Research Article

An Especial Fractional Oscillator

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We propose a peculiar fractional oscillator. By assuming that the motion takes place in a complex media where the level of fractionality is low, we find that the time rate of change of the energy of this system has an oscillatory behavior.

1. Introduction

In complex media such as glasses, liquid crystals, polymers, and biopolymers, the dynamical variable of interest often obeys fractional differential equations [1–6].

For instance, the mean squared displacement of a Brownian particle is given by $\langle x^2(t) \rangle \propto t$; this linear dependence on time is referred to as normal diffusion. In complex media this kind of behavior is often violated, leading to anomalous diffusion. For a subdiffusive process $\langle x^2(t) \rangle \propto t^\nu$, with $0 < \nu < 1$. For this process, fractional dynamic equations emerge naturally in the physical concept of continuous time random walks [6, 7].

Fractional differential equations have many applications in applied science and engineering [8–11]. Fractional differential equations have been investigated in pure sciences, such as pure mathematics [12].

As a fractional generalization of the oscillation phenomena, one can consider [13]

$$\frac{d^\alpha x(t)}{dt^\alpha} + x(t) = 0, \quad 1 < \alpha < 3, \quad t > 0. \quad (1)$$

The case with $1 < \alpha < 2$ corresponds to attenuated oscillation phenomenon [14–19]. With the initial conditions $x(0) = 1$ and $x'(0) = 0$, the solution is $x(t) = E_{\alpha,1}(-t^\alpha)$, the so-called Mittag-Leffler function.

The case with $2 < \alpha < 3$ corresponds to amplified oscillation phenomenon. With the initial conditions $x(0) = 1$, $x'(0) = 0$, and $x''(0) = 0$, the solution is $x(t) = E_{\alpha,1}(-t^\alpha)$. In [14] it has been discarded on the ground that it signifies the instability of the system. In a recent study [11] it has

been proven that the Mittag-Leffler function of this order is a Nussbaum function. Hence it may have applications in the control theory of electrical engineering. In another study [12] general equations of the type

$$\frac{d^\alpha x(t)}{dt^\alpha} + f(x(t)) = 0, \quad 2 < \alpha < 3, \quad t > 0 \quad (2)$$

have been considered; however in this work the emphasis is on the existence and uniqueness of solutions.

There are various representations of the Mittag-Leffler function:

- (i) series representation [13],
- (ii) integral representation [14],
- (iii) approximate representation for a medium with low level of fractionality [17, 19, 20].

In this work we use the approximate representation to probe fractional differential equations of order $2 < \alpha < 3$.

The plan of this paper is as follows.

In Section 2 we describe this approximate representation. In Section 3 we obtain the solution for (1) and we compare it with the exact results of [11, 14].

In Section 4 we propose a new fractional oscillator of the form

$$\frac{d^\alpha x(t)}{dt^\alpha} + b \frac{d^\beta x(t)}{dt^\beta} + x(t) = 0, \quad 2 < \alpha < 3, \quad t > 0, \quad (3)$$

where $\beta = \alpha/2$. We obtain the solution for this damped oscillator, and we discuss the time rate of change of this oscillator. Finally in Section 5 we present our conclusions.

2. The Perturbation Scheme

There exists a multitude of definitions [1–5] like Riemann-Liouville, Weyl, Reisz, and Caputo for the fractional derivative. However, the fractional operator is uniquely defined. For instance, Riemann-Liouville and Caputo formulations emerge by choice (how to include the initial values) and are fully equivalent.

In this paper we only use the Caputo fractional derivative as it is easier to apply the initial conditions in this type. The left(forward) Caputo fractional derivative for $\alpha > 0$ is defined by

$$D^\alpha f(t) = {}^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad (4)$$

$$n-1 < \alpha < n, \quad t > 0,$$

where n is an integer number and $f^{(n)}(\tau)$ denotes the n th derivative of the function $f(\tau)$.

To construct a solution for a process described by an equation with Caputo fractional derivative, one needs the initial conditions that can be written as

$$f^{(k)}(0) = c_k, \quad k = 0, 1, \dots, (n-1). \quad (5)$$

In this work we seek causal solution. Hence we require the condition $f(t) = 0$ for $t < 0$ [13]. At the time $t = 0$ the solution is determined by initial condition equation (5). And for $t > 0$ it is obtained from the fractional differential equation describing the process under consideration. Now if the order of the fractional derivative α is close to a positive integer, namely, $\alpha = n - \varepsilon$, with small positive ε [17], then we will have

$$D^{n-\varepsilon} f(t) = \frac{1}{\Gamma(\varepsilon)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{1-\varepsilon}} d\tau, \quad 0 < \varepsilon < 1, \quad t > 0. \quad (6)$$

