Research Article

On a New $I$-Convergent Double-Sequence Space

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The sequence space $BV_{\sigma}$ was introduced and studied by Mursaleen (1983). In this article we introduce the sequence space $\mathcal{2}BV_{I\sigma}$ and study some of its properties and inclusion relations.

1. Introduction and Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of all natural, real, and complex numbers, respectively. We write

$$\omega = \{ x = (x_k) : x_k \in \mathbb{C} \},$$

showing the space of all real or complex sequences.

Definition 1. A double sequence of complex numbers is defined as a function $x : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{C}$. We denote a double sequence as $(x_{ij})$ where the two subscripts run through the sequence of natural numbers independent of each other [1]. A number $a \in \mathbb{C}$ is called a double limit of a double sequence $(x_{ij})$ if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that

$$\left| (x_{ij}) - a \right| < \epsilon, \quad \forall i, j \geq N,$$

(see [2]).

Let $l_\infty$ and $c$ denote the Banach spaces of bounded and convergent sequences, respectively, with norm $\|x\|_\infty = \sup_k |x_k|$. Let $\nu$ denote the space of sequences of bounded variation; that is,

$$\nu = \left\{ x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, x_{-1} = 0 \right\},$$

(3)

where $\nu$ is a Banach space normed by

$$\|x\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}|,$$

(4)

(see [3]).

Definition 2. Let $\sigma$ be a mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional $\phi$ on $l_\infty$ is said to be an invariant mean or $\sigma$-mean if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all $k$;

(ii) $\phi(e) = 1$, where $e = \{1, 1, 1, \ldots\}$;

(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

In case $\sigma$ is the translation mapping $n \rightarrow n + 1$, a $\sigma$-mean is often called a Banach limit (see [4]), and $V_\sigma$, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences (see [5]).

If $x = (x_k)$, then $Tx = (T)x_k = (x_{\sigma(k)})$. Then it can be shown that

$$V_\sigma = \left\{ x = (x_k) : \frac{\sum_{m=1}^{\infty} t_{mk}(x)}{m} = L \text{ uniformly in } k, \right\},$$

(5)

$$L = \sigma - \lim x,$$
where \( m \geq 0, k > 0 \). Consider
\[
t_{m,k}(x) = \frac{x^k + x^{\sigma_1(k)} + \cdots + x^{\sigma_m(k)}}{m+1}, \quad t_{-1,k} = 0, \quad (6)
\]
where \( \sigma^m(k) \) denote the \( m \)th iterate of \( \sigma(k) \) at \( k \). The special case of (5) in which \( \sigma(n) = n + 1 \) was given by Lorentz [5, Theorem 1], and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on \( c \).

**Theorem 3.** A \( \sigma \)-mean extends the limit functional on \( c \) in the sense that \( \phi(x) = \lim x \) for all \( x \in c \) if and only if \( \sigma \) has no finite orbits; that is to say, if and only if, for all \( k \geq 0, j \geq 1, \) (see [3])
\[
\sigma^j(k) \neq k.
\]
Put
\[
\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x), \quad (8)
\]
assuming that \( t_{-1,k} = 0 \). A straightforward calculation shows (see [6]) that
\[
\phi_{m,k}(x) = \frac{1}{m(m+1)} \sum_{j=1}^{m} \left( x_{\sigma^j(k)} - x_{\sigma^{j-1}(k)} \right), \quad (m \geq 1),
\]
\[
\phi_{m,k}(x) = x_k, \quad (m = 0). \quad (9)
\]

For any sequence \( x, y, \) and scalar \( \lambda \), we have
\[
\phi_{m,k}(x + y) = \phi_{m,k}(x) + \phi_{m,k}(y), \quad \phi_{m,k}(\lambda x) = \lambda \phi_{m,k}(x). \quad (10)
\]

**Definition 4.** A sequence \( x \in l_\infty \) is of \( \sigma \)-bounded variation if and only if
\[
(i) \sum_{n=0}^{\infty} |\phi_{m,k}(x)| \text{ converges uniformly in } n;
(ii) \lim_{m \to \infty} \phi_{m,k}(x), \text{ which must exist, should take the same value for all } k.
\]

