(iii) the relation $\leq$ is extended to the set of all Dyck path of a given semilength as follows: if P and Q are two arbitrary Dyck paths and $P = P_1\ P_2\ \cdots\ P_r$ and $Q = Q_1\ Q_2\ \cdots\ Q_s$ are their respective decompositions into irreducible parts, then $P \leq Q$ whenever $r = s$ and $P_i \leq Q_i$ for every i.

We point out that the described order relation is a subset of the inclusion order relation defined in [9]. In the following sections, we will show that the defined relation is more suited for our studies.

3. LTR Minima and RTL Maxima of a Permutation

Some of the well-known bijections between $S_n(\tau)$, $\tau \in S_3$, and the set of Dyck paths of semilength n (see [10–12]), are based on the two notions of left-to-right minimum and right-to-left maximum of a permutation $\sigma = x_1\ x_2\ \cdots\ x_n$:

(i) the value x_i is a *left-to-right minimum* (LTR minimum for short) at position i if $x_i < x_j$ for every $j < i$;

(ii) the value x_i is a *right-to-left maximum* (RTL maximum) at position i if $x_i > x_j$ for every $j > i$.

For example, the permutation

$$\sigma = 5\ 3\ 4\ 8\ 2\ 1\ 6\ 7 \tag{1}$$

has the LTR minima 5, 3, 2, and 1 (at positions 1, 2, 5, and 6) and RTL maxima 7 and 8 (at positions 8 and 4).

We denote by $v_{\min}(\sigma)$ and $p_{\min}(\sigma)$ the sets of values and positions of the LTR minima of σ, respectively. Analogously, $v_{\max}(\sigma)$ and $p_{\max}(\sigma)$ denote the sets of values and positions of the RTL maxima of σ.

Recall that the reverse-complement of a permutation $\sigma \in S_n$ is the permutation defined by

$$\sigma^{\mathrm{rc}}(i) = n + 1 - \sigma(n + 1 - i). \tag{2}$$

For example, consider the permutation

$$\sigma = 2\ 4\ 7\ 3\ 1\ 8\ 9\ 5\ 6. \tag{3}$$

Then

$$\sigma^{\mathrm{rc}} = 4\ 5\ 1\ 2\ 9\ 7\ 3\ 6\ 8. \tag{4}$$

Note that the sets $S_n(123)$ and $S_n(1234)$ are closed under reverse-complement, namely, $\sigma \in S_n(123)$ (resp., $\sigma \in S_n(1234)$) if and only if $\sigma^{\mathrm{rc}} \in S_n(123)$ (resp. $\sigma^{\mathrm{rc}} \in S_n(1234)$).

The first assertion in the next theorem goes back to the seminal paper [12], while the second one is an immediate consequence of the straightforward fact that x is a LTR minimum of a permutation σ at position i if and only if $n + 1 - x$ is RTL maximum of σ^{rc} at position $n + 1 - i$.

Theorem 1. *A permutation $\sigma \in S_n(123)$ is completely determined by the two sets $v_{\min}(\sigma)$ and $p_{\min}(\sigma)$ of values and positions of its left-to-right minima. A permutation in $S_n(123)$ is completely determined, as well, by the two sets $v_{\max}(\sigma)$ and $p_{\max}(\sigma)$ of values and positions of its right-to-left maxima.*

Also 1234-avoiding permutations can be characterized in terms of LTR minima and RTL maxima.

This characterization can be found in [2] and is based on an equivalence relation on S_n defined as follows: $\sigma \equiv \sigma' \Leftrightarrow \sigma$ and σ' share the values and the positions of LTR minima and RTL maxima.

For example,

$$1\ 2\ 3\ 4 \equiv 1\ 3\ 2\ 4. \tag{5}$$

Straightforward arguments lead to the following result stated in [2].

Theorem 2. *Every equivalence class of the relation $\equiv$ contains exactly one 1234-avoiding permutation. In this permutation, the values that are neither LTR minima nor RTL maxima appear in decreasing order.*

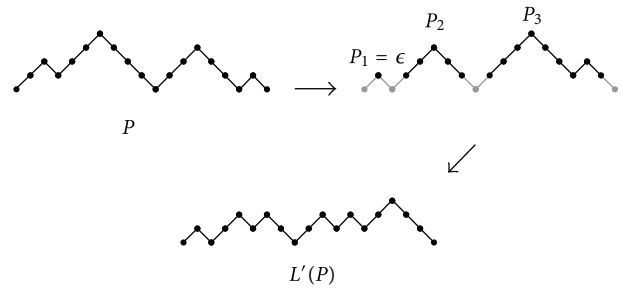


Figure 3: The map L'.

