Research Article

New Traveling Wave Solutions to the Vakhnenko-Parkes Equation

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We apply the improved (G'/G) expansion method to the Vakhnenko-Parkes equation. As a result, many new and more general exact solutions have been obtained for the equation. Comparing our solutions with those gained by other authors indicates that the improved (G'/G) expansion is more effective in solving the general solutions to differential equations.

1. Introduction

Many phenomena in physics and other fields such as biology chemistry and mechanics. are described by nonlinear partial differential equations (NLPDEs). The investigation of traveling wave solutions to nonlinear partial differential equations (NLPDEs) plays an important role in the study of nonlinear physical phenomena. In the past several decades, both mathematicians and physicists have made significant progress in this direction.

Many effective methods [1–10] have been presented such as exp-function method [1], Hirota's method [2], variational iteration method [3], the homogeneous balance method [4], Backlund and Darboux transformation method [5], the sine-cosine function method [6], the Jacobi elliptic function method [7], and auxiliary equation method [8–10].

Recently, an interesting and important discovery has been made by Vakhnenko and Parkes [11], who have demonstrated that the reduced Ostrovsky equation [12]

$$(u_t + c_0 u_x + \alpha u u_x)_x = \gamma u \quad (1)$$

can be transformed to the new integrable equation

$$u u_{xxt} - u_x u_{xt} + u^2 u_t = 0. \quad (2)$$

In the literature, the traveling wave solutions of the Vakhnenko-Parkes equation (2) are investigated by the improved tanh function method introduced in [13, 14], auxiliary equation method [10], and G'/G -expansion method [15].

The present paper is motivated by the desire to improve the work made in [4, 10, 13–15] by proposing a new improved G'/G -expansion method to construct more general exact solutions of nonlinear partial differential equations (NLPDEs). For illustration, we restrict our attention to the study of the Vakhnenko-Parkes equation (2) and successfully construct many new and more general exact solutions.

The rest of this paper is organized as follows: we give the description of the improved G'/G -expansion method in Section 2. In Section 3, we apply this method to (2). In Section 4, some conclusions are given.

2. Description of the Improved G'/G -Expansion Method

To make this paper entire, in here we enumerate the same method reported in [16]. Suppose that we have a NLPDE for $u(x, t)$ in the form

$$P(u, u_x, u_t, u_{xt}, u_{xx}, \dots) = 0, \quad (3)$$

where P is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives. The transformation $u(x, t) = u(\xi)$, $\xi = x - \omega t$ reduces (3) to the ordinary differential equation (ODE)

$$H(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \dots) = 0. \quad (4)$$

By virtue of the extended tanh-function method, we assume that the solution of (4) is in the following form

$$u(\xi) = \sum_{i=0}^m a_i F^i(\xi), \quad (5)$$

in which a_i ($i = 0, 1, 2, \dots, m$) are all real constants to be determined and the balancing number m is a positive integer which can be determined by balancing the highest derivative terms with the highest power nonlinear terms in (4), and $F(\xi)$ is

$$F(\xi) = \frac{G'(\xi)}{G(\xi)}, \quad (6)$$

where $G(\xi)$ expresses the solution of the following auxiliary ordinary differential equation:

$$GG'' = AG^2 + BGG' + C(G')^2, \quad (7)$$

where the prime denotes derivative with respect to ξ . A , B , and C are real parameters.

Using the general solutions of (7) and with the help of Maple, we have the following.

Case 1. For $B \neq 0$ and $\Delta = B^2 + 4A - 4AC \geq 0$,

$$F(\xi) = \frac{B}{2(1-C)} + \frac{B\sqrt{\Delta}}{2(1-C)} \frac{c_1 e^{(\sqrt{\Delta}/2)\xi} + c_2 e^{-(\sqrt{\Delta}/2)\xi}}{c_1 e^{(\sqrt{\Delta}/2)\xi} - c_2 e^{-(\sqrt{\Delta}/2)\xi}}. \quad (8)$$

Case 2. For $B \neq 0$ and $\Delta = B^2 + 4A - 4AC < 0$,

$$F(\xi) = \frac{B}{2(1-C)} + \frac{B\sqrt{-\Delta}}{2(1-C)} \frac{ic_1 \cos(\sqrt{-\Delta}/2)\xi - c_2 \sin(\sqrt{-\Delta}/2)\xi}{ic_1 \sin(\sqrt{-\Delta}/2)\xi + c_2 \cos(\sqrt{-\Delta}/2)\xi}. \quad (9)$$