We denote by \( \text{BV}_\sigma \), the space of all sequences of \( \sigma \)-bounded variation (see [7]):
\[
\text{BV}_\sigma = \left\{ x \in l_\infty : \sum_{m=0}^{\infty} |\phi_{m,k}(x)| < \infty, \text{uniformly in } n \right\}. \quad (11)
\]

**Theorem 5.** \( \text{BV}_\sigma \) is a Banach space normed by
\[
\|x\| = \sup_{k \to \infty} \sum_{m=0}^{\infty} |\phi_{m,k}(x)|, \quad (12)
\]
(see [8]).

Subsequently, invariant means have been studied by Ahmad and Mursaleen [9], Mursaleen et al. [3, 6, 8, 10–14], Raimi [15], Schaefer [16], Savas and Rhoades [17], Vakeel et al. [18–20], and many others [21–23]. For the first time, \( I \)-convergence was studied by Kostyrko et al. [24]. Later on, it was studied by Salat et al. [25, 26], Tripathy and Hazarika [27], Ebadullah et al. [18–20, 28], and Vakeel et al. [1, 29].

**Definition 6** (see [30, 31]). Let \( X \) be a nonempty set. Then, a family of sets \( I \subseteq 2^X \) denoting the power set of \( X \) is said to be an ideal in \( X \) if
\[
(i) \emptyset \in I;
(ii) I \text{ is additive; that is, } A, B \in I \Rightarrow A \cup B \in I;
(iii) I \text{ is hereditary that is, } A \in I, B \subseteq A \Rightarrow B \in I;
\]
An ideal \( I \subseteq 2^X \) is called nontrivial if \( I \neq 2^X \). A non-trivial ideal \( I \subseteq 2^X \) is called admissible if \( \{x : x \in X\} \subseteq I \).

A non-trivial ideal \( I \) is maximal if there cannot exist any non-trivial ideal \( J \neq I \) containing \( I \) as a subset.

For each ideal \( I \), there is a filter \( \mathcal{L}(I) \) corresponding to \( I \). That is,
\[
\mathcal{L}(I) = \{ K \subseteq N : K^c \in I \}, \quad \text{where } K^c = N - K. \quad (13)
\]

**Definition 7** (see [24, 31, 32]). A double sequence \( (x_{ij}) \in \omega \) is said to be \( I \)-convergent to a number \( L \) if for every \( \epsilon > 0 \),
\[
\left\{ i, j \in \mathbb{N} : |x_{ij} - L| \geq \epsilon \right\} \in I. \quad (14)
\]
In this case, we write \( I \lim x_{ij} = L \).

**Definition 8** (see [2]). A double sequence \( (x_{ij}) \in \omega \) is said to be \( I \)-null if \( L = 0 \). In this case, we write
\[
I \lim x_{ij} = 0. \quad (15)
\]

**Definition 9.** A double sequence \( (x_{ij}) \in \omega \) is said to be \( I \)-cauchy if for every \( \epsilon > 0 \) there exist numbers \( m = m(\epsilon), n = n(\epsilon) \) such that
\[
\left\{ i, j \in \mathbb{N} : |x_{ij} - x_{mn}| \geq \epsilon \right\} \in I. \quad (16)
\]

**Definition 10.** A double sequence \( (x_{ij}) \in \omega \) is said to be \( I \)-bounded if there exists \( M > 0 \) such that
\[
\left\{ i, j \in \mathbb{N} : |x_{ij}| > M \right\} \in I. \quad (17)
\]

**Definition 11.** A double-sequence space \( E \) is said to be solid or normal if \( (x_{ij}) \in E \) implies \( (x_i, x_{ij}) \in E \) for all sequence of scalars \( (a_{ij}) \) with \( |a_{ij}| < 1 \) for all \( i, j \in N \).

