Research Article

On Exact Sequence of Semimodules over Semirings

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We introduce the notion of exact sequence of semimodules over semirings using maximal homomorphisms and generalize some results of module theory to semimodules over semirings. Indeed, we prove, “If $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is a split exact sequence of R -semimodules and maximal R -semimodule homomorphisms with $\text{Ker}(g)$ is a strong subsemimodule of M , then $M \cong L \oplus N$ ”. Also, some results of commutative diagram of R -semimodules and maximal R -homomorphisms having exact rows are obtained.

1. Introduction

The concept of semiring was introduced by Vandiver [1] in 1934. A semiring is a nonempty set R together with two associative binary operations, addition and multiplication, which is called a semiring if (i) addition is a commutative operation, (ii) there exists $0 \in R$ such that $x + 0 = x = 0 + x$, $x0 = 0 = 0x$ for each $x \in R$, and (iii) multiplication distributes over addition both from left and right. Denote the sets of all nonnegative and residue classes of integers modulo n , respectively, by $\mathbb{Z}_0^+$ and $\mathbb{Z}_n$. The set $\mathbb{Z}_0^+$ is a semiring under usual addition and multiplication of nonnegative integers, but it is not a ring. Concepts of commutative semiring, semiring with identity 1, can be defined on the similar lines as in rings. All semirings in this paper are assumed to be commutative with identity $1 \neq 0$.

Generalizing the notion of exact sequence of modules over rings, Abuhlail [2], Al-Thani [3], and Bhambri and Dubey [4] introduced different notions of exact sequence of semimodules over semirings. In this paper, we introduce the notion of exact sequence of semimodules over semirings using maximal homomorphisms, and hence, some results of commutative diagram of R -semimodules and maximal R -homomorphisms having exact rows are obtained.

The following definitions and results will be used to prove our results.

A left R -semimodule is a commutative monoid $(M, +)$ with additive identity 0_M for which we have a function

$R \times M \rightarrow M$, defined by $(r, x) \mapsto rx$ and called scalar multiplication, which satisfies the following conditions for all elements r and r' of R and all elements x and y of M :

- (1) $(rr')x = r(r'x)$;
- (2) $r(x + y) = rx + ry$;
- (3) $(r + r')x = rx + r'x$;
- (4) $1_R x = x$;
- (5) $r0_M = 0_M = 0_R x$.

Throughout this paper, by an R -semimodule we mean a left semimodule over a semiring R . Every semiring R is a $(\mathbb{Z}_0^+, +, \cdot)$ -semimodule (see [5, page 151]). An element a of a monoid $(M, *)$ is called idempotent if $a * a = a$. If $(M, +)$ is an idempotent commutative monoid, then M is a $(\mathbb{Z}_0^+, +, \cdot)$ -semimodule with scalar multiplication defined by $rm = 0$ if $r = 0$ and $rm = m$ if $r > 0$ for all $r \in \mathbb{Z}_0^+$ and $m \in M$ (see [5, page 151]). A nonempty subset N of an R -semimodule M is called a subsemimodule of M if N is closed under addition and closed under scalar multiplication. A subsemimodule N of an R -semimodule M is called (1) a subtractive subsemimodule ($=k$ -subsemimodule) if $x, x + y \in N$, $y \in M$, then $y \in N$; (2) a strong subsemimodule if for any $x \in N$ there exists $y \in N$, such that $x + y = 0$.

Chaudhari and Bonde [6] introduced the notion of a partitioning subsemimodule and developed the quotient structure of semimodules over semirings. A subsemimodule