As noted in [17] this representation is not convenient in the limit $\varepsilon \rightarrow 0$, since

$$\frac{1}{\Gamma(\varepsilon)} = \varepsilon + O(\varepsilon^2). \quad (7)$$

However, in this case from integration by parts we find

$$D^{n-\varepsilon} f(t) = \frac{f^{(n)}(0)t^\varepsilon}{\Gamma(1+\varepsilon)} + \frac{1}{\Gamma(1+\varepsilon)} \int_0^t f^{(n+1)}(\tau)(t-\tau)^\varepsilon d\tau, \quad (8)$$

$$0 < \varepsilon < 1, \quad t > 0.$$

Now from the expansion

$$\frac{(t-\tau)^\varepsilon}{\Gamma(1+\varepsilon)} = 1 + \varepsilon [\gamma + \ln(t-\tau)]$$

$$+ \varepsilon^2 \left[\frac{1}{2} \ln^2(t-\tau) + \gamma \ln(t-\tau) + \frac{\gamma^2}{2} - \frac{\pi^2}{12} \right]$$

$$+ \dots, \quad (9)$$

where $\tau < t$ and $\gamma = 0.5552156\dots$ is the Euler constant. Hence we obtain

$$D^{n-\varepsilon} f(t) = f^{(n)}(t) + \varepsilon D_1^n f(t), \quad (10)$$

where $D_1^n f(t)$ is

$$D_1^n f(t) = f^{(n)}(0) \ln t + \gamma f^{(n)}(t) + \int_0^t f^{(n+1)}(\tau) \ln(t-\tau) d\tau. \quad (11)$$

In order to get a good approximation from (6) we find the following conditions:

- (i) $\varepsilon \ll 1$,
- (ii) $\varepsilon \ln(t-\tau) \ll 1$.

Next we generalize this expansion for the case, where $\alpha = n + \varepsilon$, with small positive ε . We note that in this case $n < \alpha < n+1$; therefore

$$D^\alpha f(t) = \frac{1}{\Gamma(n+1-\alpha)} \int_0^t \frac{f^{(n+1)}(\tau)}{(t-\tau)^{(\alpha-n)}} d\tau, \quad (12)$$

$$n < \alpha < n+1, \quad t > 0.$$

So

$$D^{n+\varepsilon} f(t) = \frac{1}{\Gamma(1-\varepsilon)} \int_0^t \frac{f^{(n+1)}(\tau)}{(t-\tau)^\varepsilon} d\tau, \quad (13)$$

$$0 < \varepsilon < 1, \quad t > 0.$$

By expansion in terms of the parameter ε we get

$$D^{n+\varepsilon} f(t) = f^{(n)}(t) - f^{(n)}(0) - \varepsilon D_1^n f(t). \quad (14)$$

But the process described by fractional derivative $D^{n+\varepsilon}$ has one extra initial condition more than the process described by $D^{n-\varepsilon}$, namely, the condition $f^{(n)}(0) = c_n$. However, if we assume $c_n = 0$, then our expansion and hence the solutions will have a continuous dependence on the parameter α in transition from $\alpha = n - \varepsilon$ to $\alpha = n + \varepsilon$. In other words if we set $c_n = 0$, then we get

$$D^\alpha f(t) = f^{(n)}(t) + (n-\alpha) D_1^n f(t), \quad (15)$$

$$\text{with } \alpha = n \mp \varepsilon, \quad 0 < \varepsilon < 1.$$

Now we consider (1) with $\alpha = n + \varepsilon$ and the initial conditions which are specified by

$$x^{(k)}(0) = \eta_k, \quad k = 0, 1, \dots, (n-1). \quad (16)$$

The perturbation expansion of the dynamical variable is given by

$$x(t) = x_0(t) + \varepsilon x_1(t), \quad (17)$$

where $x_0(t)$ is the solution of the integer case, and it is described by

$$\frac{d^n x_0(t)}{dt^n} + x_0(t) = 0, \quad t > 0, \quad (18)$$

and it is subject to

$$x_0^{(k)} = \eta_k, \quad k = 0, 1, \dots, (n-1). \quad (19)$$

The term $\varepsilon x_1(t)$ is a correction term, and it designates the deviation from the integer case. By substituting (15), (17) in (1) we obtain

$$\frac{d^n x_1(t)}{dt^n} - D_1^n x_0(t) + x_1(t) = 0, \quad t > 0, \quad (20)$$

subject to

$$x_1^{(0)} = 0, \quad k = 0, 1, \dots, (n-1). \quad (21)$$

By assuming these set of initial conditions we obtain the correct result in the limit of $\varepsilon \rightarrow 0$.