**Definition 12** (see [24, 33]). A nonempty family of sets \( \mathcal{L}(I) \subseteq 2^X \) is said to be filter on \( X \) if and only if
\[
(i) \Phi \notin \mathcal{L}(I);
(ii) \text{ for } A, B \in \mathcal{L}(I), \text{ we have } A \cap B \in \mathcal{L}(I);
(iii) \text{ for each } A \in \mathcal{L}(I) \text{ and } A \subseteq B \text{ implies } B \in \mathcal{L}(I).
\]

**Definition 13.** Let \( X \) be a linear space. A function \( g : X \to R \) is called a paranormal, if for all \( x, y, z \in X \),
\[
(i) g(x) = 0 \text{ if } x = \theta;
(ii) g(-x) = g(x);
(iii) g(x + y) \leq g(x) + g(y);
(iv) \text{ if } (\lambda_n) \text{ is a sequence of scalars with } \lambda_n \to \lambda (n \to \infty) \text{ and } x_{\lambda_n} \in X \text{ with } x_{\lambda_n} \to a (n \to \infty), \text{ in the sense that } g(x_{\lambda_n} - a) \to 0 (n \to \infty), \text{ in the sense that } g(\lambda_n x_{\lambda_n} - \lambda a) \to 0 (n \to \infty).
\]
2. Main Results

In this paper, we introduce the sequence space

$$\mathbb{2BV}_o^I := \{ x = (x_{ij}) \in \omega : \left\{ i, j \in \mathbb{N} : \left| \phi_{mnij}(x) - L \right| \geq \epsilon \right\} \in \mathcal{I} \} \quad (18)$$

for some $L \in \mathbb{C}$.

Theorem 14. $\mathbb{2BV}_o^I$ is a linear space.

Proof. Let $(x_{ij}), (y_{ij}) \in \mathbb{2BV}_o^I$ and $\alpha, \beta$ be two scalars in $\mathbb{C}$. Then for a given $\epsilon > 0$, we have

$$\left\{ i, j \in \mathbb{N} : \left| \phi_{mnij}(x) - L_1 \right| \geq \frac{\epsilon}{2} \right\} \in \mathcal{I}, \quad \text{for some} \ L_1 \in \mathbb{C},$$

and

$$\left\{ i, j \in \mathbb{N} : \left| \phi_{mnij}(y) - L_2 \right| \geq \frac{\epsilon}{2} \right\} \in \mathcal{I}, \quad \text{for some} \ L_2 \in \mathbb{C}.$$  

(19)

Now let,

$$A_1 = \left\{ i, j \in \mathbb{N} : \left| \phi_{mnij}(x) - L_1 \right| < \frac{\epsilon}{2} \right\} \in \mathcal{I},$$

for some $L_1 \in \mathbb{C}$,

$$A_2 = \left\{ i, j \in \mathbb{N} : \left| \phi_{mnij}(y) - L_2 \right| < \frac{\epsilon}{2} \right\} \in \mathcal{I},$$

for some $L_2 \in \mathbb{C}$

(20)

be such that $A_1^c, A_2^c \in \mathcal{I}$. Now consider

$$\left| \phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2) \right|$$

$$= \phi_{mnij}(\alpha x) + \phi_{mnij}(\beta y) - \alpha L_1 - \beta L_2$$

$$= \phi_{mnij}(\alpha x) - \alpha L_1 + \phi_{mnij}(\beta y) - \beta L_2$$

$$\leq \phi_{mnij}(\alpha x) - \alpha L_1 + \left| \phi_{mnij}(\beta y) - \beta L_2 \right|$$

$$= \left| \phi_{mnij}(x) - L_1 \right| + \left| \beta \right| \left| \phi_{mnij}(y) - L_2 \right|$$

$$\leq \left| \alpha \right| \frac{\epsilon}{2} + \left| \beta \right| \frac{\epsilon}{2}$$

$$= (\left| \alpha \right| + \left| \beta \right|) \frac{\epsilon}{2}$$

$$\leq \epsilon' \quad \text{(say),}$$

(21)

this implies that the sequence space

$$A_3 = \left\{ i, j \in \mathbb{N} : \left| \phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2) \right| < \epsilon' \right\} \in \mathcal{I},$$

for some $L_1, L_2 \in \mathbb{C}$.

