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Research Article

On Two Aspects of the Painlevé Analysis

Sergei Sakovich

Institute of Physics, National Academy of Sciences of Belarus, 220072 Minsk, Belarus

Correspondence should be addressed to Sergei Sakovich; saks@tut.by

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We use the Calogero equation to illustrate the following two aspects of the Painlevé analysis of nonlinear PDEs. First, if a nonlinear equation passes the Painlevé test for integrability, the singular expansions of its solutions around characteristic hypersurfaces can be neither single-valued functions of independent variables nor single-valued functionals of data. Second, if the truncation of singular expansions of solutions is consistent, the truncation not necessarily leads to the simplest, or elementary, auto-Bäcklund transformation related to the Lax pair.

1. Introduction

The Painlevé analysis is a simple and reliable tool for testing the integrability of nonlinear ODEs and PDEs [1, 2]. Concerning PDEs, there is strong empirical evidence—though not proven a theorem yet—that every nonlinear equation possessing the Painlevé property in formulation for PDEs inevitably belongs either to the class of C-integrable equations (exactly linearizable equations, including Darboux integrable ones) or to the class of S-integrable equations (Lax integrable equations, including Liouville integrable bi-Hamiltonian ones). The Painlevé analysis of nonlinear PDEs is usually performed along the so-called Weiss-Kruskal algorithm, which combines the Weiss' singular expansions of solutions around movable hypersurfaces [3] and the Kruskal's simplifying representation for singularity manifolds [4] and which follows step by step the Ablowitz-Ramani-Segur algorithm for ODEs [5].

The very first step of the Weiss-Kruskal algorithm, however, has no counterpart in the Ablowitz-Ramani-Segur algorithm: starting with the Painlevé analysis of a nonlinear PDE, one must determine which of one analytic hypersurfaces is characteristic for the tested equation, in order to perform the whole subsequent analysis of solutions around noncharacteristic hypersurfaces only. Ward [6] was the one first who stated and substantiated that the Painlevé property for PDEs must not fix any structure of solutions at characteristic hypersurfaces. Afterward, the essence of Ward's statement was mentioned as "a fact tacitly assumed by all Painlevé practitioners" [1]. Lately, however, Weiss [7, 8] declared that his result "runs counter to the observation of Ward" and that "expansions about characteristic manifolds are required to be single-valued" as functionals of data.

In the present paper, in Section 2, we show that Ward's definition of the Painlevé property for PDEs still remains well founded and that the objections of Weiss are caused by some terminological confusion. We do this via the singularity analysis of the Calogero equation as follows [9, 10]:

$$u_{xxxy} - 2u_y u_{xx} - 4u_x u_{xy} + u_{xt} = 0. \tag{1}$$

This nonlinear PDE (1) is useful to illustrate one more aspect of the Painlevé analysis. In Section 3, we find two different Bäcklund transformations of the Calogero equation (1) into itself: one follows from the truncated singular expansion for u, the other one follows from the Lax pair of (1), and the former turns out to be a special case of the square of the latter. Consequently, the Painlevé analysis does not lead to the simplest, or elementary, auto-Bäcklund transformation of (1), a phenomenon similar to what was observed in [11].

Section 4 contains concluding remarks.

2. Breaking Solitons and the Painlevé Property

Let us take the fourth-order three-dimensional nonlinear PDE (1) and assume for a while that we know nothing about its integrability and solutions. Does this (1) pass the Painlevé

test for integrability? The answer will be "yes" if we adopt the Ward's definition [6] of the Painlevé property for PDEs, but the answer will be "no" if we change the definition as proposed by Weiss [7, 8].