N of an R -semimodule M is called partitioning subsemimodule (= Q -subsemimodule) if there exists a subset Q of M such that $M = \cup\{q + N : q \in Q\}$, and if $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \emptyset \Leftrightarrow q_1 = q_2$. If N is a subsemimodule of a $(\mathbb{Z}_0^+, +, \cdot)$ -semimodule $(\mathbb{Z}_0^+, +)$ and N is generated by n , then N is a partitioning subsemimodule with $Q = \{0, 1, \dots, n-1\}$ [7, Example 2.2(2)]. If N is a partitioning subsemimodule of an R -semimodule M , then $M/N_{(Q)} = \{q + N : q \in Q\}$ forms an R -semimodule under the following addition “ $\oplus$ ” and scalar multiplication “ $\odot$ ”, $(q_1 + N) \oplus (q_2 + N) = q_3 + N$ where $q_3 \in Q$ is unique such that $q_1 + q_2 + N \subseteq q_3 + N$, and $r \odot (q_1 + N) = q_4 + N$ where $q_4 \in Q$ is unique such that $r q_1 + N \subseteq q_4 + N$. This R -semimodule $M/N_{(Q)}$ is called a quotient semimodule of M by N and denoted by $M/N_{(Q)}$. By Lemma 2.3 ([6]), there exists a unique $q_0 \in Q$ such that $q_0 + N = N$. This $q_0 + N$ is the zero element of $M/N_{(Q)}$. An R -semimodule M is said to be a direct sum of subsemimodules $N_1, N_2, \dots, N_k$ of M , if each $x \in M$ can be uniquely written as $x = x_1 + x_2 + \dots + x_k$ where $x_i \in N_i$, $1 \leq i \leq k$. It is denoted by $M = N_1 \oplus N_2 \oplus \dots \oplus N_k$ (see [5, page 184]). If R is a semiring and M, M' are R -semimodules, then a function $f : M \rightarrow M'$ is called an R -semimodule homomorphism if $f(x + y) = f(x) + f(y)$ for all $x, y \in M$ and $f(rx) = rf(x)$ for all $x \in M$ and $r \in R$. An equivalence relation ρ on an R -semimodule M is said to be an R -congruence relation if $a\rho b$ implies that $(a + c)\rho(b + c)$ for all $c \in M$ and $(ra)\rho(rb)$ for all $r \in R$. If $f : M \rightarrow M'$ is an R -semimodule homomorphism, then f induces an R -congruence relation $\equiv_f$ on M given by $m \equiv_f m' \Leftrightarrow f(m) = f(m')$ where $m, m' \in M$. If N is a subsemimodule of an R -semimodule M , then N induces an R -congruence relation $\equiv_N$ on M given by $m \equiv_N m' \Leftrightarrow$ there exist $n, n' \in N$ such that $m + n = m' + n'$ where $m, m' \in M$. If $f : M \rightarrow M'$ is an R -semimodule homomorphism, then clearly $m \equiv_{\text{Ker}(f)} m'$ implies that $m \equiv_f m'$ where $m, m' \in M$ and $\text{Ker}(f) = \{x \in M : f(x) = 0'\}$.

Definition 1. Let $f : M \rightarrow M'$ be an R -semimodule homomorphism. Then, f is called

- (1) Steady (or k -uniform [2] or k -regular [3]) if $\equiv_f$ coincides with $\equiv_{\text{Ker}(f)}$;
- (2) maximal [6] if for each $a \in f(M)$, there exists a unique $q_a \in f^{-1}(\{a\})$ such that $x + \text{Ker}(f) \subseteq q_a + \text{Ker}(f)$ for all $x \in f^{-1}(\{a\})$;
- (3) one-one if $f(x) = f(y)$ implies $x = y$ where $x, y \in M$;
- (4) isomorphism if f is one-one and onto;
- (5) i -regular [3] (or i -uniform [2]) if $\text{Im}(f)$ is a subtractive subsemimodule of M' .

Clearly, identity R -semimodule homomorphism from M onto itself is a maximal R -semimodule homomorphism.

Lemma 2 (see [8, Lemma 2.2]). *If $f : M \rightarrow M'$ is a maximal R -semimodule homomorphism, then f is steady.*

Theorem 3 (see [8, Theorem 2.1]). *Let $f : M \rightarrow M'$ be an R -semimodule homomorphism. Then, the following statements are equivalent:*

- (1) f is one-one;
- (2) $\text{Ker}(f) = \{0\}$ and f is maximal;
- (3) $\text{Ker}(f) = \{0\}$ and f is steady.

Theorem 4 (see [8, Theorem 3.3]). *Let N_1, N_2 be subsemimodules of an R -semimodule M . Then, $M = N_1 \oplus N_2$ if and only if N_1, N_2 are partitioning subsemimodules, $M = N_1 + N_2$, and $N_1 \cap N_2 = \{0\}$.*

2. Exact Sequence of Semimodules over Semirings

Generalizing the notion of exact sequence of modules over rings, Abuhlail [2], Al-Thani [3], and Bhambri and Dubey [4] introduced different notions of exact sequence of semimodules over semirings as follows.