4. The Maps λ and μ

We define two maps λ and μ between S_n and the set $\mathcal{D}_n$ of Dyck paths of semilength n. Given a permutation $\sigma \in S_n$, the path $\lambda(\sigma)$ is contructed as follows:

(i) decompose σ as $\sigma = m_1\ w_1\ m_2\ w_2\ \cdots\ m_k\ w_k$, where $m_1, m_2, \ldots, m_k$ are the left-to-right minima in σ and $w_1, w_2, \ldots, w_k$ are (possibly empty) words;

(ii) set $m_0 = n + 1$;

(iii) read the permutation from left to right and translate any LTR minimum m_i $(i > 0)$ into $m_{i-1} - m_i$ up steps and any subword w_i into $l_i + 1$ down steps, where l_i denotes the number of elements in w_i.

The statement of Theorem 1 implies that the map λ is a bijection when restricted to $S_n(123)$.

Note that the ascent-descent code (A, B) of the path $\lambda(\sigma)$ is obtained as follows:

(i) $A = n + 1 - m_1, n + 1 - m_2, \ldots, n + 1 - m_{k-1}$;

(ii) $B = p_2 - 1, p_3 - 1, \ldots, p_k - 1$, where p_i is the position of m_i.

We define a further map $\mu : S_n \to \mathcal{D}_n$:

(i) decompose σ as $\sigma = u_h\ M_h\ u_{h-1}\ M_{h-1}\ \cdots\ u_1\ M_1$, where $M_1, M_2, \ldots, M_h$ are the right-to-left maxima in σ and $u_1, u_2, \ldots, u_k$ are (possibly empty) words;

(ii) set $M_0 = 0$;

(iii) associate with M_i $(i > 0)$ the steps $U^{m_i - m_{i-1}} D$;

(iv) associate with each entry in u_i a D step.

Also in this case, the map μ is a bijection when restricted to $S_n(123)$.

The ascent-descent code (A^*, B^*) of the path $\mu(\sigma)$ is obtained as follows:

(i) $A^* = M_1, M_2, \ldots, M_{h-1}$;

(ii) $B^* = n - P_2, n - P_3, \ldots, n - P_h$, where P_i is the position of M_i.

In Figure 5 the two paths $\lambda(\sigma)$ and $\mu(\sigma)$ corresponding to $\sigma = 6\ 2\ 3\ 1\ 7\ 5\ 4$ are shown.

We can now define a map $\nu : S_n \to \mathcal{D}_n \times \mathcal{D}_n$, setting

$$\nu(\sigma) = (\lambda(\sigma), \mu(\sigma)). \tag{6}$$

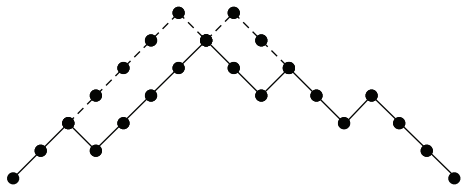


Figure 4: The dotted Dyck path covers the solid one.

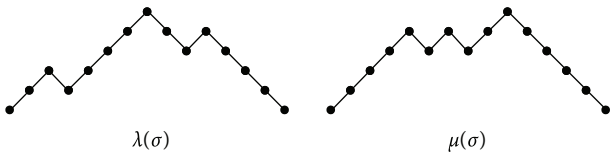


Figure 5: The Dyck paths corresponding to $\sigma = 6\ 2\ 3\ 1\ 7\ 5\ 4$.

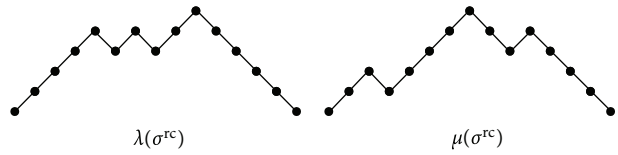


Figure 6: The Dyck paths corresponding to $\sigma^{rc} = 4\ 3\ 1\ 7\ 5\ 6\ 2$.

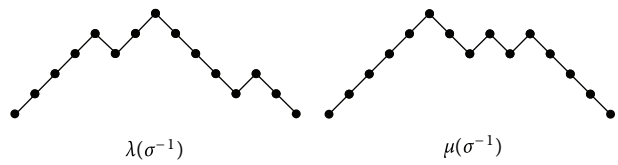


Figure 7: The Dyck paths corresponding to $\sigma^{-1} = 4\ 2\ 3\ 7\ 6\ 1\ 5$.

