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Research Article

Analysis of Transient Dynamic Behaviour of Spherical Cavity in Viscoelastic Soil Medium

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Stress, displacement, and pore pressure of a partially sealed spherical cavity in viscoelastic soil condition have been obtained in Laplace transform domain. Solutions of axisymmetric surface load and fluid pressure are derived.

1. Introduction

Biot [1, 2] presented the propagation theory of elastic waves and the general solutions for fluid-saturated porous viscoelastic medium. Akkas and Zakout [3] discussed the solution for the transient response for an axisymmetric and nontorsional load of an infinite, isotropic, elastic medium containing a spherical cavity with and without thin elastic shell embedment. In this paper, considering a viscoelastic model presented by Eringen [4], the transient response of a spherical cavity with a partially sealed shell embedded in viscoelastic soil is investigated. The solutions of stresses, displacements and pore pressure induced by axisymmetric surface load and fluid pressure are derived in Laplace transform domain. Durbin's [5] inverse Laplace transform is used to analyze the influence of partial permeable property of boundary and relative rigidity of shell and soil on the transient response of the spherical cavity. The solutions of permeable and impermeable boundary without shell are considered as two extreme cases.

2. Basic Equations and Solutions

In infinite viscoelastic saturated soil, a thin elastic shell shown in Figure 1 with inner radius a, outer radius b, and thickness $h = b - a$, has been bored. (r, θ, ϕ) are the spherical coordinates, where θ and ϕ are the meridional and circumferential angles, respectively, σ_r, σ_θ, σ_ϕ are nonvanishing components of stress tensor in case of an axisymmetric nontorsional load, that is, independent of θ and ϕ acting on the shell surface.

In spherical coordinate system (r, θ, ϕ), the equilibrium equation for soil mass is

$$\frac{\partial \sigma_r}{\partial r} + \frac{2\sigma_r - \sigma_\theta - \sigma_\phi}{r} = \frac{\partial^2}{\partial t^2}\left(\rho u_r + \rho_f w_r\right), \tag{1}$$

where u_r and w_r are radial displacement of soil skelton and displacement of pore fluid with respect to soil skelton, respectively; $\rho = (1-n)\rho_s + \eta\rho_f$, the density of soil; ρ_f and ρ_s are densities of fluid and soil grains respectively; n is porosity.

The pore fluid equilibrium equation is given by

$$-\frac{\partial p}{\partial r} = \frac{\partial^2}{\partial t^2}\left(\rho_f u_r + \frac{\rho_f}{n} w_r\right) + \frac{\eta_0}{k_d}\frac{\partial w_r}{\partial t}, \tag{2}$$

where p is excess pore pressure; η_0 is the fluid viscosity, and k_d is the intrinsic permeability of soil.

Soil is not an ideal medium. Due to overcoming the interior friction of soil, a part of energy of the propagation wave is changed into heat energy during the propagation. This

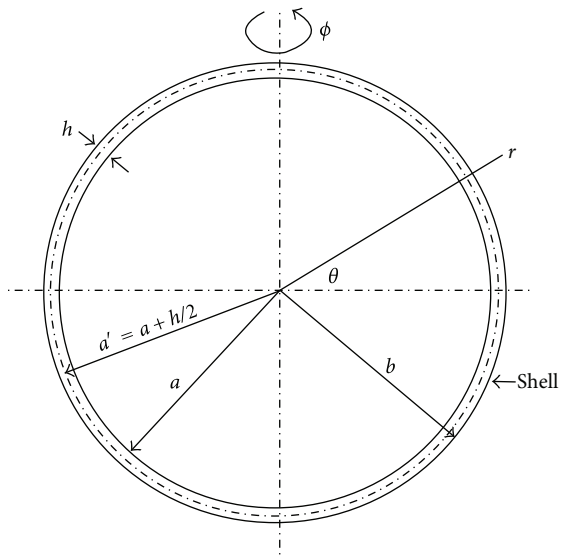

Figure 1: Geometry of the problem.

property is known as damping of material. Assuming that the viscoelastic property of soil may be simulated by Kelvin-Voigt model. Following Eringen [4], the stress-strain relationship is expressed as

$$\sigma_r = \lambda e + 2G\frac{\partial u_r}{\partial r} + \lambda'\frac{\partial e}{\partial t} + 2G'\frac{\partial}{\partial t}\left(\frac{\partial u_r}{\partial r}\right) - \alpha p, \quad (3)$$

$$\sigma_\theta = \sigma_\phi = \lambda e + 2G\frac{u_r}{r} + \lambda'\frac{\partial e}{\partial t} + 2G'\frac{\partial}{\partial t}\left(\frac{u_r}{r}\right) - \alpha p, \quad (4)$$

$$p = M\xi - \alpha M e, \quad (5)$$

where $e = \partial u_r/\partial r + 2u_r/r$ and $\xi = -(\partial w_r/\partial r + 2w_r/r)$, dilations of solid and fluid, respectively; λ and G are Lame constants of the bulk material; λ' and G' are the dilatant and shear constant of the viscoelastic soil; α and M are the compressibility parameters of the two phase medium, $0 \le \alpha \le 1$, $0 \le M \le \infty$ and $M \to \infty$, $\alpha \to 1$ for a material with incompressible constituents.