Definition 5. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of R -semimodules and R -semimodule homomorphisms is said to be exact if

- (1) $\text{Im}(f) = \text{Ker}(g)$ and g is k -uniform [2];
- (2) $\overline{I}_f = K_g$ where $\overline{I}_f = \{(c, d) \in B \times B : c + f(a) + x = d + f(b) + x \text{ for some } a, b \in A, x \in B\}$ and $K_g = \{(a, b) \in B \times B : f(a) + x = f(b) + x \text{ for some } x \in B\}$ [4];
- (3) $f(A) = \text{Ker}(g)$ and proper exact if $\text{Ker}(g) =$ subtractive closure of $f(A)$ [3].

Now, we introduce the notion of exact sequence of semimodules over semirings using maximal homomorphisms, and hence, some results of commutative diagram of R -semimodules and maximal R -homomorphisms having exact rows are obtained.

Definition 6. A sequence $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \dots \xrightarrow{f_n} M_n$ of R -semimodules and maximal R -semimodule homomorphisms is said to be exact if $\text{Im}(f_i) = \text{Ker}(f_{i+1})$ for all $1 \leq i \leq n-1$.

As every ring, R -module, and R -module homomorphism is, respectively, semiring, R -semimodule, and maximal R -semimodule homomorphism, we have every exact sequence of R -modules and R -homomorphisms is an exact sequence of R -semimodules and maximal R -semimodule homomorphisms.

If A, B , and C are R -semimodules, then by Theorem 3:

- (1) the sequence $0 \rightarrow A \xrightarrow{\psi} B$ is exact if and only if ψ is one-one;
- (2) the sequence $B \xrightarrow{\phi} C \rightarrow 0$ is exact if and only if ϕ is maximal and onto.

Now, we have the following proposition.

Proposition 7. *Let $\psi : A \rightarrow B, \phi : B \rightarrow C$ be R -semimodule homomorphisms. Then, the sequence $0 \rightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \rightarrow 0$*

is exact if and only if ψ is one-one, ϕ is maximal and onto, and $\text{Im}(\psi) = \text{Ker}(\phi)$.

Definition 8. An exact sequence $0 \rightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \rightarrow 0$ of R -semimodules and maximal R -semimodule homomorphisms will be called as a short exact sequence.

Definition 9. A short exact sequence, $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ of R -semimodules and maximal R -semimodule homomorphisms, is said to be split exact sequence if there exists a splitting map $g' : N \rightarrow M$ such that $g'(N)$ is partitioning subsemimodule of M and $gg' = \text{Id}_N$.

Example 10. Let A and B be any R -semimodules.

- (1) If $\phi : A \rightarrow B$ is any maximal R -semimodule homomorphism, then $0 \rightarrow \text{Ker}(\phi) \xrightarrow{i} A \xrightarrow{\phi} \text{Im}(\phi) \rightarrow 0$ is an exact sequence where i is an identity R -semimodule homomorphism.
- (2) Consider $(\mathbb{Z}_0^+, +, \cdot)$ -semimodules $L = (\mathbb{Z}_0^+, +)$, $M = (\mathbb{Z}_0^+ \cup \{\infty\}, +)$, and $N = (\{0, 1, \infty\}, \max)$, maximal R -semimodule homomorphism $g : M \rightarrow N$ defined by

$$g(m) = \begin{cases} 0 & \text{if } m \neq \infty, \\ \infty & \text{if } m = \infty, \end{cases} \quad (1)$$

and an identity R -semimodule homomorphism $i : L \rightarrow M$. Then, the sequence $0 \rightarrow L \xrightarrow{i} M \xrightarrow{g} N \rightarrow 0$ is not an exact sequence because g is not onto.

