5. Introduction

Throughout only finite groups are considered. Let \( \pi(G) \) stand for the set of all prime divisors of the order of a group \( G \). \( \mathcal{U} \) denotes the class of all supersolvable groups. \( H \ char G \) means \( H \) is a characteristic subgroup of \( G \). We use conventional notions and notation, as in Robinson [1].

Let \( \mathcal{F} \) be a class of groups. \( \mathcal{F} \) is called a formation provided that (1) if \( G \in \mathcal{F} \) and \( H \triangleleft G \), then \( G/H \in \mathcal{F} \) and (2) if \( G/M \) and \( G/N \) are in \( \mathcal{F} \), then \( G/M \cap N \) is in \( \mathcal{F} \) for all normal subgroups \( M, N \) of \( G \). A formation \( \mathcal{F} \) is said to be saturated if \( G/\phi(G) \in \mathcal{F} \) implies that \( G \in \mathcal{F} \).

Two subgroups \( H \) and \( K \) of a group \( G \) are said to be permutably if \( HK = KH \). \( H \) is said to be \( S \)-quasinormal in \( G \) if \( H \) permutes with every Sylow subgroup of \( G \), that is, \( HP = PH \) for any Sylow subgroup \( P \) of \( G \). This concept was introduced by Kegel in [2] and has been studied widely by many authors, such as [3,4]. An interesting question in the theory of finite groups is to determine the influence of the embedding properties of members of some distinguished families of subgroups of a group on the structure of the group. Recently, Ballester-Bolinches and Pedraza-Aguilera [5] generalized \( S \)-quasinormal subgroups to \( S \)-quasinormally embedded subgroups. \( H \) is said to be \( S \)-quasinormally embedded in \( G \) provided every Sylow subgroup of \( H \) is a Sylow subgroup of some \( S \)-quasinormal subgroup of \( G \). By applying this concept, Ballester-Bolinches and Pedraza-Aguilera got new criteria for supersolvability of groups.

A subgroup \( H \) of a group \( G \) is called to be complemented in \( G \) if \( G \) has a subgroup \( K \) such that \( G = HK \) and \( H \cap K \leq H_{\text{sec}} \), where \( H_{\text{sec}} \) denotes the subgroup of \( H \) generated by all those subgroups of \( H \) which are \( S \)-quasinormally embedded in \( G \). In this paper, we characterize \( p \)-nilpotency and supersolvability of \( G \) under the assumption that all maximal subgroups of \( P \) are \( SE \)-supplemented in \( N_{p}(P) \).
Clearly, normal subgroups, S-quasinormal subgroups, S-quasinormally embedded subgroups, and weakly S-supplemented subgroups are all SE-supplemented subgroups. But the converse does not hold in general (see [11]). Based on the observation of the above concepts, we note that supplementation of some families of subgroups of a group has a strong influence on its structure.

On the other hand, the normalizer of a Sylow subgroup of a group takes an important role in studying the structure of a group. Let $P$ be a Sylow subgroup of a group $G$; it is natural to consider the structure of $G$ if some properties of the normalizer $N_G(P)$ of $P$ are known. A classical result in this orientation is attributed to Burnside's Theorem [1, Theorem 10.1.8]. Later, Hall [12] extended it as follows: if each $p'$-element of $N_G(P)$ does commute with all elements of $P$ and if also the class size of $P$ is less than $p$, then $P$ is $p'$-nilpotent. In short, it is of significance to research into the structure of a group (see [13, Lemma 2.1]).

Lemma 2 (see [13, Lemma 2.1]). Let $H$ and $K$ be subgroups of a group $G$.

1. If $H$ is $S$-quasinormal in $G$ and $H \leq M \leq G$, then $H$ is $S$-quasinormal in $M$.
2. Let $N \triangleleft G$ and $H$ be $S$-quasinormal in $G$. Then $HN$ is $S$-quasinormal in $G$ and $HN/N$ is $S$-quasinormal in $G/N$.
3. If $H$ is $S$-quasinormal in $G$, then $H$ is subnormal in $G$.
4. If both $H$ and $K$ are $S$-quasinormal subgroups of $G$, then $H \cap K$ and $\langle H, K \rangle$ are $S$-quasinormal subgroups of $G$.