Substituting (3), (4), and (5) into (1) and (2), the governing equations of the transient response of a spherical cavity in viscoelastic solid condition can be reduced as

$$\begin{aligned}\left(\lambda + 2G + \alpha^2 M\right)\frac{\partial e}{\partial r} + \left(\lambda' + 2G'\right)\frac{\partial}{\partial t}\left(\frac{\partial e}{\partial r}\right) - \alpha M\frac{\partial \xi}{\partial r} \\ = \frac{\partial^2}{\partial t^2}\left(\rho u_r + \rho_f w_r\right),\end{aligned} \quad (6)$$

$$\alpha M\frac{\partial e}{\partial r} - M\frac{\partial \xi}{\partial r} = \frac{\partial^2}{\partial t^2}\left(\rho_f u_r + \frac{\rho_f}{n}w_r\right) + \frac{\eta_0}{k_d}\frac{\partial w_r}{\partial t}. \quad (7)$$

Now, displacements u_r and w_r are assumed to be of the forms

$$u_r = \frac{\partial}{\partial r}\left(\frac{U(r)\cos\omega t}{r}\right), \quad (8)$$

$$w_r = \frac{\partial}{\partial r}\left(\frac{W(r)\cos\omega t}{r}\right), \quad (9)$$

for solving (6) and (7). Consider

$$\begin{aligned}
e &= \frac{\partial u_r}{\partial r} + \frac{2u_r}{r}, \\
\frac{\partial e}{\partial r} &= \frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r} - \frac{2}{r^2}u_r, \\
u_r &= \frac{\partial}{\partial r}\left(\frac{U(r)\cos\omega t}{r}\right), \\
u_r &= \frac{1}{r}\frac{\partial}{\partial r}\left(U(r)\cos\omega t\right) - \frac{U(r)\cos\omega t}{r^2}, \\
\frac{\partial u_r}{\partial r} &= \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(U(r)\cos\omega t\right) - \frac{2}{r^2}\frac{\partial}{\partial r}\left(U(r)\cos\omega t\right) \\
&\quad + \frac{2}{r^3}U(r)\cos\omega t, \\
\frac{\partial^2 u_r}{\partial r^2} &= \frac{1}{r}\frac{\partial^3}{\partial r^3}\left(U(r)\cos\omega t\right) - \frac{3}{r^2}\frac{\partial^2}{\partial r^2}\left(U(r)\cos\omega t\right) \\
&\quad + \frac{6}{r^3}\frac{\partial}{\partial r}\left(U(r)\cos\omega t\right) - \frac{6}{r^4}U(r)\cos\omega t.
\end{aligned} \quad (10)$$

Then,

$$\begin{aligned}\frac{\partial e}{\partial r} &= \frac{1}{r}\frac{\partial^3}{\partial r^3}\left(U(r)\cos\omega t\right) - \frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(U(r)\cos\omega t\right) \\ &= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(U(r)\cos\omega t\right)\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\nabla^2\left(U(r)\cos\omega t\right)\right).\end{aligned} \quad (11)$$

Similarly,

$$\begin{aligned}\frac{\partial \xi}{\partial r} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(W(r)\cos\omega t\right)\right) \\ &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\nabla^2\left(W(r)\cos\omega t\right)\right).\end{aligned} \quad (12)$$

Substituting (8), (9), (11), and (12), into (6), we obtain

$$\begin{aligned}&\left(\lambda + 2G + \alpha^2 M\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\nabla^2\left(U(r)\cos\omega t\right)\right) \\ &\quad + \left(\lambda' + 2G'\right)\frac{\partial}{\partial t}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\nabla^2\left(U(r)\cos\omega t\right)\right)\right) \\ &\quad - \alpha M\frac{\partial}{\partial r}\left(\frac{1}{r}\nabla^2\left(W(r)\cos\omega t\right)\right)\end{aligned}$$

$$= \frac{\partial^2}{\partial t^2}\left(\rho\frac{\partial}{\partial r}\left(\frac{1}{r}\nabla^2\left(U(r)\cos\omega t\right)\right)\right.$$
$$\left.+\rho_f\frac{\partial}{\partial r}\left(\frac{1}{r}\nabla^2\left(W(r)\cos\omega t\right)\right)\right),$$
$$\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\lambda+2G+\alpha^2M\right)\nabla^2U(r)\cos\omega t\right.$$
$$\left.+\left(\lambda'+2G'\right)\frac{\partial}{\partial t}\nabla^2U(r)\cos\omega t+\alpha M\nabla^2W(r)\cos\omega t\right]$$
$$=\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial^2}{\partial t^2}\left(\rho U(r)\cos\omega t+\rho_fW(r)\cos\omega t\right)\right]. \tag{13}$$

Integrating w.r.t. r, we have

$$\left\{\left(\lambda+2G+\alpha^2M\right)+\left(\lambda'+2G'\right)\frac{\partial}{\partial t}\right\}\nabla^2U(r)\cos\omega t$$
$$+\alpha M\nabla^2W(r)\cos\omega t$$
$$=\frac{\partial^2}{\partial t^2}\left(\rho U(r)\cos\omega t+\rho_fW(r)\cos\omega t\right),$$
$$\left[\left(\lambda+2G+\alpha^2M\right)\nabla^2+\left(\lambda'+2G'\right)\frac{\partial}{\partial t}\nabla^2-\rho\frac{\partial^2}{\partial t^2}\right]$$
$$\times U(r)\cos\omega t+\left(\alpha M\nabla^2-\rho_f\frac{\partial^2}{\partial t^2}\right)W(r)\cos\omega t=0, \tag{14}$$

where $\nabla^2=\partial^2/\partial r^2$.