- (3) Consider $0 \rightarrow A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \rightarrow 0$, a sequence of R -semimodules where i is an identity R -semimodule homomorphism and π is defined by $\pi(a, b) = b$, for all $(a, b) \in A \oplus B$ is onto maximal R -semimodule homomorphism with $\text{Ker}(\pi) = A$. This sequence is a split exact sequence with inclusion map $t : B \rightarrow A \oplus B$ as a splitting map, since, by Theorem 4, $t(B) = B$ is a partitioning subsemimodule of $A \oplus B$.
- (4) If N is a partitioning subsemimodule of an R -semimodule A , then $f : A \rightarrow A/N_{(Q)}$ defined by $f(a) = q + N$ where $q \in Q$ is a unique element such that $a + N \subseteq q + N$ is onto maximal R -semimodule homomorphism with $\text{Ker}(f) = N$. Now, the sequence $0 \rightarrow N \xrightarrow{i} A \xrightarrow{f} A/N_{(Q)} \rightarrow 0$ is an exact sequence where i is an identity R -semimodule homomorphism.
- (5) Let $N = (2\mathbb{Z}_0^+, +)$ be a $Q = \{0, 1\}$ -subsemimodule of an $R = (\mathbb{Z}_0^+, +, \cdot)$ -semimodule $A = (\mathbb{Z}_0^+, +)$. Then, by Example 10(4), the sequence $0 \rightarrow N \xrightarrow{i} A \xrightarrow{f} A/N_{(Q)} \rightarrow 0$ is an exact sequence but it is not a split exact sequence because the only R -semimodule homomorphism from $A/N_{(Q)}$ to A is a zero R -semimodule homomorphism.

Theorem 11. If $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is a split exact sequence of R -semimodules and maximal R -semimodule homomorphisms with $\text{Ker}(g)$ is a strong subsemimodule of M , then $M \cong L \oplus N$.

Proof. Let $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ be a split exact sequence of R -semimodules, and let maximal R -semimodule homomorphisms with a splitting map g' and $\text{Ker}(g)$ be a strong subsemimodule of M . Claim that $M = f(L) \oplus g'(N) = \text{Ker}(g) \oplus g'(N)$. Since g is maximal, $\text{Ker}(g)$ is partitioning subsemimodule of M (see [6, Lemma 2.8]). Also, $g'(N)$ is a partitioning subsemimodule of M , since given sequence is a split exact sequence. Hence, by using Theorem 4, it is enough to prove that $M = \text{Ker}(g) + g'(N)$ and $\text{Ker}(g) \cap g'(N) = \{0\}$. Let $m \in M$ and $g(m) = n \in N$. Hence, $g(m) = n = gg'(n) = g(g'(n))$. By Lemma 2, g is steady. Hence, $m + k_1 = g'(n) + k_2$ for some $k_1, k_2 \in \text{Ker}(g)$. Since $\text{Ker}(g)$ is a strong subsemimodule of M , there exists $k \in \text{Ker}(g)$ such that $k_1 + k = 0$. Thus, we get $m = m + k_1 + k = g'(n) + k_2 + k \in \text{Ker}(g) + g'(N)$. So, $M = \text{Ker}(g) + g'(N) = f(L) + g'(N)$. Now, $f(l) = g'(n) \in f(L) \cap g'(N) \Rightarrow 0 = gf(l) = gg'(n) = n \Rightarrow g'(n) = 0$. Hence, $f(L) \cap g'(N) = \{0\}$. So, $M = f(L) \oplus g'(N)$. Now, define $\theta : L \oplus N \rightarrow f(L) \oplus g'(N)$ by $\theta(l + n) = f(l) + g'(n)$. Clearly, θ is an onto R -semimodule homomorphism. Since f is one-one and $gg' = \text{Id}_N$, θ is one-one. Thus, $M \cong L \oplus N$. $\square$

Corollary 12. Let $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ be a split exact sequence of R -semimodules, and let maximal R -semimodule homomorphisms with $\text{Ker}(g)$ be a strong subsemimodule of M . Then, there exists a maximal R -semimodule homomorphism $h : M \rightarrow L$ such that $hf = \text{Id}_L$.