The statement of Theorem 2 implies that the map ν is injective when restricted to $S_n(1234)$.

Note that the map ν behaves properly with respect to the reverse-complement and the inversion operators.

Proposition 3. *Let σ be a permutation in S_n. One has:*

(i) $\nu(\sigma) = (L, R) \Leftrightarrow \nu(\sigma^{rc}) = (R, L)$, *hence, the permutation σ is rc-invariant if and only if $L = R$.*

(ii) $\nu(\sigma) = (L, R) \Leftrightarrow \nu(\sigma^{-1}) = (\mathrm{rev}(L), \mathrm{rev}(R))$, *where* $\mathrm{rev}(P)$ *is the path obtained by flipping P with respect to a vertical line. Hence, the permutation σ is an involution if and only if both L and R are symmetric with respect to a vertical line.*

For example, consider $\sigma = 6\ 2\ 3\ 1\ 7\ 5\ 4$. The two paths associated with σ are shown in Figure 5. The permutation $\sigma^{rc} = 4\ 3\ 1\ 7\ 5\ 6\ 2$ is associated with the two paths in Figure 6, while the permutation $\sigma^{-1} = 4\ 2\ 3\ 7\ 6\ 1\ 5$ corresponds to the two paths in Figure 7.

Moreover, the map ν has the following further property that will be crucial in the proof of our main result.

Recall that a permutation $\sigma \in S_n$ is said to be right-connected if it does not have a suffix σ' of length $k < n$, that is a permutation of the symbols $1, 2, \ldots, k$.

For example, the permutation

$$\tau = 6\ 1\ 2\ 7\ 5\ 3\ 4\ 8 \tag{7}$$

is right-connected, while

$$\sigma = 8\ 6\ 4\ 5\ 7\ 2\ 1\ 3 \tag{8}$$

is not right-connected.

According to this definition, we can split every permutation into right-connected components:

$$\sigma = \mathbf{8}\ 6\ 4\ 5\ 7\ \mathbf{2\ 1\ 3}. \tag{9}$$

Note that, if a permutation σ is not right-connected, σ is the juxtaposition of a permutation σ'' of the set $\{t+1, \ldots, n\}$ and the permutation σ' of the set $\{1, \ldots, t\}$.

Proposition 4. *Let σ be a non right-connected permutation in S_n, with $\sigma = \sigma_1\sigma_2$, where σ_1 is a permutation of the set $\{t+1, \ldots, n\}$ and σ_2 is a permutation of set of the set $\{1, \ldots, t\}$. Then*

$$\lambda(\sigma) = P_1P_2, \qquad \mu(\sigma) = Q_1Q_2, \tag{10}$$

with $P_i = \lambda(\sigma_i)$ and $Q_i = \mu(\sigma_i)$, $i = 1, 2$.

The order relation on Dyck paths defined in Section 2 can be exploited to define two order relations on the set S_n as follows:

(i) $\sigma \leq_\lambda \tau$ if and only if $\lambda(\sigma) \leq \lambda(\tau)$;

(ii) $\sigma \leq_\mu \tau$ if and only if $\mu(\sigma) \leq \mu(\tau)$.

These order relations can be intrinsically described as follows.

Proposition 5. *Let $\sigma, \tau \in S_n$. One has $\sigma \leq_\lambda \tau$ whenever:*

(i) $v_{\min}(\tau) \subseteq v_{\min}(\sigma)$;

(ii) $p_{\min}(\tau) \subseteq p_{\min}(\sigma)$;

(iii) *setting:*

$v_{\min}(\sigma) = \{m_1, \ldots, m_h\}$ *(written in decreasing order),*

$v_{\min}(\sigma) \setminus v_{\min}(\tau) = \{m_{i_1}, m_{i_2}, \ldots, m_{i_r}\}$ *(in decreasing order),*

$p_{\min}(\sigma) \setminus p_{\min}(\tau) = \{p_{j_1}, p_{j_2}, \ldots, p_{j_r}\}$ *(in increasing order),*

then $i_k < j_k$ for every k.

