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Research Article

On the Problem of Electromagnetic Waves Propagating along a Nonlinear Inhomogeneous Cylindrical Waveguide

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Electromagnetic TE wave propagation in an inhomogeneous nonlinear cylindrical waveguide is considered. The permittivity inside the waveguide is described by the Kerr law. Inhomogeneity of the waveguide is modeled by a nonconstant term in the Kerr law. Physical problem is reduced to a nonlinear eigenvalue problem for ordinary differential equations. Existence of propagating waves is proved with the help of fixed point theorem and contracting mapping method. For numerical solution, an iteration method is suggested and its convergence is proved. Existence of eigenvalues of the problem (propagation constants) is proved and their localization is found. Conditions of k waves existence are found.

1. Introduction

Electromagnetic wave propagation in *linear* (homogeneous and inhomogeneous) waveguide plane layers and cylindrical waveguides with circular cross section is of particular interest in linear optics (see, e.g., [1, 2]). In nonlinear optics, waveguides (plane and cylindrical) filled with nonlinear medium have been the focus of a number of studies [3–11]. However, many of researches are devoted to study *homogeneous nonlinear* waveguides [6–11].

Problems of electromagnetic wave propagation in *nonlinear* waveguides (plane and cylindrical) lead to nonlinear boundary and transmission eigenvalue problems for ordinary differential equations. Eigenvalues in these problems correspond to propagation constants of the waveguides. In these problems differential equations depend nonlinearly either on sought-for functions and the spectral parameter. Boundary and/or transmission conditions depend nonlinearly on the spectral parameter. The main goal is to prove existence of eigenvalues and determine their localization. Existence and localization can be derived from the dispersion equation (DE). DE is an equation with respect to spectral parameter. There are two ways to obtain the DE. The first one is to integrate the differential equations and obtain, using boundary and/or transmission conditions, the DE. This way is of very limited applicability, as it is very rarely possible to find explicit solutions of nonlinear differential equations. However, there are some problems in which this way works (see, e.g., [10, 12, 13]). The second one is a very general approach based on reduction of the differential equations to integral equations using the Green function. This approach we call integral equation approach. Here we consider this very method. Inspite of the fact that by this method the DE is found in an implicit form, it is possible to prove existence of eigenvalues and find their localization.

Electromagnetic guided waves in a cylindrical waveguide with Kerr nonlinearity are considered in [6]. It is one of the first studies, which we know about, where electromagnetic wave propagation in nonlinear medium is considered in a rigorous electromagnetic statement. Then there were a lot of researches devoted to study Kerr nonlinearity in homogeneous plane and cylindrical waveguides. For more details, about Kerr nonlinearity and homogeneous plane and cylindrical waveguides see the following references: TE guided waves in a plane layer were investigated in [12, 14], and additional results were obtained in [13]; TM guided waves in a plane layer were investigated in [15–20]; TE guided waves in a cylindrical waveguide were investigated in [21–23]; TM guided waves in a cylindrical waveguide were investigated in [24].

In most cases it is very difficult (if at all possible) to obtain exact solutions of the equations in nonlinear waveguiding problems. However, integral equation approach can help in this case [21–25]. In this approach a problem is reduced to an integral equation whose kernel depends on the Green function of the linear part of the differential equations of the problem. Two circumstances are important for the following analysis. First, in the case of a homogeneous waveguide this Green function can be found explicitly. Second, the dispersion equation of the nonlinear homogeneous case can be written as $\mathrm{DE}_{\mathrm{lin}} + T_{\mathrm{nonlin}} = 0$, where $\mathrm{DE}_{\mathrm{lin}}$ is a linear problem term and T_{nonlin} is an extra nonlinear term. Here the linear problem term is written in an explicit form. Moreover, the equation $\mathrm{DE}_{\mathrm{lin}} = 0$ is well known and examined DE for the linear problem. Its roots are also known. All this allows to prove existence of the nonlinear problem solutions at least near to the linear problem solutions.

Here we investigate guided waves in a nonlinear inhomogeneous cylindrical waveguide filled with Kerr medium. The waveguide is placed in cylindrical coordinate system $O\rho\varphi z$, where axis z coincides with axis of the waveguide. Inhomogeneity is modeled by a function that depends on radius of the waveguide. The permittivity inside the waveguide is $\varepsilon = \varepsilon_2(\rho) + a|\mathbf{E}|^2$, where $\varepsilon_2(\rho)$ is the inhomogeneity, a is a constant in the Kerr law, and $\mathbf{E}$ is complex amplitude. If $\varepsilon_2(\rho) \equiv$ const we have a nonlinear homogeneous waveguide. The nonconstant term $\varepsilon_2(\rho)$ dramatically changes the situation. In this case we cannot find explicitly the necessary Green function, so we investigate it in an implicit form. The dispersion equation of the nonlinear inhomogeneous case can be also written as $\mathrm{DE}_{\mathrm{lin}} + T_{\mathrm{nonlin}} = 0$. However, in this case the term $\mathrm{DE}_{\mathrm{lin}}$ is written in an implicit form as opposed to the case of a homogeneous waveguide, and its roots are unknown. So, at first, we prove that the equation for the linear inhomogeneous problem $\mathrm{DE}_{\mathrm{lin}} = 0$ has roots and define localization of the roots. Then we prove that nonlinear problem has solutions.