Substituting (8) to (12), in (7), we obtain

$$\alpha M\frac{\partial}{\partial r}\left(\frac{1}{r}\nabla^2U(r)\cos\omega t\right)+M\frac{\partial}{\partial r}\left(\frac{1}{r}\nabla^2W(r)\cos\omega t\right)$$
$$=\frac{\partial^2}{\partial t^2}\left\{\rho_f\frac{\partial}{\partial r}\left(\frac{1}{r}U(r)\cos\omega t\right)+\frac{\rho_f}{n}\frac{\partial}{\partial r}\left(\frac{1}{r}W(r)\cos\omega t\right)\right\}$$
$$+\frac{\eta_0}{k_d}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}W(r)\cos\omega t\right)\right),$$
$$\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\alpha M\nabla^2U(r)\cos\omega t+M\nabla^2W(r)\cos\omega t\right)\right]$$
$$=\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\frac{\partial^2}{\partial t^2}\rho_fU(r)\cos\omega t+\frac{\partial^2}{\partial t^2}\frac{\rho_f}{n}W(r)\cos\omega t\right.\right.$$
$$\left.\left.+\frac{\eta_0}{k_d}\frac{\partial}{\partial t}W(r)\cos\omega t\right)\right]. \tag{15}$$

Integrating w.r.t. r, we have

$$\alpha M\nabla^2U(r)\cos\omega t+M\nabla^2W(r)\cos\omega t$$
$$=\frac{\partial^2}{\partial t^2}\rho_fU(r)\cos\omega t+\frac{\partial^2}{\partial t^2}\frac{\rho_f}{n}W(r)\cos\omega t$$
$$+\frac{\eta_0}{k_d}\frac{\partial}{\partial t}W(r)\cos\omega t, \tag{16}$$
$$\left[\alpha M\nabla^2-\rho_f\frac{\partial^2}{\partial t^2}\right]U(r)\cos\omega t$$
$$+\left[M\nabla^2-\frac{\rho_f}{n}\frac{\partial^2}{\partial t^2}-\frac{\eta_0}{k_d}\frac{\partial}{\partial t}\right]W(r)\cos\omega t=0.$$

Taking Laplace transform in (14) yields

$$\left[\left(\lambda+2G+\alpha^2M\right)\nabla^2\left(\frac{s}{s^2+\omega^2}\right)+\left(\lambda'+2G'\right)\nabla^2\left(\frac{-\omega^2}{s^2+\omega^2}\right)\right.$$
$$\left.-\rho\left(\frac{-\omega^2s}{s^2+\omega^2}\right)\right]U(r)$$
$$+\left(\alpha M\nabla^2\left(\frac{s}{s^2+\omega^2}\right)-\rho_f\left(\frac{-\omega^2s}{s^2+\omega^2}\right)\right)W(r)=0,$$
$$\left[\left(\lambda+2G+\alpha^2M\right)\nabla^2-\frac{\omega^2}{s}\left(\lambda'+2G'\right)\nabla^2+\rho\omega^2\right]\frac{U(r)s}{s^2+\omega^2}$$
$$+\left[\alpha M\nabla^2+\rho_f\omega^2\right]\frac{W(r)s}{s^2+\omega^2}=0. \tag{17}$$

Taking Laplace transform in (14) yields

$$\left[\alpha M\nabla^2\left(\frac{s}{s^2+\omega^2}\right)-\rho_f\left(\frac{-\omega^2s}{s^2+\omega^2}\right)\right]U(r)$$
$$+\left[M\nabla^2\left(\frac{s}{s^2+\omega^2}\right)-\frac{\rho_f}{n}\left(\frac{-\omega^2s}{s^2+\omega^2}\right)-\frac{\eta_0}{k_d}\left(\frac{-\omega^2}{s^2+\omega^2}\right)\right]$$
$$\times W(r)=0,$$
$$\left(\alpha M\nabla^2+\omega^2\rho_f\right)\frac{U(r)s}{s^2+\omega^2}+\left[\left(M\nabla^2+\frac{\rho_f}{n}\omega^2\right)+\frac{\eta_0}{k_d}\frac{\omega^2}{s}\right]$$
$$\times\frac{W(r)s}{s^2+\omega^2}=0. \tag{18}$$

In a great range of vibration frequencies, viscoelastic damp coefficient of rock and soft soil may be assumed as a constant. The dimensionless damp coefficient η is considered as

$$\eta=\frac{\lambda'}{\lambda}=\frac{G'}{G}. \tag{19}$$

Also,

$$\lambda^* = \frac{\lambda}{G}, \qquad M^* = \frac{M}{G}, \qquad \rho^* = \frac{\rho_f}{\rho},$$
$$b^* = \frac{\eta_0 a'}{k_d\sqrt{\rho G}}, \quad \frac{\rho}{G} \sim 1 \tag{20}$$

are nondimensional Lame constant, compressibility parameter, fluid density, and permeability coefficient of soil respectively. b is the radius of spherical shell, $a' = b - h/2 = a + h/2$.

Using (19) and (20) in (17) yields

$$\left[\left(\left(\lambda^* + 2\right) + \alpha^2 M^*\right)\nabla^2 - \frac{\omega^2}{s}\left(\eta\lambda' + 2\eta\right)\nabla^2 + \omega^2\right]\frac{U(r)\,s}{s^2+\omega^2}$$
$$+ \left[\alpha M^*\nabla^2 + \rho^*\omega^2\right]\frac{W(r)\,s}{s^2+\omega^2} = 0,$$
$$\left[\left\{\left(\lambda^* + 2\right)\left(1 - \frac{\omega^2}{s}\eta\right) + \alpha^2 M^* s\right\}\nabla^2 + \omega^2\right]\frac{U(r)\,s}{s^2+\omega^2}$$
$$+ \left(\alpha M^*\nabla^2 + \rho^*\omega^2\right)\frac{W(r)\,s}{s^2+\omega^2} = 0. \tag{21}$$