Lemma 3 (see [10, Lemma 2.6]). Suppose that $N$ is a normal subgroup of $G$ and $H \leq K \leq G$. Then

1. $H_{seG} \trianglelefteq H$;
2. $H_{seG} \leq H_{seK}$;
3. $H_{seG}N/N \leq (HN/N)_{se(G/N)}$;
4. If $[N, H] = 1$, then $H_{seG}N/N = (HN/N)_{se(G/N)}$.

Lemma 4 (see [10, Lemma 2.7]). Let $H$ be a subgroup of a group $G$.

1. If $H$ is SE-supplemented in $G$ and $H \leq M \leq G$, then $H$ is SE-supplemented in $M$.
2. Let $N \triangleleft G$ and $H \leq N$. If $H$ is SE-supplemented in $G$, then $H/N$ is SE-supplemented in $G/N$.
3. Let $\pi$ be a set of primes, $H$ a $\pi$-subgroup of $G$, and $N$ a normal $\pi'$-subgroup of $G$. If $H$ is SE-supplemented in $G$, then $HN/N$ is SE-supplemented in $G/N$.

Lemma 5 (see [14]). Let $G$ be a group.

1. If $P$ is an $S$-quasinormal $p$-subgroup of $G$ for some prime $p$, then $N_G(P) \geq O^p(G)$.
2. Suppose that $H$ is a $p$-subgroup of $G$ contained in $O_p(G)$. If $H$ is $S$-quasinormally embedded in $G$, then $H$ is $S$-quasinormal in $G$.

Lemma 6. Let $H$ be a subgroup of a group $G$. If $H$ is SE-supplemented in $G$ and $H \leq O_p(G)$, then $H$ is weakly SE-supplemented in $G$.

Proof. In fact, $H_{seG} = \langle H_1, H_2, \ldots, H_s \rangle$, where $H_i$ ($i = 1, 2, \ldots, s$) is $S$-quasinormally embedded in $G$. Since $H \leq O_p(G)$, we have that $H_i$ is $S$-quasinormal in $G$ by Lemma 5(2). Thus, $H_{seG} \leq H_{seG}$, where $H_{seG}$ is the largest $S$-quasinormal subgroup of $G$ contained in $H$. Consequently, $H$ is weakly SE-supplemented in $G$.

Lemma 7 (see [15, Lemma 2.1]). Let $G$ be a group. If $A$ is a normal $pi$-subgroup of $G$, then $A \leq O_p(G)$.

From Lemma 6, the following Lemma is a direct corollary of Lemma 3.1 in [7].

Lemma 8. Suppose $G = PQ$, where $P$ is a normal Sylow $p$-subgroup and $Q$ a Sylow $q$-subgroup of $G$. If $[G, p – 1] = 1$ and if also every maximal subgroup of $P$ is SE-supplemented in $G$, then $P$ is $p'$-nilpotent.

3. Main Results

Theorem 9. Let $G$ be a group and assume $p$ is a prime dividing the order of $G$ with $[G, p – 1] = 1$. If there exists a Sylow $p$-subgroup $P$ of $G$ such that every maximal subgroup of $P$ is SE-supplemented in $N_G(P)$ and if also $P'$ is $S$-quasinormal in $G$, then $G$ is $p'$-nilpotent.

Proof. Suppose that the result is not true and let $G$ be a counterexample of minimal order. We will derive a contradiction in several steps.

1. $O_p(G) \geq P' \neq 1$. Let $Q \in Syl_p(N_G(P))$, where $q$ is a prime number dividing $[N_G(P)]$ and different from $p$. We can see that all maximal subgroups of $P$ are SE-supplemented in $PQ$ by Lemma 4. Then $PQ$ meets the hypothesis of Lemma 8. It follows that $PQ$ is $p'$-nilpotent and thus $Q \leq C_G(P)$. We know that all $p'$-elements of $N_G(P)$ are contained in $C_G(P)$. If $P$ is abelian, then $N_G(P) = C_G(P)$, which yields that $G$ is $p'$-nilpotent from Burnside's theorem [1, Theorem 10.1.8], which is a contradiction. Thus we may assume that $P' \neq 1$. Since $P'$ is $S$-quasinormal in $G$, thus $P'$ is subnormal in $G$ by Lemma 2. It follows from Lemma 7 that $1 \neq P' \leq O_p(G)$.