Proof. By Theorem 11, $M = \text{Im}(f) \oplus g'(N)$. Clearly, $\pi : M \rightarrow \text{Im}(f)$ defined by $\pi(x + y) = x$ for all $x \in \text{Im}(f)$ and $y \in g'(N)$ is a maximal R -semimodule homomorphism. Let $h = f^{-1}\pi$. Then, h is maximal, since f is maximal and $hf = \text{Id}_L$. $\square$

Example 13. Consider $L = (\{\bar{0}, \bar{2}, \bar{4}\}, +_6)$, $M = (\mathbb{Z}_6, +_6)$, $N = (\{\bar{0}, \bar{3}\}, +_6)$ the $R = (\mathbb{Z}_0^+, +, \cdot)$ -semimodules, and consider that $g : M \rightarrow N$ defined by $g(\bar{m}) = \bar{n}$ where $n \equiv 3m \pmod{6}$ is an R -semimodule homomorphism, $f : L \rightarrow M$ is an identity R -semimodule homomorphism. Here, g is maximal onto R -semimodule homomorphism. Then, the sequence $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is split exact sequence and $\text{Ker}(g)$ is a strong subsemimodule of M , and hence, by Theorem 11, $M \cong L \oplus N$.

The following example shows that the condition “ $\text{Ker}(g)$ is a strong subsemimodule of M ” is essential.

Example 14. Let $L = (\mathbb{Z}_0^+, +)$, $M = (\mathbb{Z}_0^+ \cup \{\infty\}, +)$, and $N = (\{0, \infty\}, +)$ be $R = (\mathbb{Z}_0^+, +, \cdot)$ -semimodules, and let $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ be a sequence of R -semimodules and maximal R -semimodule homomorphisms where f is an inclusion map and $g : M \rightarrow N$ is defined as

$$g(m) = \begin{cases} 0 & \text{if } m \neq \infty, \\ \infty & \text{if } m = \infty. \end{cases} \quad (2)$$

Then, clearly, $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is an exact sequence. Now take $g' : N \rightarrow M$ an identity R -semimodule homomorphism. Then, $gg' = Id_N$. Here, $M = L + N$ and $M \neq L \oplus N$ because $\infty = a + \infty$ for all $a \in M$. Hence M is not isomorphic to $L \oplus N$. Clearly, $\text{Ker}(g)$ is not a strong subsemimodule of M .

Now, we extend and prove two theorems of commutative diagram of R -modules and R -homomorphisms having exact rows for commutative diagram of R -semimodules and maximal R -homomorphisms having exact rows.

Theorem 15. Suppose that the following diagram of R -semimodules and maximal R -semimodule homomorphisms is commutative and has exact rows:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array} \quad (3)$$

- (1) If β is one-one and α, g are onto, then γ is one-one.
(2) If β is onto and f', γ are one-one, then α is onto.

Proof. (1) Let $c \in \text{Ker}(\gamma)$. Since g is onto, there exists $b \in B$ such that $g(b) = c$. Now, $0 = \gamma(c) = \gamma g(b) = g'\beta(b) \Rightarrow \beta(b) \in \text{Ker}(g') = \text{Im}(f') \Rightarrow \beta(b) = f'(a')$ for some $a' \in A'$. Since α is an onto, $a' = \alpha(a)$ for some $a \in A$. Now, $\beta(b) = f'(a') = f'\alpha(a) = \beta f(a)$. Since β is one-one, $b = f(a) \in \text{Im}(f) = \text{Ker}(g)$. So $c = g(b) = 0$. Thus, $\text{Ker}(\gamma) = \{0\}$. Now, γ is a maximal R -semimodule homomorphism, and hence, by Theorem 3, γ is one-one.

(2) Let $a' \in A'$. Since β is onto, there exists $b \in B$ such that $\beta(b) = f'(a') \in \text{Im}(f') = \text{Ker}(g')$. Now, $g'\beta(b) = \gamma g(b) \Rightarrow \gamma g(b) = g'\beta(b) = 0 \Rightarrow g(b) \in \text{Ker}(\gamma) = \{0\}$ (since γ is one-one) $\Rightarrow b \in \text{Ker}(g) = \text{Im}(f) \Rightarrow b = f(a)$ for some $a \in A$. Now, $f'(a') = \beta(b) = \beta f(a) = f'\alpha(a)$. Since f' is one-one, $a' = \alpha(a)$. Hence, α is onto. $\square$

Theorem 16. Suppose that the following diagram of R -semimodules and maximal R -semimodule homomorphisms is commutative and has exact rows and middle column is exact:

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & A & \xrightarrow{\alpha} & A'' \longrightarrow 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' \\ 0 & \longrightarrow & B' & \xrightarrow{\beta'} & B & \xrightarrow{\beta} & B'' \longrightarrow 0 \\ & & \downarrow g' & & \downarrow g & & \downarrow g'' \\ 0 & \longrightarrow & C' & \xrightarrow{\gamma'} & C & \xrightarrow{\gamma} & C'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array} \quad (4)$$

Then, the last column is exact, and $g'(B')$ is subtractive if and only if the first column is exact.