Similarly, using (19) and (20) in (18) yields

$$\left(\alpha M^*\nabla^2 + \rho^*\omega^2\right)\frac{U(r)\,s}{s^2+\omega^2}$$
$$+ \left[\left(M^*\nabla^2 + \frac{\rho^*\omega^2}{n}\right) + b^*\frac{\omega^2}{s}\right]\frac{W(r)\,s}{s^2+\omega^2} = 0,$$
$$\left(\alpha M^*\nabla^2 + \rho^*\omega^2\right)\frac{U(r)\,s}{s^2+\omega^2}$$
$$+ \left[M^*\nabla^2 + \left(\frac{\rho^*}{n} + \frac{b^*}{s}\right)\omega^2\right]\frac{W(r)\,s}{s^2+\omega^2} = 0. \tag{22}$$

Solving (21) and (22) yields

$$\left[\left\{\left(\left(\lambda^* + 2\right)\left(1 - \frac{\omega^2}{s}\eta\right) + \alpha^2 M^*\right)\nabla^2 + \omega^2\right\}\right.$$
$$\left.\times\left\{M^*\nabla^2 + \left(\frac{\rho^*}{n} + \frac{b^*}{s}\right)\omega^2\right\} - \left\{\left(\alpha M^*\nabla^2 + \rho^*\omega^2\right)\right\}^2\right]$$
$$\times\left[\frac{U(r)\,s}{s^2+\omega^2}, \frac{W(r)\,s}{s^2+\omega^2}\right] = 0, \tag{23}$$

$$\left[\left(\lambda^* + 2\right)\left(1 - \frac{\omega^2}{s}\eta\right)M^*\nabla^4\right.$$
$$+ \left\{\left(\left(\lambda^* + 2\right)\left(1 - \frac{\omega^2}{s}\eta\right) + \alpha^2 M^*\right)\left(\frac{\rho^*}{n} + \frac{b^*}{s}\right)\right.$$
$$\left. + M^*\left(1 - 2\alpha\rho^*\right)\right\}\omega^2\nabla^2$$
$$\left. + \left\{\left(\frac{\rho^*}{n} + \frac{b^*}{s}\right) - \left(\rho^*\right)^2\right\}\omega^4\right](U, W)\left(\frac{s}{s^2+\omega^2}\right) = 0. \tag{24}$$

Equation (24) can be written as

$$\left(\nabla^2 - \gamma_1^2\right)\left(\nabla^2 - \gamma_2^2\right)(U, W)\left(\frac{s}{s^2+\omega^2}\right) = 0, \tag{25}$$

where γ_1 and γ_2 are the complex wave number of two dilation waves, that is,

$$\gamma_1^2 = \frac{\alpha_1 - \sqrt{\alpha_1 - 4\alpha_2}}{2},$$
$$\gamma_2^2 = \frac{\alpha_1 + \sqrt{\alpha_1 - 4\alpha_2}}{2}, \tag{26}$$

with

$$\alpha_1 = \left[\left(\left(\lambda^* + 2\right)\left(1 - \left(\omega^2/s\right)\eta\right) + \alpha^2 M^*\right)\right.$$
$$\left.\times\left(\rho^*/n + b^*/s\right) + M^*\left(1 - 2\alpha\rho^*\right)\right]\omega^2$$
$$\times\left(\left(\lambda^* + 2\right)\left(1 - \left(\omega^2/s\right)\eta\right)M^*\right)^{-1},$$
$$\alpha_2 = \frac{\left[\left(\rho^*/n + b^*/s\right) - \left(\rho^*\right)^2\right]\omega^4}{\left(\lambda^* + 2\right)\left(1 - \left(\omega^2/s\right)\eta\right)M^*}. \tag{27}$$

The general solutions of $U(r)s/(s^2+\omega^2)$ and $W(r)s/(s^2+\omega^2)$ in (25) are

$$\frac{U(r)\,s}{s^2+\omega^2} = A_1 e^{-\gamma_1 r} + A_2 e^{-\gamma_2 r} + A_3 e^{\gamma_1 r} + A_4 e^{\gamma_2 r},$$
$$\frac{W(r)\,s}{s^2+\omega^2} = B_1 e^{-\gamma_1 r} + B_2 e^{-\gamma_2 r} + B_3 e^{\gamma_1 r} + B_4 e^{\gamma_2 r}. \tag{28}$$

Considering the limitation property of radial displacement when $r \to \infty$, that is,

$$U(r) \longrightarrow \infty \quad \text{as } r \longrightarrow \infty, \qquad W(r) \longrightarrow \infty \quad \text{as } r \longrightarrow \infty. \tag{29}$$

In (28), we have, $A_3 = A_4 = B_3 = B_4$,

$$\frac{U(r)\,s}{s^2+\omega^2} = A_1 e^{-\gamma_1 r} + A_2 e^{-\gamma_2 r}, \tag{30}$$

$$\frac{W(r)\,s}{s^2+\omega^2} = B_1 e^{-\gamma_1 r} + B_2 e^{-\gamma_2 r}. \tag{31}$$

Constants A_1, A_2, B_1, and B_2 in (30) and (31) are linearly dependent and may be related by using (25) to obtain

$$B_i = \delta_i A_i, \quad i = 1, 2, \tag{32}$$

where $\delta_i = -(\alpha M \gamma_i^2 + \rho^* \omega^2)/(M^* \gamma_i^2 + (\rho^*/n + b^*/s)\omega^2)$.