It is an easy task to perform the Painlevé analysis of (1) along with the Weiss-Kruskal algorithm. A hypersurface $\phi(x, y, t) = 0$ is non-characteristic for the PDE (1) if $\phi_x^3\phi_y \neq 0$ (see, e.g., [12] for the definition and meaning of noncharacteristic hypersurfaces). The Kruskal's ansatz $\phi = x + \psi(y, t)$ with $\psi_y \neq 0$ simplifies calculations and excludes characteristic hypersurfaces from consideration. The assumption that the dominant behavior of solutions is algebraic around $\phi = 0$, that is, $u = u_0(y, t)\phi^p + \cdots$, leads to the only branch to be tested: $p = -1$ with $u_0 = -2$. (Branches with $p = 0, 1, 2, 3$, also admitted by (1), need no analysis: they are governed by the Cauchy-Kovalevskaya theorem [12] because Kovalevskaya form of the PDE (1) is analytic everywhere.) Then we substitute $u = -2\phi^{-1} + \cdots + u_r(y, t)\phi^{r-1} + \cdots$ into (1), find that u_r is not determined if $r = -1, 1, 4, 6$ ($r = -1$ corresponds to the arbitrariness of ψ), and conclude that the tested branch is generic. Finally, we substitute $u = \sum_{i=0}^{\infty} u_i(y, t)\phi^{i-1}$ into (1), find recursion relations for u_i, and check compatibility conditions at the resonances, where the arbitrary functions u_1, u_4, and u_6 appear. All the compatibility conditions turn out to be identities. Consequently, the PDE (1) has passed the Weiss-Kruskal algorithm well. Since this algorithm is sensitive to algebraic and nondominant logarithmic singularities only, we can only conjecture that the tested equation possesses the Painlevé property in the sense that all solutions of (1) are single-valued around all non-characteristic hypersurfaces. And we should expect (1) to be integrable.

The PDE (1) is integrable indeed [9, 10]. It arises as the compatibility condition for the overdetermined, linear in Φ, system

$$\Phi_{xx} + (\alpha - u_x)\Phi = 0, \tag{2}$$

$$\Phi_t + 4\Phi_{xxy} - 2u_y\Phi_x - 4u_x\Phi_y - 3u_{xy}\Phi = 0, \tag{3}$$

where the spectral parameter α is any function $\alpha(y, t)$ satisfying the following:

$$\alpha_t = 4\alpha\alpha_y. \tag{4}$$

All solutions of (4), except $\alpha =$ constant, are multivalued functions: for any nonconstant initial value $\alpha(y, 0)$, the nonlinear wave $\alpha = \alpha(y, t)$ inevitably breaks (overturns, overlaps) at some finite t [13]. Therefore, solutions of (1), obtainable by the inverse scattering transform with any nonconstant α, are multivalued functions as well. For example, the one-soliton solution of (1),

$$u = -2\lambda \tanh(\lambda x + \mu) + \beta, \tag{5}$$

where $\lambda(y, t)$, $\mu(y, t)$, and $\beta(y, t)$ are any functions satisfying equations $\lambda_t + 4\lambda^2\lambda_y = 0$ ($\lambda^2 = -\alpha$) and $\mu_t + 4\lambda^2\mu_y = 2\lambda\beta_y$, becomes a multivalued function when α breaks [14]. The N-soliton solution of (1), determined by N solutions $\alpha_1, \ldots, \alpha_N$ of (4), breaks whenever any of $\alpha_1, \ldots, \alpha_N$ breaks [14].

At first sight, such a complicated branching of solutions of (1) seems to be incompatible with the Painlevé property. Nevertheless, there is no contradiction between the fact that solutions of (1) are multi-valued functions and the fact that solutions of (1) are single-valued around all non-characteristic hypersurfaces: solutions can branch and do branch at characteristic hypersurfaces only. Indeed, it was noticed and stressed in [14] that solutions of (1) break (i.e., $u_y \to \infty$ at finite values of u) for all values of x simultaneously, and this fact means that the corresponding singularity manifolds $\phi = 0$ are characteristic for (1), $\phi_x = 0$. Consequently, the breaking solitons do not break the Painlevé property in Ward's formulation [6] because they never break at non-characteristic hypersurfaces.

Let us proceed to Weiss' objections [7, 8] against Ward's formulation of the Painlevé property for PDEs. Weiss' counterexample is "the expansion about the *characteristic manifold*" $u = u_0(t) + \sum_{i=3}^{\infty} u_i(t)\phi^i$ with $\phi = x + \psi(t)$ for the equation $u_{xxx} = (3/2)u_x^{-1}u_{xx}^2 + ku_x - u_t$, where $k =$ constant. This expansion, however, does not represent solutions around characteristic hypersurfaces: characteristic hypersurfaces for this equation are determined by the condition $\phi_x = 0$, not by $\phi_x = 1$. Actually, this Taylor expansion represents solutions around any non-characteristic hypersurface $\phi = 0$ in the important case when the Cauchy-Kovalevskaya theorem [12] does not work: in this case, we have $u_x = 0$ at $\phi = 0$, whereas the Kovalevskaya form of the equation is singular at $u_x = 0$. We can only agree with Weiss that the consideration of such special Taylor expansions is an essential part of the Painlevé analysis, but Weiss' words "the expansion about the *characteristic manifold*" turn out to be too misleading because no actual expansions around characteristic hypersurfaces can be found in the papers [7, 8].