Integral equation approach has been already used for a nonlinear inhomogeneous waveguiding problem [26]. However in study [26] authors apply integral equation approach in the way as they would solve the problem for a homogeneous waveguide. To be precise, the authors use the Green function for constant ε_2 that helps them to determine the Green function in explicit form. We pay heed that there are no theoretical results (existence of eigenvalues and their localization) in [26]. We emphasize that for inhomogeneous waveguides important and general results can be obtained with the method we use in this paper in which the Green function has implicit form.

In spite of the fact that the method here looks similar to the method in [21–24], we solve radically different problem, as we consider *inhomogeneous nonlinear* waveguide.

2. Statement of the Problem

Let us consider three-dimensional space $\mathbb{R}^3$ with cylindrical coordinate system $O\rho\varphi z$. The space is filled by isotropic medium with constant permittivity $\varepsilon_1 \geq \varepsilon_0$, where ε_0 is the permittivity of free space. In this medium a cylindrical waveguide is placed. The waveguide is filled by isotropic nonmagnetic medium and has cross section $W := \{(\rho, \varphi) : \rho^2 < R^2, 0 \leq \varphi < 2\pi\}$ and its generating line (the waveguide axis) is parallel to the axis Oz. We will consider electromagnetic waves propagating along the waveguide axis. Everywhere below $\mu = \mu_0$ is the permeability of free space.

We use Maxwell's equations in the following form [27]:

$$\begin{aligned} \operatorname{rot} \widetilde{\mathbf{H}} &= \partial_t \widetilde{\mathbf{D}}, \\ \operatorname{rot} \widetilde{\mathbf{E}} &= -\partial_t \widetilde{\mathbf{B}}, \end{aligned} \tag{1}$$

where $\widetilde{\mathbf{D}} = \varepsilon \widetilde{\mathbf{E}}$, $\widetilde{\mathbf{B}} = \mu \widetilde{\mathbf{H}}$, and $\partial_t = \partial/\partial t$. Field $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})$ is the total field.

From formulae (1), we obtain

$$\begin{aligned} \operatorname{rot} \widetilde{\mathbf{E}} &= -\partial_t \left(\mu \widetilde{\mathbf{H}}\right), \\ \operatorname{rot} \widetilde{\mathbf{H}} &= \partial_t \left(\varepsilon \widetilde{\mathbf{E}}\right). \end{aligned} \tag{2}$$

Real monochromatic field $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})$ in the medium can be written in the following form:

$$\begin{aligned} \widetilde{\mathbf{E}}(\rho, \varphi, z, t) &= \mathbf{E}^+(\rho, \varphi, z) \cos \omega t + \mathbf{E}^-(\rho, \varphi, z) \sin \omega t, \\ \widetilde{\mathbf{H}}(\rho, \varphi, z, t) &= \mathbf{H}^+(\rho, \varphi, z) \cos \omega t + \mathbf{H}^-(\rho, \varphi, z) \sin \omega t, \end{aligned} \tag{3}$$

where ω is circular frequency; $\mathbf{E}^+$, $\mathbf{E}^-$, $\mathbf{H}^+$, and $\mathbf{H}^-$ are real required vectors.

Let us form complex amplitudes $\mathbf{E}$, $\mathbf{H}$:

$$\mathbf{E} = \mathbf{E}^+ + i\mathbf{E}^-, \qquad \mathbf{H} = \mathbf{H}^+ + i\mathbf{H}^-. \tag{4}$$

It is clear that

$$\widetilde{\mathbf{E}} = \operatorname{Re}\left\{\mathbf{E}e^{-i\omega t}\right\}, \qquad \widetilde{\mathbf{H}} = \operatorname{Re}\left\{\mathbf{H}e^{-i\omega t}\right\}, \tag{5}$$

where

$$\mathbf{E} = \left(E_\rho, E_\varphi, E_z\right)^T, \qquad \mathbf{H} = \left(H_\rho, H_\varphi, H_z\right)^T, \tag{6}$$

and components in (6) depend on three spatial variables.