A_1, A_2 can be obtained from boundary conditions. Now,

$$u_r = \frac{\partial}{\partial r}\left(\frac{U(r)\cos\omega t}{r}\right). \tag{33}$$

The Laplace transformed solution of radial displacement u_r, that is, $\overline{u}_r$ is given by

$$\begin{aligned}
\overline{u}_r &= \frac{\partial}{\partial r}\left(\frac{U(r)\,s}{r\left(s^2+\omega^2\right)}\right) \\
&= \frac{\partial}{\partial r}\frac{1}{r}\left(A_1 e^{-\gamma_1 r} + A_2 e^{-\gamma_2 r}\right), \quad \text{by (30)} \\
&= -\frac{A_1}{r}\left(\gamma_1 + \frac{1}{r}\right)e^{-\gamma_1 r} - \frac{A_2}{r}\left(\gamma_2 + \frac{1}{r}\right)e^{-\gamma_2 r}.
\end{aligned} \tag{34}$$

Similarly, the Laplace transform solution of w_r is $\overline{w}_r$:

$$\begin{aligned}
w_r &= \frac{\partial}{\partial r}\left(\frac{W(r)\cos\omega t}{r}\right), \\
\overline{w}_r &= \frac{\partial}{\partial r}\left(\frac{W(r)\,s}{r\left(s^2+\omega^2\right)}\right) \\
&= \frac{\partial}{\partial r}\frac{1}{r}\left(B_1 e^{-\gamma_1 r} + B_2 e^{-\gamma_2 r}\right), \\
&= -\frac{1}{r}\left(\gamma_1 + \frac{1}{r}\right)e^{-\gamma_1 r}B_1 - \frac{1}{r}\left(\gamma_2 + \frac{1}{r}\right)e^{-\gamma_2 r}B_2 \\
&= -\frac{\delta_1}{r}\left(\gamma_1 + \frac{1}{r}\right)e^{-\gamma_1 r}A_1 \\
&\quad - \frac{\delta_2}{r}\left(\gamma_2 + \frac{1}{r}\right)e^{-\gamma_2 r}A_2, \quad \text{by (32)}.
\end{aligned} \tag{35}$$

Next, by (5),

$$\begin{aligned}
p &= M\xi - \alpha M e \\
&= -M\left(\frac{\partial w_r}{\partial r} + \frac{2w_r}{r}\right) - \alpha M\left(\frac{\partial u_r}{\partial r} + \frac{2u_r}{r}\right) \\
\frac{P}{G} &= -M^*\left(\frac{\partial w_r}{\partial r} + \frac{2w_r}{r}\right) - \alpha M^*\left(\frac{\partial u_r}{\partial r} + \frac{2u_r}{r}\right).
\end{aligned} \tag{36}$$

Taking Laplace transform and using (34) and (35), we have

$$\begin{aligned}
\left(\frac{\overline{P}}{G}\right) &= -M^*\left[\frac{\partial}{\partial r}\left\{-\sum_{i=1}^{2}\frac{\delta_i}{r}\left(\gamma_i + \frac{1}{r}\right)e^{-\gamma_i r}A_i\right\}\right. \\
&\qquad \left. + \frac{2}{r}\left\{-\sum_{i=1}^{2}\frac{\delta_i}{r}\left(\gamma_i + \frac{1}{r}\right)e^{-\gamma_i r}A_i\right\}\right] \\
&\quad - \alpha M^*\left[\frac{\partial}{\partial r}\left\{-\sum_{i=1}^{2}\frac{1}{r}\left(\gamma_i + \frac{1}{r}\right)e^{-\gamma_i r}A_i\right\}\right. \\
&\qquad \left. + \frac{2}{r}\left\{-\sum_{i=1}^{2}\frac{1}{r}\left(\gamma_i + \frac{1}{r}\right)e^{-\gamma_i r}A_i\right\}\right] \\
&= \sum_{i=1}^{2}(\alpha + \delta_i)M^* \\
&\qquad \times A_i\left[-\frac{1}{r^2}\left(\gamma_i + \frac{1}{r}\right)e^{-\gamma_i r} - \frac{1}{r^3}e^{-\gamma_i r}\right. \\
&\qquad + \frac{2}{r}\left(\gamma_i + \frac{1}{r}\right)e^{-\gamma_i r}(-\gamma_i) \\
&\qquad \left. + \frac{2}{r}\left(\gamma_i + \frac{1}{r}\right)e^{-\gamma_i r}\right], \\
\left(\frac{\overline{P}}{G}\right) &= -\sum_{i=1}^{2}(\alpha + \delta_i)\frac{1}{r}M^*\gamma_i^2 e^{-\gamma_i r}A_i.
\end{aligned} \tag{37}$$

Next, by (3),

$$\begin{aligned}
\frac{\sigma_r}{G} &= \lambda^*\left(\frac{\partial u_r}{\partial r} + \frac{2u_r}{r}\right) + 2\frac{\partial u_r}{\partial r} \\
&\quad + \lambda^*\eta\frac{\partial}{\partial t}\left(\frac{\partial u_r}{\partial r} + \frac{2u_r}{r}\right) + 2\eta\frac{\partial}{\partial t}\left(\frac{u_r}{r}\right) - \alpha\frac{P}{G}.
\end{aligned} \tag{38}$$