2. For any Normal Subgroup $N$ of $G$ Contained in $P$, $G/N$ is $p'$-Nilpotent and $G$ is Solvable. It is clear that $[G/N, p – 1] = 1$. For any maximal subgroup $P_i/N$ of $P/N$, $P_i$ is a maximal subgroup of $P$. From this hypothesis, $P_i$ is SE-supplemented in $N_G(P)$ and $P'$ is $S$-quasinormal in $G$. It follows by Lemma 4
that $P_1/N$ is SE-supplemented in $N_{G}(P)/N = N_{G}(P/N)$, and $(P/N)' = P'/N$ is S-quasinormal in $G/N$ by Lemma 2. As a result, $G/N$ meets the hypothesis of the theorem. The choice of $G$ yields that $G/N$ is $p$-nilpotent. Let $M/N$ be a normal $p$-complement of $G/N$. If $p = 2$, then $M/N$ is a group of odd order. It follows from the Feit-Thompson theorem which asserts that every group of odd order is solvable, so is $M$. We note that $G/M$ is a 2-group, and so $G$ is solvable. If $p \neq 2$, then $G$ is a group of odd order by $(|G|, p - 1) = 1$. Similarly, it deduces that $G$ is solvable, too.

$3 | G | = p^a q^b$ for Some Prime $q \neq p$. Since $G$ is solvable, there exists a Sylow system $P_1 = \{P_1, P_2, \ldots, P_s\}$ of $G$ with $G_i = P_iP_j$ for $2 \leq i \leq s$. By Lemmas 2 and 4, the hypothesis still holds for every $G_i$. If $|\pi(G_i)| > 2$, then $G_i < G$ and thus $G_i$ is $p$-nilpotent by the choice of $G$, whence $P_i \leq G_i$, thereby meaning that $P$ normalizes $P_i$ for each $2 \leq i \leq s$. Therefore $G$ is $p$-nilpotent and $K = P_1P_2 \cdots P_s$ is a normal $p$-complement of $G$, which is a contradiction. Now, we may assume that $|G| = p^a q^b$.

(4) The Final Contradiction. Let $N$ be a minimal normal subgroup of $G$. Because $P$ is $S$-quasinormal in $G$ and $N \leq G$, $P^2N/N$ is $S$-quasinormal in $G/N$ by Lemma 2. Now we consider the quotient group $G/N$. If $N$ is a $q$-group, then $PN/N \in Syl_q(G/N)$, and for any maximal subgroup $M/N$ of $PN/N$, we have $M = P_1N$, where $P_1$ is a maximal subgroup of $P$. From this hypothesis, $P_1$ is SE-supplemented in $N_{G}(P)$. Then there is a subgroup $T$ of $G$ such that $N_{G}(P) = PT$ and $P_1 \cap T \leq (P_1)_{x \in N, t \in T}$. Since $P \in Syl_p(G)$, we know that

$$N_{G/N}(PN/N) = N_{G}(P/N) = (P/N)(TN/N).$$

As $|P_1|, |N| = 1$,

$$|P_1 \cap TN| = \frac{|P_1| \cdot |TN|}{|T|} = \frac{|P_1| \cdot |T|}{|G|} = \frac{|P_1| \cdot |T|}{|G|} = |P_1 \cap T|.$$

This means that $P_1 \cap TN = P_1 \cap T$, and thus we have

$$\left(\frac{P_1N}{N}\right) \cap \left(\frac{TN}{N}\right) = \left(\frac{P_1N \cap TN}{N}\right) = \left(\frac{P_1 \cap T}{N}\right).$$

By Lemma 3, it follows that $(P_1)_{x \in N, t \in T}N/N = (P_1N/N)_{x \in N, t \in T}$, and thus $G/N$ meets the hypothesis of our theorem. The choice of $G$ implies that $G/N$ is $p$-nilpotent, and so is $G$, contradicting the fact that $G$ is a counterexample of minimal order. Consequently, we may assume that $N$ is a $p$-group and thus $N \leq P$. It follows by (2) that $G/N$ is $p$-nilpotent. Hence we may assume that $N$ is the unique minimal normal subgroup of $G$ and $N \notin \Phi(G)$. These mean that $\Phi(G) = 1$ and $O_P(G) = F(G) = N$.