Proof. Suppose that the last column is exact and $g'(B')$ is subtractive. We claim that $\text{Ker}(f') = \{0\}$, $\text{Im}(f') = \text{Ker}(g')$, and g' is onto.

(i) Let $a' \in \text{Ker}(f')$. Now, $f\alpha'(a') = \beta'f'(a') = \beta'(0) = 0$ implies that $\alpha'(a') \in \text{Ker}(f) = \{0\}$. Since α' is one-one, $a' = 0$. Hence, $\text{Ker}(f') = \{0\}$.

(ii) Let $b' \in \text{Ker}(g')$. Now, $g\beta'(b') = \gamma'g'(b') = \gamma'(0) = 0 \Rightarrow \beta'(b') \in \text{Ker}(g) = \text{Im}(f)$. So, $\beta'(b') = f(a)$ for some $a \in A$. As $\text{Im}(\beta') = \text{Ker}(\beta)$ and $\beta f = f''\alpha$, we get $0 = \beta\beta'(b') = \beta f(a) = f''\alpha(a)$. Hence, $\alpha(a) \in \text{Ker}(f'') = \{0\}$ implies that $a \in \text{Ker}(\alpha) = \text{Im}(\alpha')$. So $a = \alpha'(a')$ for some $a' \in A'$. Now, $\beta'(b') = f(a) = f\alpha'(a') = \beta'f'(a')$. Since β' is one-one, $b' = f'(a') \in \text{Im}(f')$. Hence, $\text{Ker}(g') \subseteq \text{Im}(f')$. Now, let $x = f'(a') \in \text{Im}(f')$ for some $a' \in A'$. Therefore, $\beta'(x) = \beta'f'(a') = f\alpha'(a')$ implies that $\beta'(x) \in \text{Im}(f) = \text{Ker}(g)$. Now, $\gamma'g'(x) = g\beta'(x) = 0$. Since γ' is one-one, $g'(x) = 0$. Now $x \in \text{Ker}(g')$. Thus, $\text{Im}(f') \subseteq \text{Ker}(g')$. Hence, we get the required equality.

(iii) Let $c' \in C'$. Therefore, $\gamma'(c') \in C$ and $\gamma'(c') = g(b)$ for some $b \in B$, since g is onto. Now, $g''\beta(b) = \gamma g(b) = \gamma\gamma'(c') = 0$ implies that $\beta(b) \in \text{Ker}(g'') = \text{Im}(f'')$ implies $\beta(b) = f''(a'')$ for some $a'' \in A''$. As α is onto, we get $\beta(b) = f''(a'') = f''\alpha(a) = \beta f(a)$ for some $a \in A$. Since β is maximal and hence steady, so there exist $k_1, k_2 \in \text{Ker}(\beta)$ such that $b + k_1 = f(a) + k_2$. As $\text{Ker}(\beta) = \text{Im}(\beta')$, we have $b + \beta'(b'_1) = f(a) + \beta'(b'_2) \Rightarrow g(b) + g\beta'(b'_1) = gf(a) + g\beta'(b'_2) \Rightarrow g(b) + \gamma'g'(b'_1) = 0 + \gamma'g'(b'_2) \Rightarrow \gamma'(c' + g'(b'_1)) = \gamma'g'(b'_2) \Rightarrow c' + g'(b'_1) = g'(b'_2)$, since γ' is one-one. Now, $c' \in g'(B')$, as $g'(B')$ is subtractive. Hence, g' is onto. So,

$0 \rightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \rightarrow 0$ is an exact sequence.