Taking Laplace transform of both sides

$$\begin{aligned}
\left(\frac{\overline{\sigma}_r}{G}\right) &= \lambda^*\left(\frac{\partial \overline{u}_r}{\partial r} + \frac{2\overline{u}_r}{r}\right) + 2\frac{\partial \overline{u}_r}{\partial r} \\
&\quad + \lambda^*\eta\left(\frac{\partial}{\partial r}\left(\frac{-\omega^2}{s}\overline{u}_r\right) - \frac{2\omega^2}{s}\frac{\overline{u}_r}{r}\right) \\
&\quad + 2\eta\frac{\partial}{\partial r}\left(\frac{-\omega^2}{s}\overline{u}_r\right) - \alpha\frac{\overline{P}}{G} \\
&= (\lambda^* + 2)\left(1 - \frac{\omega^2}{s}\eta\right)\frac{\partial \overline{u}_r}{\partial r} \\
&\quad + 2\lambda^*\left(1 - \frac{\omega^2}{s}\eta\right)\frac{\overline{u}_r}{r} - \alpha\frac{\overline{P}}{G}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2}(\lambda^*+2)\left(1-\frac{\omega^2}{s}\eta\right)\frac{\partial}{\partial r}\left(-\frac{1}{r}\left(\gamma_i+\frac{1}{r}\right)e^{-\gamma_i r}\right)\\
&\quad+\frac{2\lambda^*}{r}\left(1-\frac{\omega^2}{s}\eta\right)\sum_{i=1}^{2}\left(-\frac{1}{r}\left(\gamma_i+\frac{1}{r}\right)e^{-\gamma_i r}A_i\right)\\
&\quad+\alpha\sum_{i=1}^{2}(\alpha+\delta_i)A_iM^*\gamma_i^2e^{-\gamma_i r},\quad \text{by (34) and (35)}\\
&= -\sum_{i=1}^{2}(\lambda^*-2)\left(1-\frac{\omega^2}{s}\eta\right)\\
&\qquad\times A_i\left\{-\frac{1}{r^2}\left(\gamma_i+\frac{1}{r}\right)e^{-\gamma_i r}-\frac{1}{r^3}e^{-\gamma_i r}\right.\\
&\qquad\qquad\left.+\frac{1}{r}\left(\gamma_i+\frac{1}{r}\right)e^{-\gamma_i r}(-\gamma_i)\right\}\\
&\quad-2\lambda^*\sum_{i=1}^{2}\left(1-\frac{\omega^2}{s}\eta\right)\frac{1}{r^2}\left(\gamma_i+\frac{1}{r}\right)e^{-\gamma_i r}A_i\\
&\quad+\alpha\sum_{i=1}^{2}\frac{1}{r}(\alpha+\delta_i)A_iM^*\gamma_i^2e^{-\gamma_i r}\\
&= \sum_{i=1}^{2}(\lambda^*+2)\left(1-\frac{\omega^2}{s}\eta\right)e^{-\gamma_i r}A_i\gamma_i^2\\
&\quad+2\sum_{i=1}^{2}(\lambda^*+2)\left(1-\frac{\omega^2}{s}\eta\right)e^{-\gamma_i r}\frac{A_i}{r^2}\left(\gamma_i+\frac{1}{r}\right)\\
&\quad-2\sum_{i=1}^{2}\lambda^*\left(1-\frac{\omega^2}{s}\eta\right)+e^{-\gamma_i r}\frac{A_i}{r^2}\left(\gamma_i+\frac{1}{r}\right)\\
&\quad+\alpha\sum_{i=1}^{2}\frac{1}{r}(\alpha+\delta_i)A_iM^*\gamma_i^2e^{-\gamma_i r}\\
\left(\frac{\overline{\sigma}_r}{G}\right) &= A_i\sum_{i=1}^{2}\left[\frac{1}{r}(\lambda^*+2)\left(1-\frac{\omega^2}{s}\eta\right)\gamma_i^2\frac{4}{r^2}\left(1-\frac{\omega^2}{s}\eta\right)\right.\\
&\qquad\left.\times\left(\gamma_i+\frac{1}{r}\right)+\frac{\alpha}{r}\gamma_i^2(\alpha+\delta_i)M^*\right].
\end{aligned}
\tag{39}
$$

Similarly, by (4), (35), and (37), we have

$$
\begin{aligned}
\left(\frac{\sigma_\theta}{G}\right) &= \left(\frac{\sigma_\phi}{G}\right)\\
&= \lambda^*e+2\frac{u_r}{r}+\eta\lambda^*\frac{\partial e}{\partial t}+2\eta\frac{\partial}{\partial t}\left(\frac{u_r}{r}\right)-\alpha\frac{p}{G}\\
&= \lambda^*\left(\frac{\partial u_r}{\partial r}+\frac{2u_r}{r}\right)+\frac{2u_r}{r}+\eta\lambda^*\frac{\partial}{\partial t}\left(\frac{\partial u_r}{\partial r}+\frac{2u_r}{r}\right)\\
&\quad+2\eta\frac{\partial}{\partial t}\left(\frac{u_r}{r}\right)-\alpha\frac{p}{G}.
\end{aligned}
\tag{40}
$$

Then

$$
\begin{aligned}
&L\left(\frac{\sigma_\theta}{G}\right)\\
&= \left(\frac{\overline{\sigma}_\theta}{G}\right)\\
&= \lambda^*\left(\frac{\partial\overline{u}_r}{\partial r}+\frac{2\overline{u}_r}{r}\right)+\frac{2\overline{u}_r}{r}\\
&\quad+\eta\lambda^*\left(-\frac{\omega^2}{s}\frac{\partial\overline{u}_r}{\partial r}-\frac{2\omega^2}{s}\frac{\overline{u}_r}{r}\right)\\
&\quad+2\eta\left(-\frac{\omega^2}{s}\frac{\overline{u}_r}{r}\right)-\alpha\frac{\overline{p}}{G}\\
&= \lambda^*\left(1-\frac{\omega^2}{s}\eta\right)\frac{\partial\overline{u}_r}{\partial r}\\
&\quad+2(\lambda^*+1)\left(1-\frac{\omega^2}{s}\eta\right)\frac{\overline{u}_r}{r}-\alpha\left(\frac{\overline{p}}{G}\right)\\
&= -\lambda^*\left(1-\frac{\omega^2}{s}\eta\right)\sum_{i=1}^{2}\frac{\partial}{\partial r}\frac{1}{r}\left(\gamma_i+\frac{1}{r}\right)e^{-\gamma_i r}A_i\\
&\quad-2(\lambda^*+1)\left(1-\frac{\omega^2}{s}\eta\right)\\
&\quad\times\sum_{i=1}^{2}\frac{1}{r^2}\left(\gamma_i+\frac{1}{r}\right)e^{-\gamma_i r}A_i-\alpha\left(\frac{\overline{p}}{G}\right)\\
&= -\lambda^*\left(1-\frac{\omega^2}{s}\eta\right)\\
&\quad\times\sum_{i=1}^{2}\left[-\frac{1}{r^2}\left(\gamma_i+\frac{1}{r}\right)-\frac{1}{r^3}-\frac{\gamma_i}{r}\left(\gamma_i+\frac{1}{r}\right)\right]A_ie^{-\gamma_i r}\\
&\quad-2(\lambda^*+1)\left(1-\frac{\omega^2}{s}\eta\right)\\
&\quad\times\sum_{i=1}^{2}\frac{1}{r^2}\left(\gamma_i+\frac{1}{r}\right)e^{-\gamma_i r}A_i-\alpha\left(\frac{\overline{p}}{G}\right)\\
&= \sum_{i=1}^{2}\frac{\lambda^*}{r}\left(1-\frac{\omega^2}{s}\eta\right)\gamma_i^2e^{-\gamma_i r}A_i\\
&\quad-\sum_{i=1}^{2}\left(1-\frac{\omega^2}{s}\eta\right)\\
&\qquad\times\left[-\frac{2\lambda^*}{r^2}\left(\gamma_i+\frac{1}{r}\right)+\frac{2\lambda^*}{r^2}\left(\gamma_i+\frac{1}{r}\right)+\frac{2}{r^2}\right]\\
&\quad-\alpha\left(\frac{\overline{p}}{G}\right)
\end{aligned}
$$