As $P'$ is $S$-quasinormal in $G$, we know that $N_{G}(P') \geq O_P(G)$ by Lemma 5. Since $P$ normalizes $P'$, we get that $P'' \subseteq G$, then $P'' = O_P(G) = N$ since $N$ is the unique minimal normal subgroup of $G$. It follows from (2) that $G/O_P(G)$ is $p$-nilpotent; thus $O_{P}(G)/Q \subseteq G$, where $Q \subseteq Syl_P(G)$. Since $O_{P}(G)/Q \cap P = O_P(G) = P' \leq \Phi(P)$, $O_{P}(G)/Q$ is $p$-nilpotent by Tate's Theorem [16, Theorem 4.4.7]. Thus $Q$ char $O_{P}(G)/Q \subseteq G$ which yields that $Q \subseteq G$; that is, $G$ is $p$-nilpotent, a contradiction. The final contradiction completes the proof.

\[\square\]

Remark 10. In Theorem 9, the condition that $P'$ is $S$-quasinormal in $G$ cannot be removed.

Example II. Let $G = PSL_2(q)$, where $q > 1$ and $q = \pm 1 (\text{mod } 8)$. Let $P$ be a Sylow $2$-subgroup of $G$. By [16, II, Theorem 8.27], we know that $P$ is self-normalizing in $PSL_2(q)$. Clearly, every maximal subgroup of $P$ is normal in $N_{G}(P) = P$, and thus they are all SE-supplemented in $N_{G}(P)$. However, $G$ is not $2$-nilpotent.

Remark 12. The hypothesis in Theorem 9 that $P'$ is $S$-quasinormal in $G$ still cannot be left out when $G$ is solvable and $p$ an odd prime number.

Example 13. Let $H$ be the elementary abelian $3$-group of order 27. Then $H = Z_3 \times Z_3 \times Z_3$. It is easy to see that a subgroup of Aut($H$) is isomorphic to $Z_{13} \times Z_3$. Now assume that $G = (Z_3 \times Z_3 \times Z_3) \times (Z_3 \times Z_3 \times Z_3)$. Let $P_3$ be a Sylow $3$-subgroup of $G$. It is clear that $N_{G}(P_3) = P_3$ and thus every maximal subgroup of $P_3$ is SE-supplemented in $N_{G}(P_3)$, however, $G$ is not $3$-nilpotent.

Theorem 14. Let $G$ be a group, $H$ a normal subgroup of $G$ such that $G/H$ is $p$-nilpotent, and $P$ a Sylow $p$-subgroup of $H$, where $p$ is a prime dividing the order of $G$ with $(|G|, p - 1) = 1$. If there exists a Sylow $p$-subgroup $P$ of $H$ such that every maximal subgroup of $P$ is SE-supplemented in $N_{H}(P)$ and such that $P'$ is $S$-quasinormal in $G$, then $G$ is $p$-nilpotent.

Proof. Assume that the result is not true and let $G$ with subgroup $H$ be a minimal counterexample to the theorem in respect to $|G| + |H|$. By Lemmas 2 and 4, we can see that every maximal subgroup of $P$ is SE-supplemented in $N_{H}(P)$ and $P'$ is $S$-quasinormal in $H$. It follows that $H$ is $p$-nilpotent by Theorem 9. Let $M$ be the normal $p$-complement of $H$; then $M \leq G$. If $M \neq 1$, we consider the factor group $G/M$ with subgroup $H/M$. It is clear that $H = PM$ and $(|P|, |M|) = 1$. With a similar argument as in step (4) of Theorem 9, we obtain that the hypothesis still holds for $G/M$ with subgroup $H/M$. Thereby the choice of $G$ implies that $G/M$ is $p$-nilpotent. Consequently, $G$ is $p$-nilpotent, which is a contradiction. Now we may suppose that $M = 1$; that is, $H = P$ is a $p$-group. Let $T/P$ be the normal $p$-complement of $G/P$; this makes sense as $G/P = G/H$ is $p$-nilpotent. It is easy to see that every maximal subgroup of $P$ is SE-supplemented in $N_{P}(P)$ and $P'$ is $S$-quasinormal in $T$, whence $T$ is $p$-nilpotent by Theorem 9. As a result, $T_{p'}$ char $T$ is $G$ implying that $T_{p'}$ is also a normal Hall $p'$-subgroup of $G$; that is, $G$ is
$p$-nilpotent, a contradiction. The proof of the theorem is now complete.

The following result now follows directly from Theorem 14.

**Corollary 15.** Let $G$ be a group and assume $p$ is a prime dividing the order of $G$ with $(|G|, p - 1) = 1$. Suppose that $H$ is a normal subgroup of $G$ such that $G/H$ is $p$-nilpotent. If there exists a Sylow $p$-subgroup $P$ of $H$ such that every maximal subgroup of $P$ is $S$-quasinormal in $N_G(P)$ and such that $P$ is $S$-quasinormal in $G$, then $G$ is $p$-nilpotent.