Now, let us return to the Calogero equation (1) and see what really happens at the characteristic hypersurfaces $\phi(x, y, t) = 0$ with $\phi_x\phi_y = 0$. When, for example, $\phi_x = 0$ and $\phi_y \neq 0$, we take $\phi = y + \psi(t)$ and $u = u_0(x, t)\phi^p + \cdots$, $p =$ constant, and find from (1) that any value of p is admissible. Therefore, the expansions will not be single-valued functionals of ϕ for noninteger p. For example, if $p = -1/2$, we get the expansion

$$u = \sum_{i=0}^{\infty} u_i(x, t)\,\phi^{(i-1)/2} \tag{6}$$

with the coefficients u_i determined by the recursion relations

$$\sum_{i=0}^{n} (i-1)\left[u_i(u_{n-i})_{xx} + 2(u_i)_x(u_{n-i})_x\right] - \frac{1}{2}(n-2)\left[(u_{n-1})_{xxx} + \psi_t(u_{n-1})_x\right] - (u_{n-3})_{xt} = 0, \tag{7}$$

where $n = 0, 1, 2, \ldots$ and $u_i = 0$ at $i < 0$. The structure of (7) differs from the habitual structure of recursion relations for non-characteristic hypersurfaces very considerably. There are no resonances in (7), but the expansion (6) contains infinitely many arbitrary functions of t in addition to $\psi(t)$: they arise

pair by pair as "constants" of integration of (7) because (7) is a second-order ODE in u_n for every n. Namely, $u_0 = (\sigma_0 x + \tau_0)^{1/3}$, $u_1 = (5/36)\sigma_0(\sigma_0 x + \tau_0)^{-1} + \sigma_1(\sigma_0 x + \tau_0)^{1/3} + \tau_1 + (1/4)\psi_t x$, $u_2 = \sigma_2(\sigma_0 x + \tau_0)^{1/3} + \tau_2(\sigma_0 x + \tau_0)^{2/3}$, and so forth, where $\sigma_i(t)$ and $\tau_i(t)$ are arbitrary functions, $i = 0, 1, 2, \ldots$. We see that the expansion (6) is multi-valued both as a function of x, y, t (via coefficients u_i and noninteger degrees of ϕ) and as a functional of ϕ. Consequently, if we accept Weiss' formulation [7, 8] of the Painlevé property, the integrable Calogero equation (1) will not pass the Painlevé test for integrability. Evidently, Weiss' formulation asks too much of the tested equation.

3. Two Auto-Bäcklund Transformations

Let us try to find a Bäcklund transformation of the Calogero equation (1) into itself. Two different methods will lead us to two different transformations. Then, we will find a relation between the two results.

First, we employ the method of truncated singular expansions of Weiss [15] and the new expansion function $\chi = (\phi^{-1}\phi_x - (1/2)\phi_x^{-1}\phi_{xx})^{-1}$ of Conte [16] which simplifies computations very considerably (note also that the Kruskal's ansatz is not used for ϕ in what follows). We substitute $u = g(x, y, t)\chi^{-1} + f(x, y, t)$ into the Calogero equation (1) and find that $g = -2$ and that ϕ and f must satisfy the following system of four equations:

$$d - 2c\left(s + 2f_x\right) + 2f_y = 0, \tag{8}$$

$$d_x - \frac{1}{2}c\left(s_x + 2f_{xx}\right) - 2c_x\left(s + 2f_x\right) + 2f_{xy} = 0, \tag{9}$$

$$\begin{aligned} d_{xx} + ds - c_x\left(s_x + 2f_{xx}\right) - 2\left(c_{xx} + cs\right)\left(s + 2f_x\right) \\ - s_{xy} + 2sf_y = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} s_{xxy} + f_{xxxy} + \left(c_{xx} + cs\right)\left(s_x + 2f_{xx}\right) - 2f_y\left(s_x + f_{xx}\right) \\ - 4\left(s + f_x\right)\left(s_y + f_{xy}\right) + 2ss_y + s_t + f_{xt} = 0, \end{aligned} \tag{11}$$