$$= \sum_{i=1}^{2} \left[\frac{\lambda^*}{r} \left(1 - \frac{\omega^2}{s}\eta\right) \gamma_i^2 - \frac{2}{r^2} \left(1 - \frac{\omega^2}{s}\eta\right) \left(\gamma_i + \frac{1}{r}\right) + \frac{\alpha}{r} \gamma_i^2 (\alpha + \delta_i) M^* \right] e^{-\gamma_i r} A_i. \tag{41}$$

3. Solution of Shell Embedment and Axisymmetric Loading

Dynamic loads applied on the surface of shell considered herein are an axially symmetric radial traction and axially symmetric fluid pressure with the step style shown in Figure 2, where T^* is the nondimensional step load time ($T^* = T\sqrt{G/\rho}/b$), T is actual step load time; ($t^* = t\sqrt{G/\rho}/b$), the nondimensional time; t is actual time; q_0 is maximum of the step load. In the domain of Laplace transform, the load can be expressed as

$$\bar{q}(s) = \frac{q_0}{T^*} \left(\frac{1 - e^{-T^* s}}{s^2} \right), \quad r = b. \tag{42}$$

Here, the case of a thin, elastic shell embedded in infinite viscoelastic saturated soil subjected to axisymmetric surface load and fluid pressure is considered. The equation of motion of this shell under nontorsional axisymmetric loading is

$$2(1 + \mu_1)\mu_r' + \gamma_0^2 \frac{\partial^2 \mu_r'}{\partial t^2} = q_0(t) \frac{(a')(1 - \mu_1^2)}{E_1 h}, \tag{43}$$

where $\gamma_0^2 = c_1^2/c_p^2$; $c_1 = \sqrt{(\lambda + 2G)/\rho}$ and $c_p = \sqrt{E_1/(\rho_f(1 - \mu^2))}$ are the dilatational wave velocity and plate velocity, respectively; E_1, μ_1 are the modulus and Poission's rate of shell, respectively; $q_0(t)$ is the net outward radial pressure. For a thin shell, the thick $h/2$ can be omitted without significant error. The interface shell and soil can be defined as $r = a + h/2 = a'$. The stress and displacement condition at the interface is expressed as

$$q_0(t) = q(t) - \sigma_r, \quad r = a + \frac{h}{2} = a', \tag{44}$$

$$u_r = -u_r', \quad r = a + \frac{h}{2} = a', \tag{45}$$

where $q(t)$ is the radial stress applied at the inner surface of the shall; σ_r is the stress exerted by the soil on the shell and can be given by (39).

In practical situation, the condition is frequently found in two extreme cases; permeable and impermeable.

The partial permeable flow boundary condition is

$$\frac{\partial p}{\partial r} = \frac{kp}{a'} \quad \text{at } r = a + \frac{h}{2} = a', \tag{46}$$

where $k = (k_1/k_d)(1/\log(b/a))$ is a dimensionless permeability parameter that defines the flow capacity of the shell. The parameter k depends on the relative permeability of the shell and soil as well as the geometry of the shell, that is,

(1) when the spherical shell is impermeable, that is, $k_1 = 0$, k tends to zero and

(2) when the shell is permeable, that is, $k_1 = \infty$, k tends to infinity.

Substituting (34), (39), (44) into (43) and (37) into (46), we obtain

$$m_1 A_1 + m_2 A_2 = \frac{(a')^2 (1 - \mu_1^2)}{E_1^* h} \frac{\bar{q}(s)}{G}, \qquad n_1 A_1 + n_2 A_2 = 0, \tag{47}$$

where

$$\begin{aligned} m_i &= \left[2(1 + \mu_1) + \gamma_0^2 \omega^2\right] \frac{1}{a'} \left(\gamma_i + \frac{1}{a'}\right) e^{-\gamma_i a'} \\ &\quad + \left[\frac{1}{a'} (\lambda^* + 2) \left(1 - \frac{\omega^2}{s}\eta\right) \times \gamma_i^2 \frac{4}{(a')^2} \left(1 - \frac{\omega^2}{s}\eta\right) \left(\gamma_i + \frac{1}{a'}\right) + \frac{\alpha}{a'} \gamma_i^2 (\alpha + \delta_i) M^* \right] \frac{(a')^2 (1 - \mu_1^2)}{E_1^* h} e^{-\gamma_i a'} \\ n_i &= \frac{1}{a'} \left(\gamma_i + \frac{k+1}{a'}\right) (\alpha + \delta_i) M^* \gamma_i^2 e^{-\gamma_i a'}, \quad i = 1, 2, \\ E_i^* &= \frac{E_1}{G}, \\ A_1 &= -\frac{n_2}{n_1} A_2, \\ A_2 &= -\left(\frac{n_1}{n_1 m_2 - n_2 m_1}\right) \frac{(a')^2 (1 - \mu_1^2)}{E_1^* h} \frac{\bar{q}(s)}{G}. \end{aligned} \tag{48}$$