Similarly, $\sigma \leq_\mu \tau$ whenever:

(i) $v_{\max}(\tau) \subseteq v_{\max}(\sigma)$;

(ii) $p_{\max}(\tau) \subseteq p_{\max}(\sigma)$;

(iii) *setting:*

$v_{\max}(\sigma) = \{M_1, \ldots, M_t\}$ *(written in increasing order),*

$v_{\max}(\sigma) \setminus v_{\max}(\tau) = \{M_{i_1}, M_{i_2}, \ldots, M_{i_q}\}$ *(in increasing order),*

$p_{\max}(\sigma) \setminus p_{\max}(\tau) = \{P_{j_1}, P_{j_2}, \ldots, P_{j_q}\}$ *(in decreasing order),*

then $i_k < j_k$ *for every* k.

For example, consider the permutation

$$\sigma = 6\ 8\ 7\ 3\ 2\ 5\ 9\ 1\ 4. \tag{11}$$

We have $v_{\min}(\sigma) = \{6, 3, 2, 1\}$, $p_{\min}(\sigma) = \{1, 4, 5, 8\}$, $v_{\max}(\sigma) = \{4, 9\}$, and $p_{\max}(\sigma) = \{9, 7\}$. The permutation

$$\tau = 3\ 4\ 9\ 2\ 6\ 8\ 7\ 1\ 5 \tag{12}$$

is such that $v_{\min}(\tau) = \{3, 2, 1\}$ and $p_{\min}(\tau) = \{1, 4, 8\}$, hence, $\sigma \leq_\lambda \tau$. Moreover, the permutation

$$\rho = 2\ 7\ 1\ 3\ 4\ 6\ 5\ 8\ 9 \tag{13}$$

is such that $v_{\max}(\rho) = \{9\}$ and $p_{\max}(\rho) = \{9\}$, hence, $\sigma \leq_\mu \rho$.

5. Main Results

We say that a pair of Dyck paths (P, Q) is admissible if there exists a permutation α such that $P = \lambda(\alpha)$ and $Q = \mu(\alpha)$. Needless to say, the set of admissible pairs is in bijection with the set of 1234-avoiding permutations.

In the case when the two paths P and Q are irreducible, if the pair (P, Q) is admissible, then the peaks of the two paths have different x and y coordinates. We observe that this is not a sufficient condition. For example, consider the pair $P = UUUDDUUDDD$ and $Q = UUUUDUDDDD$. The unique permutation $\sigma = 3\ 2\ 1\ 5\ 4$ having LTR-minima and RTL-maxima at the positions prescribed by P and Q has an extra LTR-minimum at position 2. Hence, (P, Q) is not admissible.

We want to show that the operator L' on Dyck paths allows us to characterize the set of admissible pairs. We begin with a preliminary result concerning the pairs of Dyck paths corresponding to 123-avoiding permutations:

Theorem 6. *For every* $\sigma \in S_n(123)$, *one has:*

$$\mu(\sigma) = L'(\lambda(\sigma)). \tag{14}$$

Proof. Proposition 4, together with the definition of the map L', allows us to restrict our attention to the right-connected case.

Recall (see [12]) that a permutation σ avoids 123 if and only if the set $v_{\min}(\sigma) \cup v_{\max}(\sigma) = [n]$. It is simple to check that, if σ is right-connected, the sets of LTR minima and RTL maxima are disjoint.

Consider now a permutation σ with LTR minima $m_1, \ldots, m_{k-1}, m_k = 1$ and RTL maxima $M_1, \ldots, M_{h-1}, M_h = n$. Denote by (A, B) the ascent-descent code of the path $P = \lambda(\sigma)$ and by (A^*, B^*) the ascent-descent code of the path $\mu(\sigma)$.

As noted before, the ascent code A' of $L'(P)$ is obtained by computing the integers $\overline{A}_i = A_i - 1$ and then considering the set $\widehat{A} = [n-2] \setminus \{\overline{A}_1, \ldots, \overline{A}_{k-1}\}$, which can be written as

$$\begin{aligned}\widehat{A} = \{&n-(n-1), n-(n-2), \ldots, n-2\} \\ &\setminus \{n-m_1, \ldots, n-m_{k-1}\}.\end{aligned} \tag{15}$$

Since $\{m_1, \ldots, m_{k-1}\} \cup \{M_1, \ldots, M_{h-1}\} = \{2, 3, \ldots, n-1\}$, we have

$$\widehat{A} = \{n - M_1, \ldots, n - M_{h-1}\}. \tag{16}$$

Hence, $A' = A^*$.