(22)

Hence, $(\alpha x + \beta y) \in \mathbb{2BV}_o^I$. Therefore, $\mathbb{2BV}_o^I$ is a linear space.

Theorem 15. The space $\mathbb{2BV}_o^I$ is a paranormed space, paranormed by

$$g(x_{ij}) = \sup_{ij} \phi_{mnij}(x).$$  

(23)

Proof. For $x = (x_{ij}) = 0, g(x_{ij}) = 0$ is trivial.

For $x = (x_{ij}) \neq 0, g(x_{ij}) \neq 0$, we have

(i) $g(x) = \sup_{ij} \phi_{mnij}(x) \geq 0$ for all $x \in \mathbb{2BV}_o^I$.

(ii) $g(-x) = \sup_{ij} \phi_{mnij}(-x) = \sup_{ij}(-\phi_{mnij}(x)) = -\sup_{ij} \phi_{mnij}(x) = g(x)$ for all $x \in \mathbb{2BV}_o^I$.

(iii) $g(x + y) = \sup_{ij} \phi_{mnij}(x + y) \leq \sup_{ij} \phi_{mnij}(x) + \sup_{ij} \phi_{mnij}(y) = g(x) + g(y)$.

(iv) Let $(\lambda_{ij})$ be a sequence of scalars with $\lambda_{ij} \rightarrow \lambda$ (ij $\rightarrow \infty$) and $(x) \in \mathbb{2BV}_o^I$ such that

$$\phi_{mnij}(x) \rightarrow L \quad (ij \rightarrow \infty),$$

(24)

in the sense that

$$g(\phi_{mnij}(x) - L) \rightarrow 0 \quad (ij \rightarrow \infty).$$

(25)

Therefore,

$$g \left( \lambda_{ij} \phi_{mnij}(x) - \lambda L \right) \leq g \left( \lambda_{ij} \phi_{mnij}(x) \right) - g \left( \lambda L \right)$$

$$= \lambda_{ij} g \left( \phi_{mnij}(x) \right) - \lambda g(L) \rightarrow 0 \quad \text{as} \quad (ij \rightarrow \infty).$$

(26)

Hence, $\mathbb{2BV}_o^I$ is a paranormed space.

Theorem 16. $\mathbb{2BV}_o^I$ is a closed subspace of $\mathbb{2BV}_o^I$.

Proof. Let $(x_{ij}^{(p,q)})$ be a cauchy sequence in $\mathbb{2BV}_o^I$ such that $x_{ij}^{(p,q)} \rightarrow x$. We show that $x \in \mathbb{2BV}_o^I$. Since $(x_{ij}^{(p,q)}) \in \mathbb{2BV}_o^I$, then there exists $d_{pq}$ such that

$$\left\{ i, j \in \mathbb{N} : \left| \phi_{mnij}(x_{ij}^{(p,q)}) - a_{pq} \right| \geq \epsilon \right\} \in \mathcal{I}. \quad (27)$$

We need to show that

(i) $(a_{pq})$ converges to $a$.

(ii) If $U = \{ i, j \in \mathbb{N} : |x_{ij} - a| < \epsilon \},$ then $U^c \in \mathcal{I}$. 

Since \((x_{ij}^{pq})\) is a Cauchy sequence in \(\mathcal{B}V^I\), then for a given \(\varepsilon > 0\), there exists \(k_0 \in \mathbb{N}\) such that
\[
\sup_{\tilde{q}} |\phi_{mn,ij}(x_{ij}^{pq}) - \phi_{mn,ij}(x_{ij}^{rs})| < \frac{\varepsilon}{3}, \quad \forall p, q, r, s \geq k_0.
\]
(28)

For a given \(\varepsilon > 0\), we have
\[
B_{pqrs} = \left\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x_{ij}^{pq}) - \phi_{mn,ij}(x_{ij}^{rs})| < \frac{\varepsilon}{3} \right\},
\]
\[
B_{pq} = \left\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x_{ij}^{pq}) - a_{pq}| < \frac{\varepsilon}{3} \right\},
\]
\[
B_{rs} = \left\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x_{ij}^{rs}) - a_{rs}| < \frac{\varepsilon}{3} \right\}.
\]
(29)