The following two theorems study the supersolvability of a group.

**Theorem 16.** Let $G$ be a group. Suppose that for any prime $p$ dividing $|G|$, there exists a Sylow $p$-subgroup $P$ of $G$ such that every maximal subgroup of $P$ is $SE$-supplemented in $N_G(P)$ and such that $P$ is $S$-quasinormal in $G$. Then $G$ is supersolvable.

**Proof.** Let $G$ be a minimal counterexample. According to Theorem 9, it is easy to see that $G$ is $p$-nilpotent for the minimal prime $p$ dividing $|G|$. Let $K$ be the normal $p$-complement of $G$. Then by Lemmas 2 and 4, it is clear that $K$ meets the hypothesis of the theorem, and then $K$ is supersolvable by the choice of $G$. Let $q = \max n(K)$ and $Q \leq Syl_q(K)$. Then $Q \in Syl_q(G)$ and $Q \leq G$. Let $N$ be a minimal normal subgroup of $G$ contained in $Q$. Thus $N$ is an elementary abelian $q$-subgroup of $G$. Now we consider the factor group $G/N$. By Lemmas 2 and 4, we know that every maximal subgroup $Q_1/N$ of $Q/N$ is $SE$-supplemented in $N_G(N)/N = N_G(Q)/N = G/N$ and $(Q/N)^{q} = Q'/N$ is $S$-quasinormal in $G/N$. Let $R/N \in Syl_q(G/N)$ for any $r \neq q$. Then for any maximal subgroup $T/N$ of $RN/N$, we know that $T = R_1N$, where $R_1$ is a maximal subgroup of $R$. From the hypothesis, $R_1$ is $SE$-supplemented in $N_G(R)$. Then there is a subgroup $K$ of $N_G(R)$ such that $N_G(R) = R_1K$ and $R_1 \cap K \leq (R_1)_{seN(G,R)}$. Because $R \in Syl_q(G)$, we know that

$$N_{G/N}(R/N) N = N_G(Q) N = (R_1N)(KN) / N$$

Since $[R_1, N] = 1$,

$$[R_1 \cap KN] = [R_1, KN] \cdot [R_1, K] / N_G(R)N = [R_1, K] / N_G(R)N$$

This means that $R_1 \cap KN = R_1 \cap K$; thus

$$\left( \frac{R_1N}{N} \right) \cap \left( \frac{KN}{N} \right) = \left( \frac{R_1N \cap KN}{N} \right) = \left( \frac{R_1 \cap KN}{N} \right) N / N$$

Note that $(R_1)_{seN(G,Q)}^{R/Q} = (R_1N/N)_{seN(G,N/RN/N)}$ by Lemma 3. Consequently, $G/N$ satisfies the hypothesis of the theorem. It follows that the choice of $G/N$ implies that $G$ is supersolvable. As the class of all supersolvable subgroups is a saturated formation, we may assume that $N$ is the unique minimal normal subgroup of $G$ contained in $Q$ with $N \leq \Phi(G)$, so there exists a maximal subgroup $M$ of $G$ such that $G = MN$ and $M \cap N = 1$. Since $Q = O_2(G) \leq F(G) \leq C_G(N)$, $N$ and $M$ normalize $Q \cap M$, whence $Q \cap M \leq G$. Thus $Q \cap M = 1$ or $N \leq Q \cap M$. If the latter case holds, then $N \leq M$; namely, $G = NM = M$, a contradiction. Hence $Q \cap M = 1$ and $Q = N$ is a minimal normal subgroup of $G$.

Let $Q_1$ be a maximal subgroup of $Q$ and $T$ be a supplement of $Q_1$ in $G$; then $Q_1T = G$ and so $Q = Q \cap Q_1T = Q_1(Q \cap T)$. This means that $Q \cap T = 1$. However, since $Q \cap N = T$ is normal in $G$ and $Q$ is a minimal normal subgroup of $G$, we have $Q \cap T = Q$. Hence $T = G$ is the unique supplement of $Q_1$ in $G$. Then $Q_1 \cap T \leq (Q_1)_{seN(G,Q)}$ by Lemma 6; namely, $(Q_1)_{seN(G,Q)} = S$-quasinormal in $G$. Thus $N_{G}(Q_1) \geq O^p(G)$ by Lemma 5. Since $Q_1 \lneq Q$, we have $Q_1 \lneq O^p(G) = G$. It follows from the minimal normality of $Q$ in $G$ that $Q_1 = 1$ and so $Q = q$. Since $G/Q$ is supersolvable, $G$ is supersolvable, a contradiction. The proof of the theorem is now complete.