3. Amplified Oscillations

The phenomena of amplified oscillations correspond to the case $2 < \alpha < 3$, in (1). It has been discussed in [11, 12, 14].

In control theory, a branch of electrical engineering, one utilizes a closed feedback loop to guarantee the stability of the system. In [11] it is proved that the solution of (1) is a Nussbaum function. Hence when it is used as a part of a general system, it can guarantee the asymptotic stability of this larger system. But this discussion is restricted to a specific field of applied science, namely, control theory.

In [12] the general properties of (2) such as uniqueness and existence have been investigated. But it does not consider the behavior of the solution.

In [14] (1) is briefly discussed; however it was discarded on the grounds that it signals the instability of the system under considerations.

Our strategy to overcome this difficulty is

- (1) to consider the media with low-level fractionality,
- (2) to add a damping term to remove this instability.

In this section we investigate the limit, where $\alpha = 2 + \varepsilon$, so

$$\frac{d^{2+\varepsilon} x(t)}{dt^{2+\varepsilon}} + x(t) = 0, \quad t > 0. \quad (22)$$

For the normalized initial conditions $x_0(0) = 1$, $x_0'(0) = 0$, and $x_0''(0) = 0$ one obtains [19]

$$x(t) = \cos(t) + \varepsilon x_1(t), \quad \varepsilon \ll 1, \quad (23)$$

where

$$x_1(t) = \frac{1}{2} [1 - \cos(t) + t \text{Si}(t) \cos(t) - t \text{Ci}(t) \sin(t)]. \quad (24)$$

Here $\text{Si}(t)$ and $\text{Ci}(t)$ are sine and cosine integral functions, respectively [21],

$$\text{Si}(t) = \int_0^t \frac{\sin(u)}{u} du, \quad \text{Ci}(t) = - \int_t^\infty \frac{\cos(u)}{u} du. \quad (25)$$

For this process we define the momenta by

$$p(t) = \frac{d^\beta x(t)}{dt^\beta}, \quad \text{where } \beta = 1 + \frac{\varepsilon}{2}. \quad (26)$$

From (26) we find [19]

$$p(t) = -\sin(t) + \varepsilon p_1(t), \quad (27)$$

where

$$p_1(t) = \frac{1}{2} [\sin(t) - t \text{Si}(t) \sin(t) - t \text{Ci}(t) \cos(t)]. \quad (28)$$

Hence the total energy of this oscillator is

$$E_F = \frac{1}{2} (x^2 + p^2) = \frac{1}{2} - \frac{\varepsilon}{2} [1 - \cos(t) - t \text{Si}(t)]. \quad (29)$$

It is possible to calculate the time rate of change of the energy. The result is

$$\frac{dE_F}{dt} = \frac{\varepsilon}{2} \text{Si}(t), \quad \varepsilon > 0, \quad t > 0. \quad (30)$$

Therefore, the time rate of change of the energy is always positive. Hence the total energy of the system is a monotonous increasing function of time, which means that the system absorbs energy from the environment. For instance, if we take $\varepsilon = 10^{-10}$, then we have about one percent increase in the value of E_F in a year. For $\varepsilon = 10^{-5}$ we have about one percent increase in the value of E_F in a day. These values are chosen in a way that the condition $\varepsilon t \ll 1$ is satisfied [17].

Thus, our study suggests an absorption interpretation for fractional derivative of order $\alpha = 2 + \varepsilon$ for the solution of (1), to be contrasted with the common dissipation interpretation for fractional derivative of order $\alpha = 2 - \varepsilon$.

3.1. Comparison with the Exact Result. With our choice of the initial condition, the exact result is

$$x(t) = E_\alpha(-t^\alpha) = f_\alpha(t) + g_\alpha(t), \quad (31)$$

where the one parameter Mittag-Leffler function is

$$E_\alpha = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (32)$$

$$x(t) = E_\alpha(-t^\alpha) = f_\alpha(t) + g_\alpha(t), \quad (33)$$

where

$$f_\alpha(t) = \frac{1}{\pi} \int_0^\infty \exp(-rt) \frac{r^{\alpha-1} \sin(\alpha\pi)}{r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1} dr. \quad (34)$$

The function $g_\alpha(t)$ is given by

$$g_\alpha(t) = \frac{2}{\alpha} \exp\left(t \cos\left(\frac{\pi}{\alpha}\right)\right) \cos\left[t \sin\left(\frac{\pi}{\alpha}\right)\right], \quad 2 < \alpha < 3. \quad (35)$$

Using $\alpha = 2 + \varepsilon$ and the condition $\varepsilon t \ll 1$, we obtain

$$\begin{aligned} f_\alpha(t) &= \frac{\varepsilon}{2} \left[1 - t \sin(t) \text{Ci}(t) + t \cos(t) \text{Si}(t) - t \frac{\pi}{2} \cos(t) \right] \\ &\quad + \dots \\ g_\alpha(t) &= \cos(t) - \varepsilon \left[\frac{1}{2} - t \frac{\pi}{4} \right] \cos(t). \end{aligned} \quad (36)$$

By substituting (36) into (33) we recover the result expressed in (23).