Then \(B_{pqrs}, B_{pq}^c,\) and \(B_{rs}^c\) are in \(\mathcal{B}V^I\). Let \(B^c = B_{pqrs}^c \cap B_{pq}^c \cap B_{rs}^c\), where \(B = \{ i, j \in \mathbb{N} : |a_{pq} - a_{rs}| < \varepsilon \}\). Then \(B^c \in 1\). If \(k_0 \in B^c\), then for each \(p, q, r, s \geq k_0\), we have
\[
\left\{ i, j \in \mathbb{N} : |a_{pq} - a_{rs}| < \varepsilon \right\}
\]
\[
\sup_{\tilde{q}} |\phi_{mn,ij}(x_{ij}^{pq}) - \phi_{mn,ij}(x_{ij}^{rs})| < \frac{\varepsilon}{3}, \quad \forall p, q, r, s \geq k_0.
\]
(30)

Then \((a_{pq})\) is a Cauchy sequence of scalars in \(C\), so there exists a scalar \(a \in C\) such that \((a_{pq}) \rightarrow a\), as \(p, q \rightarrow \infty\).

For the next step, let \(0 < \delta < 1\) be given. Then, we show that if
\[
U = \left\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x_{ij}^{pq}) - a| < \delta \right\},
\]
(31)

then \(U \in 1\). Since \(\phi_{mn,ij}(x_{ij}^{pq}) \rightarrow \phi_{mn,ij}(x)\), then there exists \(p_0, q_0 \in \mathbb{N}\) such that
\[
P = \left\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x_{ij}^{pq}) - \phi_{mn,ij}(x)| < \frac{\delta}{3} \right\},
\]
(32)

which implies that \(P^c \in 1\). The number \(p_0, q_0\) can be so chosen that together with (32), we have
\[
Q = \left\{ i, j \in \mathbb{N} : |a_{pq_{0,0}} - a| < \frac{\delta}{3} \right\},
\]
(33)

such that \(Q^c \in 1\). Since \(\phi_{mn,ij}(x_{ij}^{pq_{0,0}}) - a_{pq_{0,0}} \geq \delta\) \(\in 1\), then we have a subset \(S\) of \(\mathbb{N}\) such that \(S^c \in 1\), where
\[
S = \left\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x_{ij}^{pq_{0,0}}) - a_{pq_{0,0}}| < \frac{\delta}{3} \right\}.
\]
(34)

Let \(U^r = P^c \cap Q^c \cap S^c\), where \(U = \{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x) - a| < \delta \}\).

Therefore, for each \(i, j \in U^r\), we have
\[
\left\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x) - a| < \delta \right\}
\]
\[
\sup_{\tilde{q}} |\phi_{mn,ij}(x_{ij}^{pq}) - \phi_{mn,ij}(x_{ij}^{rs})| < \frac{\delta}{3}, \quad \forall p, q, r, s \geq k_0.
\]
(35)

Hence, the result \(\mathcal{B}V^I_a \subset \mathcal{B}V^I_{co}\) follows.

\textbf{Theorem 17.} The space \(\mathcal{B}V^I_a\) is nowhere dense subset of \(\mathcal{B}V^I_{co}\).

\textbf{Proof.} Proof of the result follows from the previous theorem.

\textbf{Theorem 18.} The space \(\mathcal{B}V^I_a\) is solid and monotone.