where $s = \phi_x^{-1}\phi_{xxx} - (3/2)\phi_x^{-2}\phi_{xx}^2$, $c = -\phi_x^{-1}\phi_y$, and $d = -\phi_x^{-1}\phi_t$. Substituting (8) into (9), we get $s_x + 2f_{xx} = 0$ which leads to $s + 2f_x = 2\alpha$, where the function $\alpha(y, t)$ appears as a "constant" of integration. Then, (8) changes into $d - 4\alpha c + 2f_y = 0$, (10) is satisfied identically, and $\alpha_t = 4\alpha\alpha_y$ follows from (11) (for this reason we use the same letter α as for the spectral parameter). Consequently, the system (8)–(11) is equivalent to the system of two equations

$$\begin{aligned} \phi_{xxx} - \frac{3}{2}\phi_x^{-1}\phi_{xx}^2 + 2\phi_x f_x - 2\alpha\phi_x = 0, \\ \phi_t - 2\phi_x f_y - 4\alpha\phi_y = 0, \end{aligned} \tag{12}$$

where $\alpha(y, t)$ is any solution of (4). The truncated expansion

$$u = \phi_x^{-1}\phi_{xx} - 2\phi^{-1}\phi_x + f \tag{13}$$

is a Miura transformation of system (12) into (1). One more Miura transformation of (12) into (1), namely,

$$v = \phi_x^{-1}\phi_{xx} + f, \tag{14}$$

where v satisfies (1), follows from (13) automatically [15, 16]. This chain of two Miura transformations (13) and (14) generates an auto-Bäcklund transformation for (1). Indeed, eliminating ϕ and f from (13) and (14) by means of (12) and differentiations, we get the following system:

$$w_{xx} - \frac{1}{2}w^{-1}w_x^2 - wz_x + \frac{1}{8}w^3 + 2\alpha w = 0, \tag{15}$$

$$z_{xy} - w^{-1}\left(w_x z_y + 4\alpha w_y - w_t\right) - \frac{1}{2}ww_y - 4\alpha_y = 0, \tag{16}$$

where $w = u - v$, $z = u + v$, and $\alpha(y, t)$ is any solution of (4). Direct but tedious computations prove that the system (15)-(16) is compatible in v if u satisfies (1), and that the system is compatible in u if v satisfies (1) (i.e., one gets (1) for u when eliminates v from (15)-(16) by differentiations and vice versa). Therefore, according to the definition in [17], the system (15)-(16) is a Bäcklund transformation of the PDE (1) into itself.

It looks strange, however, that the x-part (15) of the obtained Bäcklund transformation is a second-order ODE, whereas (2) of the associated linear problem for the PDE (1) is the same as for the potential KdV equation $u_t = u_{xxx} - 3u_x^2$. Let us apply the method of Chen [18] to the linear problem (2)-(3) and find that the PDE (1) does admit one more auto-Bäcklund transformation with the same first-order x-part as for the potential KdV equation. We rewrite (2) as $u_x = (\Phi_x/\Phi)_x + (\Phi_x/\Phi)^2 + \alpha$, introduce the new variable ω such that $\omega_x = (\Phi_x/\Phi)^2 + \alpha$, and get in this way $u = \omega \pm \varepsilon$, where $\varepsilon = (\omega_x - \alpha)^{1/2}$. Then (3) gives us the following fourth-order PDE for ω:

$$\varepsilon_{xxy} - 2\left(\omega_y\varepsilon\right)_x - 4\alpha\varepsilon_y + \varepsilon_t = 0. \tag{17}$$

It is very essential that (17) is the one and the same equation for both choices of the sign in $u = \omega \pm \varepsilon$. Owing to this fact, we have two Miura transformations of (17) into (1), namely,

$$\begin{aligned} u &= \omega + \left(\omega_x - \alpha\right)^{1/2}, \\ v &= \omega - \left(\omega_x - \alpha\right)^{1/2}, \end{aligned} \tag{18}$$

where u and v are solutions of (1) if ω satisfies (17). Eliminating ω from (18) and (17), we get

$$z_x - \frac{1}{2}w^2 - 2\alpha = 0, \tag{19}$$

$$w_{xxy} - w_x z_y - \left(w^2 + 4\alpha\right)w_y + w_t - 2\alpha_y w = 0, \tag{20}$$

where $w = u - v$, $z = u + v$, and $\alpha(y, t)$ is any solution of (4). One can check that the system (19)-(20) is compatible in v if u satisfies (1) and vice versa. Therefore, (19) and (20) constitute a Bäcklund transformation of the PDE (1) into itself.