Conversely, assume that the first column is exact, then, by a similar argument as previously mentioned we can show that $\text{Ker}(f'') = \{0\}$, $\text{Im}(f'') = \text{Ker}(g'')$. Now, let $c'' \in C''$. Since γ and g are onto, there exists $b \in B$ such that $c'' = \gamma g(b) = g''\beta(b) \in \text{Im}(g'')$. So, g'' is onto. Hence, the last column is exact. Also, g' is onto which implies that $g'(B') = C'$ which is a subtractive subsemimodule. $\square$

Now, we extend and prove the well-known “5-lemma [9]” of R -modules and R -homomorphisms for R -semimodules and maximal R -semimodule homomorphisms.

Lemma 17 (the 5-lemma). Given commutative diagram of R -semimodules and maximal R -semimodule homomorphisms with exact rows:

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 & \xrightarrow{g_3} & M_4 & \xrightarrow{g_4} & M_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3 & \xrightarrow{h_3} & N_4 & \xrightarrow{h_4} & N_5 \end{array} \quad (5)$$

Then,

- (i) if f_1 is onto and f_2, f_4 are one-one, then f_3 is one-one;
(ii) if f_5 is one-one, f_2, f_4 are onto, and $\text{Im}(f_3)$ is a subtractive subsemimodule of N_3 , then f_3 is onto;

(iii) if f_1, f_2, f_4, f_5 are isomorphisms and $\text{Im}(f_3)$ is a subtractive subsemimodule of N_3 , then so is f_3 .

Proof. (i) Let $m_3 \in \text{Ker}(f_3)$. Now, $f_4 g_3(m_3) = h_3 f_3(m_3) = 0 \Rightarrow g_3(m_3) \in \text{Ker}(f_4) = \{0\} \Rightarrow m_3 \in \text{Ker}(g_3) = \text{Im}(g_2) \Rightarrow m_3 = g_2(m_2)$ for some $m_2 \in M_2$. Now, $0 = f_3(m_3) = f_3 g_2(m_2) = h_2 f_2(m_2) \Rightarrow f_2(m_2) \in \text{Ker}(h_2) = \text{Im}(h_1) \Rightarrow f_2(m_2) = h_1(n_1)$ for some $n_1 \in N_1$. Since f_1 is onto, $n_1 = f_1(m_1)$ for some $m_1 \in M_1$. Thus, $f_2(m_2) = h_1 f_1(m_1) = f_2 g_1(m_1)$. So, $m_2 = g_1(m_1)$, since f_2 is one-one. Now, $m_3 = g_2(m_2) = g_2 g_1(m_1) = 0$. So, $\text{Ker}(f_3) = \{0\}$. Since f_3 is maximal, f_3 is one-one.

(ii) Let $n_3 \in N_3$. Since f_4 is onto, $h_3(n_3) = f_4(m_4)$ for some $m_4 \in M_4$. Now, $0 = h_4 h_3(n_3) = h_4 f_4(m_4) = f_5 g_4(m_4) \Rightarrow g_4(m_4) = 0$ since f_5 is one-one. So, $m_4 \in \text{Ker}(g_4) = \text{Im}(g_3) \Rightarrow m_4 = g_3(m_3)$ for some $m_3 \in M_3$. Now, $h_3(n_3) = f_4(m_4) = f_4 g_3(m_3) = h_3 f_3(m_3)$ and h_3 is maximal and hence steady which implies that there exist $k_1 = h_2(n'_2), k_2 = h_2(n''_2) \in \text{Ker}(h_3) = \text{Im}(h_2)$ for some $n'_2, n''_2 \in N_2$ such that $n_3 + h_2(n'_2) = f_3(m_3) + h_2(n''_2)$. Now $n'_2 = f_2(m'_2)$ and $n''_2 = f_2(m''_2)$ for some $m'_2, m''_2 \in M_2$, since f_2 is onto. Now, $n_3 + h_2(n'_2) = f_3(m_3) + h_2(n''_2) \Rightarrow n_3 + h_2 f_2(m'_2) = f_3(m_3) + h_2 f_2(m''_2) \Rightarrow n_3 + f_3 g_2(m'_2) = f_3(m_3) + f_3 g_2(m''_2) \in \text{Im}(f_3)$. Since $\text{Im}(f_3)$ is subtractive subsemimodule of N_3 , $n_3 \in \text{Im}(f_3)$. Hence, f_3 is onto.

(iii) Follows from (i) and (ii). $\square$

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