\textbf{Proof.} Let \((x_{ij}) \in \mathcal{B}V^I_a\) and \(x_j\), be a sequence of scalars with \(|x_{ij}| \leq 1\) for all \(i, j \in \mathbb{N}\). Then, we have
\[
\left| a_{ij} \phi_{mn,ij}(x_{ij}) \right| \leq |a_{ij}| \left| \phi_{mn,ij}(x_{ij}) \right| \leq \phi_{mn,ij}(x_{ij}), \quad \forall i, j \in \mathbb{N}.
\]
(36)

The space \(\mathcal{B}V^I_a\) is solid follows from the following inclusion relation:
\[
\left\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x_{ij})| \geq \varepsilon \right\} \supset \left\{ i, j \in \mathbb{N} : |a_{ij}\phi_{mn,ij}(x_{ij})| \geq \varepsilon \right\}.
\]
(37)

Also a sequence space is solid implies monotone. Hence, the space \(\mathcal{B}V^I_a\) is monotone.

\textbf{Theorem 19.} \(\mathcal{B}V^I_{co} \subset \mathcal{B}V^I_{a}\) and the inclusions are proper.

\textbf{Proof.} Let \(x = (x_{ij}) \in \mathcal{B}V^I_{co}\). Then, we have \(\{ i, j \in \mathbb{N} : |x_{ij}| \geq \varepsilon \} \in 1\). Since \(2\mathcal{B}V^I_a \subset \mathcal{B}V^I_{co}\), \(x = (x_{ij}) \in \mathcal{B}V^I_{co}\) implies
\[
\left\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x_{ij})| \geq \varepsilon \right\} \supset \left\{ i, j \in \mathbb{N} : |a_{ij}\phi_{mn,ij}(x_{ij})| \geq \varepsilon \right\} \subset \mathcal{B}V^I_a.
\]
(38)

Now let,
\[
A_1 = \left\{ i, j \in \mathbb{N} : |x_{ij}| < \varepsilon \right\},
\]
(39)

\[
A_2 = \left\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x_{ij})| < \varepsilon \right\}
\]
be such that \(A_1^c, A_2^c \in 1\). As \(\mathcal{B}V^I_{co} \subset \mathcal{B}V^I_a\), taking supremum over \(i, j\) we get \(A_1^c \subset A_2^c\). Hence, \(2\mathcal{B}V^I_{co} \subset 2\mathcal{B}V^I_a\).

Next we show that the inclusion is proper

(i) First for \(2\mathcal{B}V^I_{co} \subset 2\mathcal{B}V^I_a\). Consider \(x \in 2\mathcal{B}V^I_a\), then by the definition
\[
\left\{ i, j \in \mathbb{N} : |\phi_{mn,ij}(x) - L| \geq \varepsilon \right\} \in 1
\]
(40)

for some \(L \in C\),
we have
\[ \phi_{mn,ij}(x) = t_{mn,ij}(x) - t_{(m-1)(n-1),ij}(x), \] (41)
where
\[ t_{mn,ij}(x) = \frac{x_{ij} + x_{\sigma(ij)} + \cdots + x_{\sigma^{mn}(ij)}}{mn}. \] (42)

Therefore,
\[
\begin{align*}
t_{mn,ij}(x) - t_{(m-1)(n-1),ij}(x) &= \frac{x_{ij} + x_{\sigma(ij)} + \cdots + x_{\sigma^{mn}(ij)}}{mn} \\
&\quad - \frac{x_{(i+1)(j+1)} + \cdots + x_{(i+m)(j+n)}}{mn(m-1)(n-1)} \\
&= \frac{(m-1)(n-1)(x_{ij} + x_{\sigma(ij)} + \cdots + x_{\sigma^{mn}(ij)})}{mn(m-1)(n-1)} \\
&\quad - \frac{mn(x_{ij} + x_{\sigma(ij)} + \cdots + x_{\sigma^{mn}(ij)})}{mn(m-1)(n-1)} \\
&= \frac{(m-1)(n-1)x_{ij} + x_{\sigma(ij)} + \cdots + x_{\sigma^{mn}(ij)}}{mn(m-1)(n-1)},
\end{align*}
\] (43)

On solving, we get
\[
\phi_{mn,ij}(x) = \frac{mnx_{\sigma^{mn}(ij)}}{mn(m-1)(n-1)} (1 - m - n) + \frac{x_{ij} + x_{\sigma(ij)} + \cdots + x_{\sigma^{mn}(ij)}}{mn(m-1)(n-1)}.
\] (44)