It is known (see, e.g., [3, 6, 28]) that Kerr law in isotropic medium for a monochromatic wave $\mathbf{E}e^{-i\omega t}$ has the form $\varepsilon = \varepsilon_2 + a|\mathbf{E}|^2$, where $\mathbf{E}$ is complex amplitude, ε_2 is a constant part of the permittivity ε, a is the coefficient of nonlinearity.

We obtain that in this case dependence of Maxwell's equations on t is the same as in the case of constant ε inside the waveguide. This allows us to write Maxwell's equations (2) in the form

$$\begin{aligned} \operatorname{rot}\left(\mathbf{E}e^{-i\omega t}\right) &= i\omega\mu\mathbf{H}e^{-i\omega t}, \\ \operatorname{rot}\left(\mathbf{H}e^{-i\omega t}\right) &= -i\omega\varepsilon\mathbf{E}e^{-i\omega t}. \end{aligned} \tag{7}$$

Complex amplitudes (4) satisfy the Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}, \\ \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \end{aligned} \tag{8}$$

the continuity condition for the tangential components on the media interfaces (on the boundary of the waveguide) and

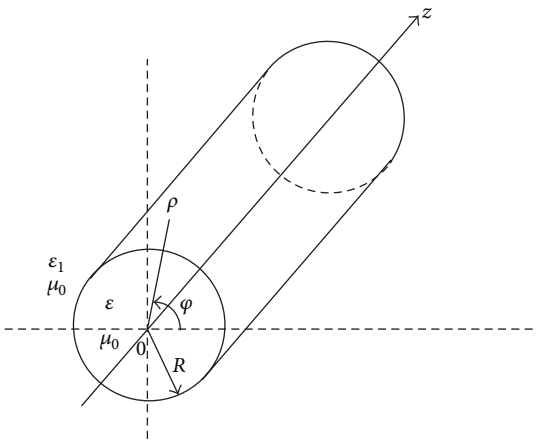

Figure 1: Geometry of the problem.

the radiation condition at infinity: the electromagnetic field exponentially decays as $\rho \to \infty$.

The permittivity in the entire space has the form

$$\varepsilon = \varepsilon_0 \begin{cases} \varepsilon_1, & \rho > R \\ \varepsilon_2(\rho) + a|\mathbf{E}|^2, & \rho < R, \end{cases} \tag{9}$$

where a is a real positive value, $\varepsilon_2(\rho) > \varepsilon_1$. Here $\varepsilon_2(\rho)$ is a linear part of the permittivity.

The solutions to the Maxwell equations are sought in the entire space.

Thereby, passing from time-dependent equations (1) to time-independent equations (8) is grounded on previous consideration.

Geometry of the problem is shown in Figure 1. The waveguide is infinite along axis Oz.

Let us consider TE waves with harmonical dependence on time

$$\mathbf{E}e^{-i\omega t} = e^{-i\omega t}\left(0, E_\varphi, 0\right)^T, \qquad \mathbf{H}e^{-i\omega t} = e^{-i\omega t}\left(H_\rho, 0, H_z\right)^T, \tag{10}$$

where $\mathbf{E}, \mathbf{H}$ are the complex amplitudes.

Substituting the complex amplitudes into Maxwell equations (8), we obtain

$$\begin{aligned} &\frac{1}{\rho}\frac{\partial H_z}{\partial \varphi} = 0, \qquad \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = -i\omega\varepsilon E_\varphi, \\ &\frac{1}{\rho}\frac{\partial H_\rho}{\partial \varphi} = 0, \qquad \frac{\partial E_\varphi}{\partial z} = -i\omega\mu H_\rho, \\ &\frac{1}{\rho}\frac{\partial\left(\rho E_\varphi\right)}{\partial \rho} = i\omega\mu H_z. \end{aligned} \tag{11}$$

It is obvious from the first and the third equations of this system that H_z and H_ρ do not depend on φ. This implies that E_φ does not depend on φ.

Independence of the components on φ can be explained if we chose dependence on φ in the form $e^{in\varphi}$ with $n = 0$.

Waves propagating along waveguide axis Oz depend harmonically on z. This means that the fields components have the form

$$\begin{aligned} &E_\varphi = E_\varphi(\rho)\, e^{i\gamma z}, \qquad H_\rho = H_\rho(\rho)\, e^{i\gamma z}, \\ &H_z = H_z(\rho)\, e^{i\gamma z}, \end{aligned} \tag{12}$$

where γ is the unknown spectral parameter of the problem (propagation constant).