Case 3. For $B = 0$ and $A(C-1) \geq 0$,

$$F(\xi) = \frac{\sqrt{A(C-1)}}{(1-C)} \times \frac{c_1 \cos(\sqrt{A(C-1)}\xi) + c_2 \sin(\sqrt{A(C-1)}\xi)}{c_1 \sin(\sqrt{A(C-1)}\xi) - c_2 \cos(\sqrt{A(C-1)}\xi)}. \quad (10)$$

Case 4. For $B = 0$ and $A(C-1) < 0$,

$$F(\xi) = \frac{\sqrt{-A(C-1)}}{(1-C)} \times \frac{ic_1 \cosh(\sqrt{-A(C-1)}\xi) - c_2 \sinh(\sqrt{-A(C-1)}\xi)}{ic_1 \sinh(\sqrt{-A(C-1)}\xi) - c_2 \cosh(\sqrt{-A(C-1)}\xi)}. \quad (11)$$

Herein, A , B , C and c_1 , c_2 are real parameters.

3. Traveling Wave Solutions for the Vakhnenko-Parkes Equation

We now consider the improved G'/G -expansion method to obtain the traveling wave solutions of (2); substituting

$$u = u(\xi) = u(x - wt) \quad (12)$$

into (2), integrating once with respect to ξ , and setting the integration constant equal to zero yield

$$3uu'' - 3u'^2 + u^3 = 0. \quad (13)$$

Balancing u^3 with uu'' gives the leading order $m = 2$, so we take the ansatz

$$u(\xi) = a_0 + a_1 F(\xi) + a_2 F^2(\xi), \quad (14)$$

where a_0 , a_1 , and a_2 are constant and need to be determined and $F(\xi)$ express the solution of (6). Substituting (14) and (7) along with (6) into (13) and using Maple yield a system of equations of F^i ; setting the coefficients of F^i ($i = 0, 1, 2, \dots, 6$) in the obtained system of equations to zero, we can deduce the following set of algebraic polynomials with respect to unknown a_0 , a_1 , and a_2 ; namely,

$$\begin{aligned} &6a_2^2 + 6a_2^2 C^2 - 12a_2^2 C + a_2^3 = 0, \\ &6a_2^2 BC - 6a_2^2 B + 3a_2^2 a_1 + 12a_1 a_2 C^2 - 24a_1 a_2 C + 12a_1 a_2 = 0, \\ &-6a_1^2 C + 15a_1 a_2 CB - 15a_1 a_2 B + 18a_0 a_2 C^2 - 36a_0 a_2 C \\ &+ 3a_0 a_2^2 + 3a_1^2 + 3a_1^2 a_2 + 18a_0 a_2 + 3a_2^2 C^2 = 0, \\ &-12a_0 a_1 C + 6a_1 a_2 AC - 3a_1^2 B + 3a_1^2 BC - 6a_2^2 AB \\ &+ 6a_0 a_1 a_2 + 3a_1 a_2 B^2 + 6a_0 a_1 + 30a_0 a_2 BC \\ &- 6a_1 a_2 A - 30a_0 a_2 B + a_1^3 + 6a_0 a_1 C^2 = 0, \\ &-6a_2^2 A^2 - 9a_0 a_1 B - 3a_1 a_2 AB + 3a_0 a_1^2 + 3a_0^2 a_2 \\ &+ 24a_0 a_2 AC + 9a_0 a_1 BC - 24a_0 a_2 A + 12a_0 a_2 B^2 = 0, \\ &3a_0 a_1 B^2 + 6a_0 a_1 AC - 3a_1^2 AB - 6a_0 a_1 A + 3a_0^2 a_1 \\ &- 6a_1 a_2 A^2 + 18a_0 a_2 AB = 0, \\ &6a_0 a_2 A^2 + 3a_0 a_1 AB - 3a_1^2 A^2 + a_0^3 = 0. \end{aligned} \quad (15)$$

Solving the set of algebraic equations by the use of Maple, we get the following results:

$$\begin{aligned} a_0 &= -6AC + 6A; & a_1 &= -6B(C-1); \\ a_2 &= -6(C-1)^2. \end{aligned} \quad (16)$$

Substituting (16) and (12) along with $F(\xi)$ in (8), (9), (10), and (11) into (14), we obtain the following exponential

function solutions, hyperbolic function solutions, and triangular function solutions of (2). These solutions are as follows:

- (1) when we choose $B \neq 0$ and $\Delta = B^2 + 4A - 4AC \geq 0$, then the exponential function solutions can be found as

$$u(x, t) = -6AC + 6A + \frac{3}{2}B^2 - \frac{3}{2}\Delta B^2 \left[\frac{c_1 e^{(\sqrt{\Delta}/2)(x-\omega t)} + c_2 e^{-(\sqrt{\Delta}/2)(x-\omega t)}}{c_1 e^{(\sqrt{\Delta}/2)(x-\omega t)} - c_2 e^{-(\sqrt{\Delta}/2)(x-\omega t)}} \right]^2; \quad (17)$$

- (2) when we choose $B \neq 0$ and $\Delta = B^2 + 4A - 4AC < 0$, the triangular function solutions will be

$$u(x, t) = -6AC + 6A + \frac{3}{2}B^2 + \frac{3}{2}\Delta B^2 \times \left[\frac{ic_1 \cos(\sqrt{-\Delta}/2)(x-\omega t) - c_2 \sin(\sqrt{-\Delta}/2)(x-\omega t)}{ic_1 \sin(\sqrt{-\Delta}/2)(x-\omega t) + c_2 \cos(\sqrt{-\Delta}/2)(x-\omega t)} \right]^2; \quad (18)$$

- (3) if we choose $B = 0$ and $A(C - 1) \geq 0$, then the triangular function solutions are

$$u(x, t) = -6AC + 6A - 6A(C - 1) \times \left[\frac{c_1 \cos(\sqrt{A(C-1)}(x-\omega t)) + c_2 \sin(\sqrt{A(C-1)}(x-\omega t))}{c_1 \sin(\sqrt{A(C-1)}(x-\omega t)) - c_2 \cos(\sqrt{A(C-1)}(x-\omega t))} \right]^2; \quad (19)$$

- (4) again, when we choose $B = 0$ and $A(C - 1) < 0$, then hyperbolic function solutions are

$$u(x, t) = -6AC + 6A + 6A(C - 1) \times \left[\frac{ic_1 \cosh(\sqrt{-A(C-1)}(x-\omega t)) - c_2 \sinh(\sqrt{-A(C-1)}(x-\omega t))}{ic_1 \sinh(\sqrt{-A(C-1)}(x-\omega t)) - c_2 \cosh(\sqrt{-A(C-1)}(x-\omega t))} \right]^2, \quad (20)$$

where A, B, C and c_1, c_2 are real parameters.

4. Conclusion and Discussion

In this work, we have presented an improved G'/G -expansion method and applied it to obtain new traveling wave solutions of the Vakhnenko-Parkes equation. In contrast to other G'/G -expansion methods, some benefits are available for this method.

First, all the nonlinear PDEs which can be solved by other G'/G -expansion methods can be solved easily by this method. We have successfully obtained many new exact traveling wave solutions. To our knowledge, these solutions have not been reported in the former literature.

Second, if we used the special value of parameters c_1, c_2 and A, B, C , we can obtain some traveling wave solutions which have been found by Yaşar [13], such as the following.

If we suppose that $B^2 = 1, A - AC = 3/4$, the exponential function solutions (17) at $c_1 = c_2$ can be rewritten as follows:

$$u(x, t) = 6 - 6\coth^2(x - \omega t), \quad (21)$$

while at $c_1 = -c_2$, they yield

$$u(x, t) = 6 - 6\tanh^2(x - \omega t). \quad (22)$$

Similarly, when we choose $A - AC = -1$, it is easy to see that the trigonometric function solutions (19) at $c_1 \neq 0, c_2 = 0$ can be rewritten as follows:

$$u(x, t) = -6 - 6\cot^2(x - \omega t), \quad (23)$$

while at $c_1 = 0, c_2 \neq 0$, one can obtain

$$u(x, t) = -6 - 6\tan^2(x - \omega t). \quad (24)$$

In [3], the authors yield (21), (22), (23), and (24) by using tanh function method.

On the other hand, in [10], Kangalgil and Ayaz have derived a traveling wave solution

$$u(x, t) = \frac{3}{2}a \left[2(1 + \varepsilon \coth \eta) - (1 + \varepsilon \coth \eta)^2 \right] \quad (25)$$

by the auxiliary equation method, where $\eta = \sqrt{a}/2, a > 0$.

Choosing $B^2 = 1, \Delta = a$, and $c_1 = c_2$ in (17) and after some simplifications, we obtain (25) at $\varepsilon = 1$.

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