Similarly, the descent code B' of $L'(P)$ is obtained by considering the set

$$\widehat{B} = [n-2] \setminus \{B_1, \ldots, B_{k-1}\} = [n-2] \setminus \{p_2 - 1, \ldots, p_k - 1\}. \tag{17}$$

Since $\{p_1, \ldots, p_{k-1}\} \cup \{P_1, \ldots, P_{h-1}\} = \{2, 3, \ldots, n-1\}$, we have

$$\widehat{B} = \{P_2 - 1, \ldots, P_{h-1} - 1\}. \tag{18}$$

Hence, $B' = B^*$. □

For example, the 123-avoiding permutation $\sigma = 8\ 5\ 9\ 7\ 6\ 2\ 4\ 3\ 1$ corresponds to the pair of Dyck paths $(P, L'(P))$ in Figure 3.

We are now in position to state our main result.

Theorem 7. *A pair* (P, Q) *is admissible if and only if* $P \geq L'(Q)$ *and* $Q \geq L'(P)$.

Proof. Consider a permutation $\sigma \in S_n(1234)$ and let σ' be the unique permutation in $S_n(123)$ with the same LTR minima as σ, at the same positions. Obviously, $\sigma' \leq_\mu \sigma$, since in σ' every element that is not a LTR minimum is a RTL maximum (see Proposition 5). Recalling that $\mu(\sigma') = L'(\lambda(\sigma)) = L'(P)$, we get the first inequality. The other inequality follows from the fact that the pair (P, Q) is admissible whenever the pair (Q, P) is admissible.

Consider now a pair of Dyck paths (P, Q) such that $P \geq L'(Q)$ and $Q \geq L'(P)$. Proposition 4 allows us to restrict to the case P, Q irreducible. Denote by σ and τ the permutations in $S_n(123)$ corresponding via ν to the pairs $(P, L'(P))$ and $(L'(Q), Q)$, respectively. Since $P \geq L'(Q)$ and $Q \geq L'(P)$, we have $\tau \leq_\lambda \sigma$ and $\sigma \leq_\mu \tau$.

We define a permutation $\alpha \in S_n$ as follows:

(i) $\alpha(x) = \sigma(x)$ if $x \in p_{\min}(\sigma)$;

(ii) $\alpha(x) = \tau(x)$ if $x \in p_{\max}(\tau)$;

(iii) if $x \notin p_{\min}(\sigma) \cup p_{\max}(\tau)$, we have $x \in p_{\max}(\sigma) \setminus p_{\max}(\tau) = p_{\min}(\tau) \setminus p_{\min}(\sigma) = \{p_{j_1}, \ldots, p_{j_r}\}$, written in increasing order. Set

$$\alpha\left(p_{j_k}\right) = m_{i_k}, \tag{19}$$

where $m_{i_1}, m_{i_2}, \ldots, m_{i_r}$ are the elements in $v_{\min}(\tau) \setminus v_{\min}(\sigma) = v_{\max}(\sigma) \setminus v_{\max}(\tau)$, written in decreasing order.

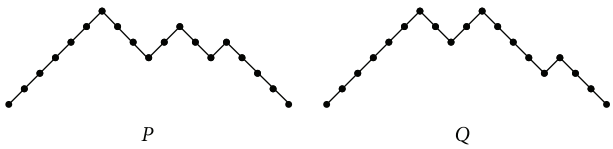


Figure 8

The permutation α is obtained as the concatenation of three decreasing sequences. Hence, α avoids 1234. We have to prove that $v_{\min}(\alpha) = v_{\min}(\sigma)$ and $v_{\max}(\alpha) = v_{\max}(\tau)$.

It is immediate that $v_{\min}(\sigma) \subseteq v_{\min}(\alpha)$. In order to prove that $v_{\min}(\sigma) = v_{\min}(\alpha)$ it remains to show that the values $m_{i_1}, m_{i_2}, \ldots, m_{i_r}$ are not LTR minima of α.

In fact, for every k, consider $\alpha(p_{j_k}) = m_{i_k} = \tau(p_{i_k})$. Consider the sets $A = \{p_1, p_2, \ldots, p_{i_k}\}$, $B = \{m_1, m_2, \ldots, m_{i_k}\}$, and their subsets $A' = \{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}$ and $B' = \{m_{i_1}, m_{i_2}, \ldots, m_{i_k}\}$. The k elements in B' do not belong to $v_{\min}(\sigma)$ (and hence, the $i_k - k$ elements in $B \setminus B'$ are the largest elements in $v_{\min}(\sigma)$). Proposition 5 ensures that each of them occupies in α a position that is strictly greater than the position occupied in τ. This implies that $p_{j_k} < p_{i_k}$ and that at most $k-1$ elements in B' occupy in τ a position that belongs to A. Hence, in α, at least $i_k - k + 1$ positions in A are occupied by entries belonging to $v_{\min}(\sigma)$. This implies that there is in α a position preceding p_{j_k} occupied by a value less than m_{i_k}. Hence, m_{i_k} is not a LTR minimum of α.