As \( \sigma \) is a translation map, that is, \( \sigma(n) = n + 1 \), we have
\[
\phi_{mn,ij}(x) = \frac{mnx_{i+m}(j+n)}{mn(m-1)(n-1)} (1 - m - n) + \frac{x_{ij} + x_{i+1}(j+1) + \cdots + x_{i+m(n)(j+n)}}{mn(m-1)(n-1)}.
\] (45)

Taking \( \lim i, j \to \infty \), we have
\[
\lim_{(i,j)\to\infty} \phi_{mn,ij}(x) = \lim_{(i,j)\to\infty} \left[ \frac{mnx_{i+m}(j+n) + (1 - m - n)}{mn(m-1)(n-1)} \right] (x_{ij} + x_{i+1}(j+1) + \cdots + x_{i+m(n)(j+n)}) \times (mn(m-1)(n-1))^{-1},
\] (46)

Since \( m, n, L \neq 0 \), therefore \( \lim_{i, j \to \infty} \phi_{mn,ij}(x) \neq 0 \) which implies that \( x \notin (2c^l) \). Hence, we get that the inclusion is proper.

(ii) Second for \( 2BV_{\sigma} \subset 2_{\infty} \).

The result of this part follows from the proof of Theorem 18.

Theorem 20. \( 2c^l \subset 2BV_{\sigma} \subset 2_{\infty} \) and the inclusions are proper.

Proof. Let \( x = (x_{ij}) \in 2c^l \). Then, we have
\[
\{i, j \in N : |x_{ij} - L| \geq \varepsilon \} \in I.
\] (47)

Since \( c \subset BV_{\sigma} \subset \infty \), which implies \( x = (x_{ij}) \in 2BV_{\sigma} \) implies
\[
\{i, j \in N : |\phi_{mn,ij}(x) - L| \geq \varepsilon \} \in I.
\] (48)

Now let,
\[
B_1 = \{i, j \in N : |\phi_{ij} - L| < \varepsilon \}, \quad B_2 = \{i, j \in N : |\phi_{mn,ij}(x) - L| < \varepsilon \}
\] (49)

be such that \( B_1, B_2 \in I \). As
\[
2_{\infty} = \left\{ x = (x_{ij}) : \sup_{ij} |x_{ij}| < \infty \right\},
\] (50)

taking \( \lim \sup \) over \( i, j \), we get \( B_1^c \subset B_2^c \). Hence, \( 2c^l \subset 2BV_{\sigma} \subset 2_{\infty} \).

Next, we show that the inclusion is proper

(i) First for \( 2c^l \subset 2BV_{\sigma} \). We show that \( 2c^l \not\subset 2BV_{\sigma} \).

Let \( x = (x_{ij}) \in 2BV_{\sigma} \), then by the definition
\[
2BV_{\sigma} := \left\{ x = (x_{ij}) \in \omega : \left\{ i, j \in N : |\phi_{mn,ij}(x) - L| \geq \varepsilon \right\} \in I \right\}
\] (51)

for some \( L \in C \).
We have,

\[ |\phi_{m,n,i,j}(x) - L| \geq \epsilon. \]

We say that the \( I \)-\( \lim \phi_{m,n,i,j}(x) = L \).

Now considering the case when \( \|t_{m,n,i,j}(x) - L\| < \epsilon \), then

\[ |t_{m,n,i,j}(x) - t_{(m-1)(n-1),i,j}(x) - L| < \epsilon, \]

(52)

when \( m,n = 0 \), then we have \( \phi_{m,n,i,j}(x) = t_{i,j}(x) = x_{i,j} \).

Therefore we get,

\[ x_{i,j} - L < \epsilon \quad \forall i, j \in \mathbb{N}. \]  

(53)

Hence, \( x \notin J^{c} = \{i,j \in \mathbb{N} : |x_{i,j} - L| \geq \epsilon\} \in I \). Hence, the inclusion is proper.

(ii) Second for \( J^{c} \subset J^{c} \).

The result follows from the proof of Theorem 18. \( \Box \)

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References


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