So we obtain from system (11) that

$$\begin{aligned} &i\gamma H_\rho(\rho) - H_z'(\rho) = -i\omega\varepsilon E_\varphi(\rho), \\ &i\gamma E_\varphi(\rho) = -i\omega\mu H_\rho(\rho), \\ &\frac{1}{\rho}\left(\rho E_\varphi(\rho)\right)' = i\omega\mu H_z(\rho), \end{aligned} \tag{13}$$

where $(\,\cdot\,)' \equiv d/d\rho$.

Then $H_z(\rho) = (1/i\omega\mu)(1/\rho)(\rho E_\varphi(\rho))'$ and $H_\rho(\rho) = -(\gamma/\omega\mu)E_\varphi(\rho)$. From the first equation of the latter system, we obtain

$$\left(\frac{1}{\rho}\left(\rho E_\varphi(\rho)\right)'\right)' + \left(\omega^2\mu\varepsilon - \gamma^2\right)E_\varphi(\rho) = 0. \tag{14}$$

Denoting by $u(\rho) := E_\varphi(\rho)$, we obtain

$$u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u + \left(k_0^2\tilde{\varepsilon} - \gamma^2\right)u = 0 \tag{15}$$

and $\varepsilon = \tilde{\varepsilon}\varepsilon_0$, where

$$\tilde{\varepsilon} = \begin{cases} \varepsilon_1, & \rho > R, \\ \varepsilon_2(\rho) + au^2, & \rho < R, \end{cases} \tag{16}$$

and $k_0^2 = \omega^2\mu\varepsilon_0$.

Also we assume that function u is sufficiently smooth:

$$u(\rho) \in C[0,+\infty) \cap C^1[0,+\infty) \cap C^2(0,R) \cap C^2(R,+\infty). \tag{17}$$

Physical nature of the problem implies these conditions.

We will seek γ under conditions $k_0^2\varepsilon_1 < \gamma^2 < k_0^2 \min_{\rho\in[0,R]}\varepsilon_2(\rho)$.

In the domain $\rho > R$, we have $\tilde{\varepsilon} = \varepsilon_1$. From (15), we obtain the equation

$$u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u + k_1^2 u = 0, \tag{18}$$

where $k_1^2 = k_0^2\varepsilon_1 - \gamma^2$. It is the Bessel equation.

In the domain $\rho < R$, we have $\tilde{\varepsilon} = \varepsilon_2(\rho) + au^2$. From (15), we obtain the equation

$$u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u + k^2(\rho)u + \alpha u^3 = 0, \tag{19}$$

where $k^2(\rho) = k_2^2(\rho) - \gamma^2$, $k_2^2(\rho) = k_0^2\varepsilon_2(\rho)$, and $\alpha = ak_0^2$.

Tangential components of electromagnetic field are known to be continuous at media interfaces. Hence we obtain

$$E_\varphi(R+0) = E_\varphi(R-0), \qquad H_z(R+0) = H_z(R-0). \tag{20}$$

Further, we have $H_z(\rho) = (1/i\omega\mu)(1/\rho)E_\varphi(\rho) + E'_\varphi(\rho))$. Since $E_\varphi(\rho)$ and $H_z(\rho)$ are continuous at the point $\rho = R$, therefore, $E'_\varphi(\rho)$ is continuous at $\rho = R$. These conditions imply the transmission conditions for functions $u(\rho)$ and $u'(\rho)$

$$[u]_{\rho=R} = 0, \qquad \left[u'\right]_{\rho=R} = 0, \tag{21}$$

where $[f]_{x=x_0} = \lim_{x\to x_0-0} f(x) - \lim_{x\to x_0+0} f(x)$.

Let us formulate the transmission eigenvalue problem (problem P). It is necessary to find eigenvalues γ and correspond to them nonzero eigenfunctions $u(\rho)$ such that $u(\rho)$ satisfy (18), (19); transmission conditions (21) and the radiation condition at infinity: eigenfunctions exponentially decay as $\rho \to \infty$.

The general solution of (18) is taken in the following form $u(\rho) = bH_1^{(1)}(k_1\rho) + b_1H_1^{(2)}(k_1\rho)$, where $H_1^{(1)}$ and $H_1^{(2)}$ are the Hankel functions of the first and the second kinds, respectively. In accordance with the radiation condition we obtain that $b_1 = 0$; then the solution has the form $u(\rho) = bH_1^{(1)}(k_1\rho)$, $\rho > R$, where b is a constant. If $\operatorname{Re} k_1 = 0$, then

$$u(\rho) = \tilde{b}K_1(|k_1|\rho), \quad \rho > R, \tag{22}$$

as $H_1^{(1)}(iz) = -(2/\pi)K_1(z)$ and $K_1(z)$ is the Macdonald function.

The radiation condition is fulfilled since $K_1(|k_1|\rho) \to 0$ as $\rho \to \infty$.