Analogous arguments can be used to prove that $v_{\max}(\alpha) = v_{\max}(\tau)$. Hence, $\nu(\alpha) = (P, Q)$, as desired. □

For example, consider the pair of Dyck paths in Figure 8.

It can be checked that $P \geq L'(Q)$ and $Q \geq L'(P)$. The permutations $\sigma = \nu^{-1}((P, L'(P)))$ and $\tau = \nu^{-1}((L'(Q), Q))$ are as follows:

$$\begin{aligned} \sigma &= 4\ 9\ 8\ 2\ 7\ 1\ 6\ 5\ 3, \\ \tau &= 7\ 5\ 9\ 4\ 3\ 2\ 8\ 1\ 6. \end{aligned} \tag{20}$$

We have $v_{\min}(\sigma) = \{4, 2, 1\}$, $p_{\min}(\sigma) = \{1, 4, 6\}$, $v_{\min}(\tau) = \{7, 5, 4, 3, 2, 1\}$, $p_{\min}(\tau) = \{1, 2, 4, 5, 6, 8\}$, $v_{\max}(\sigma) = \{3, 5, 6, 7, 8, 9\}$, $p_{\max}(\sigma) = \{9, 8, 7, 5, 3, 2\}$, $v_{\max}(\tau) = \{6, 8, 9\}$, and $p_{\max}(\tau) = \{9, 7, 3\}$.

The permutation $\alpha = \nu^{-1}((P, Q))$ is

$$\alpha = 4\ 7\ 9\ 2\ 5\ 1\ 8\ 3\ 6. \tag{21}$$

As expected, $v_{\min}(\alpha) = v_{\min}(\sigma)$, $p_{\min}(\alpha) = p_{\min}(\sigma)$, $v_{\max}(\alpha) = v_{\max}(\tau)$, and $p_{\max}(\alpha) = p_{\max}(\tau)$.

References

[1] D. E. Knuth, *The Art of Computer Programming. Volume 3*, Addison-Wesley, Reading, Mass, USA, 1973.

[2] M. Bóna, *Combinatorics of Permutations*, Discrete Mathematics and Its Applications, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2004.

[3] A. Claesson and S. Kitaev, "Classification of bijections between 321- and 132-avoiding permutations," in *Proceedings of the 20th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC '08)*, vol. 60, p. 30, 2008.

[4] I. M. Gessel, "Symmetric functions and P-recursiveness," *Journal of Combinatorial Theory. Series A*, vol. 53, no. 2, pp. 257–285, 1990.

[5] G. Kreweras, "Sur les éventails de segments," *Cahiers du B.U.R.O*, vol. 15, pp. 3–41, 1970.

[6] D. Callan, "Bijections from Dyck paths to 321-avoiding permutations revisited," http://arxiv.org/abs/0711.2684.

[7] J.-C. Lalanne, "Une involution sur les chemins de Dyck," *European Journal of Combinatorics*, vol. 13, no. 6, pp. 477–487, 1992.

[8] J.-C. Lalanne, "Sur une involution sur les chemins de Dyck," *Theoretical Computer Science*, vol. 117, no. 1-2, pp. 203–215, 1993.

[9] E. Barcucci, A. Bernini, L. Ferrari, and M. Poneti, "A distributive lattice structure connecting Dyck paths, noncrossing partitions and 312-avoiding permutations," *Order*, vol. 22, no. 4, pp. 311–328, 2005.

[10] S. C. Billey, W. Jockusch, and R. P. Stanley, "Some combinatorial properties of Schubert polynomials," *Journal of Algebraic Combinatorics*, vol. 2, no. 4, pp. 345–374, 1993.

[11] C. Krattenthaler, "Permutations with restricted patterns and Dyck paths," *Advances in Applied Mathematics*, vol. 27, no. 2-3, pp. 510–530, 2001.

[12] R. Simion and F. W. Schmidt, "Restricted permutations," *European Journal of Combinatorics*, vol. 6, no. 4, pp. 383–406, 1985.