We have obtained two auto-Bäcklund transformations for Calogero equation (1): (15)-(16) and (19)-(20). The two

transformations are both different in their forms, which is evident, and in solutions u they generate from a given solution v. For example, if v is any function $\gamma(y,t)$, we find from (19)-(20) that the corresponding u is the one-soliton solution (5) with $\beta = \gamma$, whereas the transformation (15)-(16) leads for this v either to the soliton (5) with $\beta = \gamma - 2\lambda$ or to the more complicated solution

$$u = -8\lambda\left[\cosh\left(\lambda x + \mu\right)\right]^2 \times \left\{\sinh\left[2\left(\lambda x + \mu\right)\right] + 2\left(\lambda x + \nu\right)\right\}^{-1} + \gamma, \tag{21}$$

where the functions $\lambda(y,t)$, $\mu(y,t)$, and $\nu(y,t)$ are any solutions of the equations $\lambda_t + 4\lambda^2\lambda_y = 0$ ($\lambda^2 = -\alpha$), $\mu_t + 4\lambda^2\mu_y = 2\lambda\gamma_y$ and $\nu_t + 4\lambda^2\nu_y - 8\lambda\lambda_y\nu = 2\lambda\gamma_y - 8\lambda^2\mu_y$. (For completeness, we should also mention the generated solutions u with $u_x = 0$: $u = \gamma - 2\lambda$ for (19)-(20), and $u = \gamma$ and $u = \gamma - 4\lambda$ for (15)-(16), $\lambda^2 = -\alpha$.) Nevertheless, these two different auto-Bäcklund transformations are related to each other: the transformation (15)-(16) is nothing but a special case of the square of the transformation (19)-(20). More precisely, if functions a, b, and q are taken such that $u = a$ and $v = q$ satisfy system (19)-(20) with some spectral parameter α, and $u = q$ and $v = b$ satisfy system (19)-(20) with the same α, then $u = a$ and $v = b$ satisfy system (15)-(16) with the same spectral parameter α. Indeed, by eliminating q from the relations $a_x + q_x = (1/2)(a-q)^2 + 2\alpha_1$ and $q_x + b_x = (1/2)(q-b)^2 + 2\alpha_2$, we get (15) for $u = a$ and $v = b$ if and only if $\alpha_1 = \alpha_2 = \alpha$, and (16) follows from (20) in the same way. Therefore, our words "a special case of the square" mean that the transformation (15)-(16) is composed of two transformations (19)-(20) with equal spectral parameters.

We have shown that the method of truncated singular expansions does not lead to the simplest auto-Bäcklund transformation of Calogero equation (1), related to the equation's Lax pair, and one may only guess that the transformation (19)-(20) can be derived from the transformation (15)-(16).

4. Conclusion

In this paper, we used the Calogero equation to illustrate the following two aspects of the Painlevé analysis of nonlinear PDEs.

First, if a nonlinear equation passes the Painlevé test for integrability, the singular expansions of its solutions around characteristic hypersurfaces can be neither single-valued functions of independent variables nor single-valued functionals of data. Of course, if the Painlevé property is considered as an abstract analytic property, one may give any definition of it. However, if the Painlevé property is defined to be used as an indicator of integrability of nonlinear equations, the adequacy of its definition becomes an experimental result. By the singularity analysis of the Calogero equation, we have shown that Ward's definition of the Painlevé property for PDEs is well founded.

Second, if the truncation of singular expansions of solutions is consistent, the truncation not necessarily leads to the simplest, or elementary, auto-Bäcklund transformation related to the Lax pair. We have found two different Bäcklund transformations of the Calogero equation into itself: one follows from the truncated singular expansion, the other one follows from the Lax pair, and the former turns out to be a special case of the square of the latter. In other words, the way from the truncated singular expansions to Bäcklund transformations and Lax pairs is not so straightforward as it is sometimes stated in the literature.

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