3. Nonlinear Integral Equation and Dispersion Equation

Consider nonlinear equation (19) written in the form

$$\left(\rho u'\right)' + \left(k^2(\rho)\rho - \frac{1}{\rho}\right)u + \alpha\rho u^3 = 0 \tag{23}$$

and the linear equation

$$\left(\rho u'\right)' + \left(k^2(\rho)\rho - \frac{1}{\rho}\right)u = 0. \tag{24}$$

The latter equation can be written in the operator form as

$$L_k u = 0, \qquad L_k = \frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + \left(k^2(\rho)\rho - \frac{1}{\rho}\right) \tag{25}$$

(here we place index k in order to stress that the operator and the Green function depend on $k(\rho)$).

Suppose that the Green function $G_k(\rho, \rho_0; \lambda)$ exists for the following boundary value problem

$$L_k G_k = -\delta(\rho - \rho_0), \quad G|_{\rho=0} = G'\big|_{\rho=R} = 0 \quad (0 < \rho_0 < R). \tag{26}$$

In this case the Green function has the representation (see, e.g., [29, 30])

$$G_k(\rho, \rho_0; \lambda) = -\frac{v_i(\rho)\, v_i(\rho_0)}{\lambda - \lambda_i} + G_1(\rho, \rho_0; \lambda) \tag{27}$$

in the vicinity of eigenvalue λ_i. Here $\lambda := \gamma^2$ and $G_1(\rho, \rho_0; \lambda)$ is regular with respect to λ in the vicinity of λ_i; λ_n, $v_n(\rho)$ are complete orthonormal (real) eigenvalues and eigenfunctions systems of boundary eigenvalue problem

$$\begin{gathered}\left(\rho v_n'\right)' + \left(k_2^2(\rho)\rho - \frac{1}{\rho}\right)v_n = \lambda_n\rho v_n, \\ v_n\big|_{\rho=0} = v_n'\big|_{\rho=R} = 0.\end{gathered} \tag{28}$$

The Green function exists if $\lambda \neq \lambda_i$.

For $\varepsilon_2 \equiv$ const explicit form of the Green function is given in [21].

Let us write (19) in the operator form

$$L_k u + \alpha B(u) = 0, \qquad B(u) = \rho u^3(\rho). \tag{29}$$

Using the second Green formula [31]

$$\begin{aligned}\int_0^R (vL_k u - uL_k v)\, d\rho &= \int_0^R \left(v\left(\rho u'\right)' - u\left(\rho v'\right)'\right) d\rho \\ &= R\left(u'(R)\ v(R) - v'(R)\, u(R)\right)\end{aligned} \tag{30}$$

and assuming that $v = G$, we obtain that

$$\begin{aligned}&\int_0^R (G_k L_k u - uL_k G_k)\, dp \\ &\quad = R\left(u'(R-0)\, G_k(R, \rho_0) - G_k'(R, \rho_0)\, u(R-0)\right) \\ &\quad = Ru'(R-0)\, G_k(R, \rho_0).\end{aligned} \tag{31}$$

From the previous formulae, we obtain

$$\begin{gathered}\int_0^R uL_k G_k d\rho = -\int_0^R u(\rho)\,\delta(\rho - \rho_0)\, d\rho = -u(\rho_0), \\ \int_0^R G_k L_k u d\rho = -\alpha\int_0^R G_k(\rho, \rho_0)\,\rho u^3(\rho)\, d\rho.\end{gathered} \tag{32}$$

Taking into account these results and using (29), we obtain the nonlinear integral representation of solution $u(\rho_0)$ of (19) on the segment $[0, R]$

$$\begin{aligned}u(\rho_0) = \alpha&\int_0^R G_k(\rho, \rho_0)\,\rho u^3(\rho)\, d\rho \\ &+ Ru'(R-0)\, G_k(R, \rho_0), \quad 0 \le \rho_0 \le R.\end{aligned} \tag{33}$$

Using transmission conditions $u'(R-0) = u'(R+0)$, we can rewrite (33)

$$u(\rho_0) = \alpha\int_0^R G_k(\rho, \rho_0)\,\rho u^3(\rho)\, d\rho + f(\rho_0), \quad 0 \le \rho_0 \le R, \tag{34}$$

where $f(\rho_0) = Ru'(R+0)G_k(R, \rho_0)$.

Using (34) and transmission condition $u(R-0) = u(R+0)$, we obtain the dispersion equation (DE) with respect to the propagation constant

$$u(R+0) = \alpha \int_0^R G_k(\rho, R)\, \rho u^3(\rho)\, d\rho + R u'(R+0)\, G(R,R), \tag{35}$$

Let us denote by $N(\rho, \rho_0; \lambda) := \alpha G_k(\rho, \rho_0; \lambda)\rho$ and consider integral equation (34)

$$u(\rho_0) = \int_0^R N(\rho, \rho_0)\, u^3(\rho)\, d\rho + f(\rho_0) \tag{36}$$

in $C[0,R]$ [32]. It is assumed that $f \in C[0,R]$ and $\lambda \neq \lambda_i$.