4. An Especial Fractional Oscillator

In this section we investigate the damped fractional oscillator described by (3). For simplicity we consider a case where the damping coefficient is $b = 2 \cos(\pi/\alpha)$. In the media with low-level fractionality $\alpha = 2 + \varepsilon$. By inserting (17) in (3) we get

$$\frac{d^2 x_0(t)}{dt^2} + x_0(t) = 0, \quad t > 0. \quad (37)$$

With the initial conditions $x_0(0) = 1$ and $x'_0(0) = 0$, we will have

$$x(t) = \cos(t) + \varepsilon x_1(t), \quad t > 0. \quad (38)$$

The perturbation term $x_1(t)$ satisfies

$$\frac{d^2 x_1(t)}{dt^2} - D_1^2 x_0(t) + \frac{\pi}{2} \frac{dx_0(t)}{dt} + x_1(t) = 0, \quad t > 0, \quad (39)$$

with the initial conditions $x_1(0) = 0$ and $x'_1(0) = 0$. By assuming these initial conditions we obtain the correct result in the limit of $\varepsilon \rightarrow 0$. From (39) and after some calculations we obtain

$$\begin{aligned} x_1(t) &= \frac{1}{2} [1 - \cos(t) + t \text{Si}(t) \cos(t) - t \text{Ci}(t) \sin(t)] \\ &\quad + \frac{\pi}{4} [\sin(t) - t \cos(t)]. \end{aligned} \quad (40)$$

For this process we define the momenta by

$$p(t) = \frac{d^\beta x(t)}{dt^\beta}, \quad \text{where } \beta = 1 + \frac{\varepsilon}{2}. \quad (41)$$

From (41) we find

$$p(t) = -\sin(t) + \varepsilon p_1(t), \quad (42)$$

where

$$\begin{aligned} p_1(t) &= \frac{1}{2} [\sin(t) - t \text{Si}(t) \sin(t) - t \text{Ci}(t) \cos(t)] \\ &\quad + \frac{\pi}{4} t \sin(t). \end{aligned} \quad (43)$$

Hence the total energy of this oscillator is

$$\begin{aligned} E_F &= \frac{1}{2} (x^2 + p^2) = \frac{1}{2} - \frac{\varepsilon}{2} [1 - \cos(t) - t \text{Si}(t)] \\ &\quad + \frac{\varepsilon \pi}{4} [\sin(t) \cos(t) - t]. \end{aligned} \quad (44)$$

It is possible to calculate the time rate of change of the energy. The result is

$$\frac{dE_F}{dt} = \frac{\varepsilon}{2} \text{Si}(t) - \frac{\varepsilon \pi}{2} \sin^2(t), \quad \varepsilon > 0, \quad t > 0. \quad (45)$$

The first term and the second term in the right hand side of (45) are due to the first and the second term of (3), respectively. The first term is positive and denotes absorption of energy from the environment, and the second term is always negative and shows the dissipation of energy from the system to the environment. For large values of t we have $\text{Si}(t) \rightarrow \pi/2$. Hence from (45)

$$\frac{dE_F}{dt} \approx \frac{\varepsilon \pi}{4} \cos(2t), \quad \varepsilon > 0. \quad (46)$$

We see that the time average of time rate of change of energy of this system is zero; namely,

$$\left\langle \frac{dE_F}{dt} \right\rangle = 0, \quad \varepsilon > 0. \quad (47)$$

5. Conclusions

Previous studies of fractional oscillation were limited to the case of $0 < \alpha < 2$; see [13–19]. The solutions portray damped motion. At the limit of $t \rightarrow \infty$ we have the algebraic decay of the solutions. These solutions are of relevance in vibration of mechanical systems or oscillations of electrical network where the energy is dissipated in the form of heat. Hence in this domain of the values of the parameter α , fractional derivative is a convenient tool for modeling of damping.

The results of Section 3 indicate that we may use fractional derivative to model amplification process as well.

In Section 4 we introduced a damped oscillator for a case where the order of fractional derivative is $2 + \varepsilon$. The model presented in Section 4 has not been considered for general values of $2 < \alpha < 3$. It will be of interest to go beyond the small epsilon expansion. We conjecture that analytical solution for this model exists. We plan to report on these and other related issues in the future.

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