**Theorem 17.** Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{H}$, and assume that $G$ is a group with a normal subgroup $H$ satisfying $G/H \in \mathcal{F}$. Suppose that for any prime $p$ dividing $|H|$, there exists a Sylow $p$-subgroup $P$ of $H$ such that every maximal subgroup of $P$ is $SE$-supplemented in $N_G(P)$ and such that $P$ is $S$-quasinormal in $G$; then $G \in \mathcal{F}$.

**Proof.** Suppose that the result is not true and let $G$ with subgroup $H$ be a minimal counterexample to the theorem in respect to $|G| + |H|$. By Lemmas 2 and 4, it is clear that for any prime $p$ dividing $|H|$, there exists a Sylow $p$-subgroup $P$ of $H$ such that every maximal subgroup of $P$ is $SE$-supplemented in $N_G(P)$ and that $P$ is $S$-quasinormal in $H$. Then $H$ meets the hypothesis of Theorem 16, and thus $H$ is supersolvable. Let $q = \max n(H)$ and $Q \in Syl_q(H)$; then $Q \lneq G$. Let $N$ be a minimal normal subgroup of $G$ contained in $Q$, and consider the quotient group $G/N$. First, $G/N)(H/N) = G/H \in \mathcal{F}$. With a similar argument as in the proof of Theorem 16, we can obtain that $G/N$ with subgroup $H/N$ meets the hypothesis; hence $G/N \in \mathcal{F}$ by the choice of $G$. Consequently, we may assume that $H = Q = N$ is a minimal normal subgroup of $G$. Since $\mathcal{F}$ constitutes a saturated formation, $G \lneq \Phi(G)$. Then there exists a maximal subgroup $M$ of $G$ such that $G = NM$ and $N \cap M = 1$. Let $M_\alpha$ be a Sylow $q$-subgroup of $M$. Then $G_\alpha = QM_\alpha$ is a Sylow $q$-subgroup of $G$. Let $Q_1 \leq Q \cap Q_\alpha$, where $Q_\alpha$ is a maximal subgroup of $G_\alpha$ containing $M_\alpha$. Then $Q_1$ is a maximal subgroup of $Q$ and $Q_1 \lneq G_\alpha$. From this hypothesis, $Q_1$ is $SE$-supplemented in $N_G(Q_1) = G$. Let $T$ be any supplement of $Q_1$ in $G$; then $Q_1T = G$ and $Q = Q \cap Q_1T = Q_1(Q \cap T)$. This means that $Q \cap T \neq 1$. We note that $Q \cap T$ is normal in $G$ and $Q$ is a minimal normal subgroup of $G$, $Q \cap T = Q$. Thus $T = G$ is the unique supplement of $Q_1$ in $G$. As a result, we know that $Q_1 = Q_1 \cap T \leq (Q_1)_{seN(G,Q)}$ by
Lemma 6. Then \((Q_1)_{sG} = Q_1\) is S-quasinormal in \(G\). It follows that \(N_G(Q_1) \supseteq O^s(G)\) by Lemma 5. As \(Q_1 \nsubseteq G_p\), it is easy to see that \(N_G(Q_1) \supseteq G_pO^s(G) = G\). By the minimal normality of \(Q\) in \(G\), it is clear that \(Q_1 = 1\) and thus \(Q\) is a cyclic group of order \(q\). It follows that \(G \in \mathcal{F}\) by [6, Lemma 2.16], a contradiction. The proof is completed.

From Theorem 17 the following corollary is immediate.

**Corollary 18.** Let \(\mathcal{F}\) be a saturated formation containing the class of all supersolvable groups \(\mathcal{U}\), and let \(G\) be a group with a normal subgroup \(H\) satisfying \(G/H \in \mathcal{F}\). If for any prime \(p\) dividing \(|H|\), there exists a Sylow \(p\)-subgroup \(P\) of \(H\) such that every maximal subgroup of \(P\) is \(S\)-quasinormally embedded or weakly \(SE\)-supplemented in \(N_G(P)\) and such that also \(P'\) is \(S\)-quasinormal in \(G\), then \(G \in \mathcal{F}\).

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