The kernel $N(\rho, \rho_0)$ is continuous in the square $0 \leq \rho, \rho_0 \leq R$.

Let us consider linear integral operator $Nw = \int_0^R N(\rho, \rho_0) w(\rho) d\rho$ in $C[0,R]$. It is bounded, completely continuous, and $\|N\| = \max_{\rho_0 \in [0,R]} \int_0^R |N(\rho, \rho_0)| d\rho$.

Since nonlinear operator $B_0(u) = u^3(\rho)$ is bounded and continuous in $C[0,R]$, therefore, nonlinear operator $F(u) = \int_0^R N(\rho, \rho_0) u^3(\rho) d\rho + f(\rho_0)$ is completely continuous in any bounded set in $C[0,R]$.

The following theorems (about existence of a unique solution and continuous dependence of the solution on the parameter) can be proved in the same way as for the case of a homogeneous nonlinear cylindrical waveguide (for details of proofs, see [22, 33]).

Proposition 1. *If $\alpha \leq A^2$, where*

$$A = \frac{2}{3} \frac{1}{\|f\| \sqrt{3\|N_1\|}}, \quad \|N_1\| = \max_{\rho_0 \in [0,R]} \int_0^R |\rho G_k(\rho, \rho_0)|\, d\rho, \tag{37}$$

then (36) *has a unique continuous solution $u \in C[0,R]$ such that $\|u\| \leq r_*$, where*

$$r_* = -\frac{2}{\sqrt{3\|N\|}} \cos\left(\frac{1}{3} \arccos\left(\frac{3\sqrt{3}}{2} \|f\| \sqrt{\|N\|}\right) - \frac{2\pi}{3}\right) \tag{38}$$

is a root of the equation $\|N\| r^3 + \|f\| = r$.

Note that $A > 0$ does not depend on α.

Proposition 2. *Let the kernel N and the right-hand side f of equation* (36) *depend continuously on the parameter $\lambda \in \Lambda_0$, $N(\rho, \rho_0; \lambda) \subset C([0,R] \times [0,R] \times \Lambda_0)$, $f(\rho_0; \lambda) \subset C([0,R] \times \Lambda_0)$ on some segment Λ_0 of the real number axis. Let also*

$$0 < \|f(\lambda)\| < \frac{2}{3\sqrt{3\|N(\lambda)\|}}. \tag{39}$$

Then, for $\lambda \in \Lambda_0$, a unique solution $u(\rho; \lambda)$ of (36) exists and depends continuously on λ, $u(\rho; \lambda) \subset C([0,R] \times \Lambda_0)$.

4. Iteration Method

Approximate solutions u_n of integral equation (36) represented in the form $u = W(u)$ can be found by means of the iteration process $u_{n+1} = W(u_n)$, $n = 0, 1, \ldots$,

$$u_0 = 0, \qquad u_{n+1} = \alpha \int_0^R G_k(\rho, \rho_0)\, \rho u_n^3 d\rho + f, \qquad n = 0, 1, \ldots. \tag{40}$$

The sequence u_n converges uniformly to solution u of (36) by virtue of the fact that $F(u)$ is a contracting operator. The estimate of the convergence rate of iteration process (40) is also known. Let us formulate these results as the following (for proof see [22]).

Proposition 3. *The sequence of approximate solutions u_n of* (36)*, obtained by means of iteration process* (40)*, converges in the norm of space $C[0,R]$ to (unique) exact solution u of this equation. The following estimate of the convergence rate is valid $\|u_n - u\| \leq (q^n/(1-q)) f(u_0)$, $n \to \infty$, where $q := 3Nr_*^2 < 1$ is the coefficient of contraction of mapping F.*

5. Theorem of Existence

Taking into account formula (22), DE (35) can be represented in the form

$$K_1(|k_1| R) - |k_1| R K_1'(|k_1| R)\, G_k(R, R; \lambda) = \frac{\alpha}{\tilde{b}} \int_0^R G_k(\rho, R; \lambda)\, \rho u^3(\rho)\, d\rho. \tag{41}$$

As it can be seen DE (41) depends on $\tilde{b}$. Here $\tilde{b}$ is an initial condition. This is the peculiarity of this (and not only this) nonlinear problem. For the linear problem (if $\alpha = 0$), we obtain, as it is expected, the DE that does not depend on the initial condition.

From the properties of Bessel functions, it follows that

$$-|k_1| R K_1'(|k_1| R) = |k_1| R K_0(|k_1| R) + K_1(|k_1| R). \tag{42}$$

Now we can rewrite DE (41) in the following form:

$$g(\lambda) = \alpha F(\lambda), \tag{43}$$

where

$$\begin{aligned} g(\lambda) &= K_1(|k_1| R) \\ &\quad + \Big(|k_1| R K_0(|k_1| R) + K_1(|k_1| R)\Big) G_k(R, R; \lambda), \\ F(\lambda) &= \int_0^R G_k(\rho, R; \lambda)\, \rho u^3(\rho)\, d\rho. \end{aligned} \tag{44}$$

We should note that DE (41) depends on frequency ω implicitly. If one obtains λ^* for chosen R^* (radius of the inner core) such that $g(\lambda^*) = \alpha F(\lambda^*)$ is satisfied, then one can calculate ω^* which satisfies the propagation constants $\gamma^* = \sqrt{\lambda^*}$ using formulae in the beginning of this section.

The zeros of the function $\Phi(\gamma) \equiv g(\lambda) - \alpha F(\lambda)$ are those values of λ for which a nonzero solution of the problem P exists. The following assertion gives us sufficient conditions for the existence of zeros of the function Φ.

Let us consider the question about existence of solutions of the linear problem $g(\lambda) = 0$.

This equation can be rewritten in the form

$$G_k(R, R; \lambda) = -\frac{K_1(|k_1| R)}{|k_1| R K_0(|k_1| R) + K_1(|k_1| R)}. \quad (45)$$

From expression $G_k(R, R; \lambda) = -v_i^2(R)/(\lambda - \lambda_i) + G_1(R, R; \lambda)$, it follows that $G_k(R, R; \lambda)$ continuously varies from $-\infty$ to $+\infty$ when λ varies from λ_i to λ_{i+1}.

As value $K_1(|k_1|R)/(|k_1|RK_0(|k_1|R) + K_1(|k_1|R))$ is bounded, then there is at least one root of equation $g(\lambda) = 0$, and this root lies between λ_i and λ_{i+1}.

Finally it is necessary to prove that term $v_i(R)$ does not vanish in expression $G_k(R, R; \lambda)$. We prove this fact by contradiction. Let $v_i(R) = 0$. Consider a Cauchy problem for equation $\rho v_i'' + v_i' + (k_2^2(\rho)\rho - 1/\rho)v_i = \lambda_i \rho v_i$ with initial conditions $v_i|_{\rho=R} = v_i'|_{\rho=R} = 0$ as $\rho \in [\delta, R]$, where $\delta > 0$. From the general theory of ordinary differential equations (see, e.g., [34]) it is known that solution $v_i(\rho)$ of considered Cauchy problem exists and is unique as $\rho \in [\delta, R]$. In this case, this solution coincides with function $v_i(R)$ as $\rho \in [\delta, R]$. Function $v_i(R)$ is the function, which is contained in Green's function representation (27). On the other hand, a solution of the Cauchy problem for a linear equation with zero initial condition is the trivial solution. This contradicts with representation (27) of Green's function $G_k(\rho, \rho_0; \lambda)$ in the vicinity of $\lambda = \lambda_i$.

Consider nonlinear problem. Let inequalities

$$\varepsilon_1 < \lambda_0 < \lambda_1 < \cdots < \lambda_{k-1} < \lambda_k < \varepsilon_2 \quad (46)$$

hold, where $k \geq 1$ and $\varepsilon_2 = \min_{\rho \in [0,R]} \varepsilon_2(\rho)$.

We can choose sufficiently small $\delta_i > 0$ such that the Green function $G_k(\rho, \rho_0; \lambda)$ exists and is continuous on $\Gamma := \bigcup_{i=1}^{k} \Gamma_i$, where

$$\Gamma_i := \left[\sqrt{\lambda_{i-1}} + \delta_{i-1}, \sqrt{\lambda_i} - \delta_i\right], \quad i = \overline{1, k} \quad (47)$$

and the following inequality $g(\sqrt{\lambda_{i-1}} + \delta_{i-1}) g(\sqrt{\lambda_i} - \delta_i) < 0$ is satisfied.

It follows from the choice of δ_i that $F(\lambda)$ is bounded. Moreover, product $\alpha F(\lambda)$ can be made sufficiently small by choosing appropriate α (the estimation is given at the end of this section). Let us consider DE $\Phi(\lambda) = 0$. As it is shown before function $g(\lambda)$ is continuous, and reverse sign when λ varies from $\lambda_{i-1} + \delta_{i-1}$ to $\lambda_i - \delta_i$. As function $F(\lambda)$ is bounded then it is clear that equation $\Phi(\lambda) = 0$ has at least k roots $\widetilde{\lambda}_i$, $i = \overline{1, k}$ if we choose appropriate α. Here $\widetilde{\lambda}_i \in (\lambda_{i-1} + \delta_{i-1}, \lambda_i - \delta_i)$, $i = \overline{1, k}$.

On the basis of previous consideration, we can formulate the main result of this paper.

Theorem 4. *Let the values $\varepsilon_1, \varepsilon_2 = \min_{\rho \in [0,R]} \varepsilon_2(\rho), \alpha$ satisfy condition $\varepsilon_2 > \varepsilon_1 > 0$, and let the following inequalities $\varepsilon_1 < \lambda_0 < \lambda_1 < \cdots < \lambda_{k-1} < \lambda_k < \varepsilon_2$ hold, where $k \geq 1$ is an integer. Then there is a value $\alpha_0 > 0$ such that for any $\alpha \leq \alpha_0$ at least k values $\gamma_i, i = \overline{1, k}$ exist such that the problem P has a nonzero solution and $\gamma_i \in (\sqrt{\lambda_{i-1}} + \delta_{i-1}, \sqrt{\lambda_i} - \delta_i)$.*

Proof. The Green function exists for all $\gamma \in \Gamma$. It is also clear that function $A(\gamma) = 2/3\|f(\gamma)\| \sqrt{3\|N_1(\gamma)\|}$ is continuous as $\gamma \in \Gamma$. Let $A_1 = \min_{\gamma \in \Gamma} A(\gamma)$ and $\alpha < A_1^2$. In accordance with Proposition 1, there is a unique solution $u = u(\gamma)$ of (36) for any $\gamma \in \Gamma$. This solution is continuous and $\|u\| \leq r_* = r_*(\gamma)$. Let $r_{00} = \max_{\gamma \in \Gamma} r_*(\gamma)$. The following estimation $|F(\lambda)| \leq C r_{00}^3$ is valid, where C is a constant.

Function $g(\gamma)$ is continuous and equation $g(\gamma) = 0$ has at least one root $\widetilde{\gamma}_i$ inside segment Γ_i, that is, $\sqrt{\lambda_{i-1}} + \delta_{i-1} < \widetilde{\gamma}_i < \sqrt{\lambda_i} - \delta_i$. Let us denote $M_1 = \min_{0 \leq i \leq k-1} |g(\sqrt{\lambda_i} + \delta_i)|$, $M_2 = \min_{1 \leq i \leq k} |g(\sqrt{\lambda_i} - \delta_i)|$. Value $\widetilde{M} = \min\{M_1, M_2\}$ is positive and does not depend on α.

If $\alpha \leq \widetilde{M}/C r_{00}^3$, then

$$\left(g\left(\sqrt{\lambda_{i-1}} + \delta_{i-1}\right) - \alpha F\left(\sqrt{\lambda_{i-1}} + \delta_{i-1}\right)\right) \times \left(g\left(\sqrt{\lambda_i} - \delta_i\right) - \alpha F\left(\sqrt{\lambda_i} - \delta_i\right)\right) < 0. \quad (48)$$

As $g(\lambda) - \alpha F(\lambda)$ is continuous, it follows that equation $g(\lambda) - \alpha F(\lambda) = 0$ has a root γ_i inside Γ_i, that is $\sqrt{\lambda_i} + \delta_i < \gamma_i < \sqrt{\lambda_{i+1}} - \delta_{i+1}$. We can choose $\alpha_0 = \min\{A_1^2, \widetilde{M}/C r_{00}^3\}$.

From Theorem 4, it follows that, under the previous assumptions, there exist axially symmetrical propagating TE waves in cylindrical dielectric waveguides of circular cross-section filled with a nonmagnetic isotropic inhomogeneous medium with Kerr nonlinearity. This result generalizes the well-known similar statement for dielectric waveguides of circular cross-section filled with a linear medium (i.e., $\alpha = 0$).

It should be noticed that the value α_0 can be effectively estimated. □

6. Conclusion

In this study, we suggest and develop a method to investigate the problem of existence of electromagnetic waves that propagate along axis of an inhomogeneous nonlinear cylindrical waveguide. The nonlinearity inside the waveguide is described by the Kerr law; the inhomogeneity is described by a function that depends on radius of the waveguide.

Here we show that the integral equation approach allows us to investigate quite general problem for nonlinear inhomogeneous waveguides.

We should say that this method can be used to prove existence of guided waves in a nonlinear inhomogeneous waveguide for TM waves.

Numerical results can be obtained with the help of iteration procedure from Section 4.

A separate paper will be devoted to development of a couple of numerical methods for this problem.

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