Under the fluid pressure on shell surface, the displacement and stress components are continuous at the kinematic interface between the spherical shell and soil. In this case, flow boundary conditions are

$$q_0(t) = -\sigma_r \quad \text{at } r = a + \frac{h}{2} = a', \tag{49}$$

$$u_r = -u_r' \quad \text{at } r = a + \frac{h}{2} = a', \tag{50}$$

$$\frac{\partial p}{\partial r} = \frac{k}{a} (p + q(t)) \quad \text{at } r = a + \frac{h}{2} = a'. \tag{51}$$

Substituting (34), (39), (49), (50) into (43) and (37) into (51) yields

$$m_1 A_1 + m_2 A_2 = 0, \qquad n_1 A_1 + n_2 A_2 = \frac{k}{a'} \frac{\bar{q}(s)}{G}, \tag{52}$$

Table 1: Parameters used in computation.

Quantity	Notation	Value
Elastic modulus of soil (MPa)	E_s	7
Poisson rate of soil	μ	0.35
Poisson rate of shell	μ_l	0.09
Dimensionless shell density	ρ_l^*	1.4
Dimensionless shell thickness	h/a	0.04
Dimensionless fluid density	ρ^*	0.45
Compressible parameter of material	α	0.96
Dimensionless compressible parameter	M^*	18
Dimensionless permeability coefficient	b^*	9
Viscoelastic damp coefficient	η	0.35
Porosity	n	0.35
Gradually applied step load time	T^*	1

or

$$A_1 = -\frac{m_1}{m_2} A_2,$$
$$A_2 = \frac{k}{a'} \left(\frac{m_1}{m_1 n_2 - m_2 n_1} \right) \frac{\overline{q}(s)}{G}. \tag{53}$$

Using inverse Laplace transform and numerical computation, the final solutionin time domain can be obtained after determining A_1 and A_2.

4. Results and Discussion

In this paper, we will discuss the influences of partial permeable property of boundary and relative rigidity of shell and soil (defined as RR $= E_l/E_s$) on the transient response of the spherical cavity. The numerical results are presented for the material and geometric parameters which are listed in Table 1.

4.1. Solutions Corresponding to Fluid Pressure. The histories of dimensionless radial displacement under fluid pressure are shown in Figure 2 when parameters $k = 0.5$. It is noted that at a certain time instant as shown in Figure 2, there exists maximum displacement at the interface of shell and soil. With the increase of time, radial displacement decreased vibrationally and finally to an asymptotic value of zero. Radial displacement decreased obviously with increasing relative rigidity, and increased with increasing of parameter k (Figure 3).

The excess pore pressures induced by fluid pressure are shown in Figure 4. However, the influence of parameter k (Figure 4) is significant. When the shell boundary became almost impermeable ($k \to 0$), almost no excess pore pressure existed, whereas with the increasing of time, the excess pore pressure at the interface equaled the fluid pressure ($P = -q_0$) when the shell boundary became almost permeable ($k \to \infty$).

4.2. Solutions Corresponding to Radial Load. The histories of dimensionless radial displacement at the interface of shell and soil induced by axially symmetric radial surface load are shown in Figure 5 when the parameter $k = 0.5$. With the increase of the dimensionless time ($t^* = t\sqrt{G/\rho/a}$), radial displacement increases to maximum value, then decreases and is noted once again. Eventually, it tends to an asymptotic value. When relative rigidity RR $= 0$, the shell is complete flexible, there is the maximum radial displacement at the interface of shell and soil. The value of radial displacement

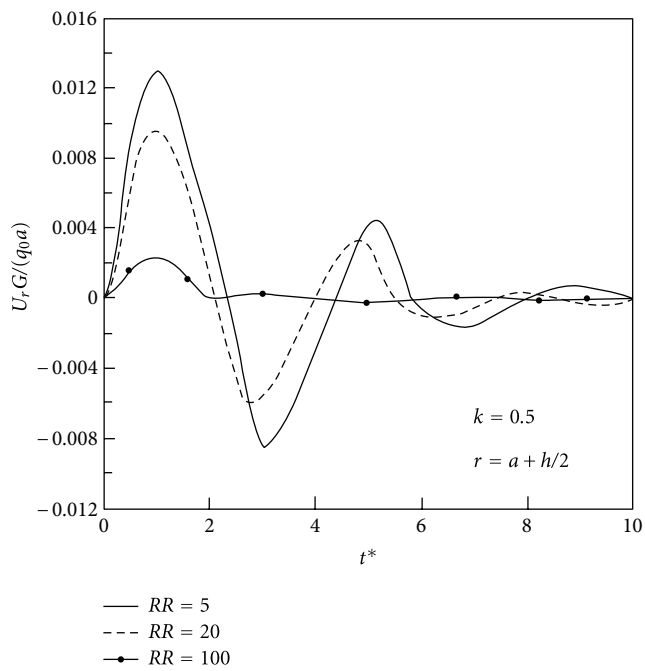

Figure 2: Curve of radial displacement versus relative rigidity (fluid pressure).

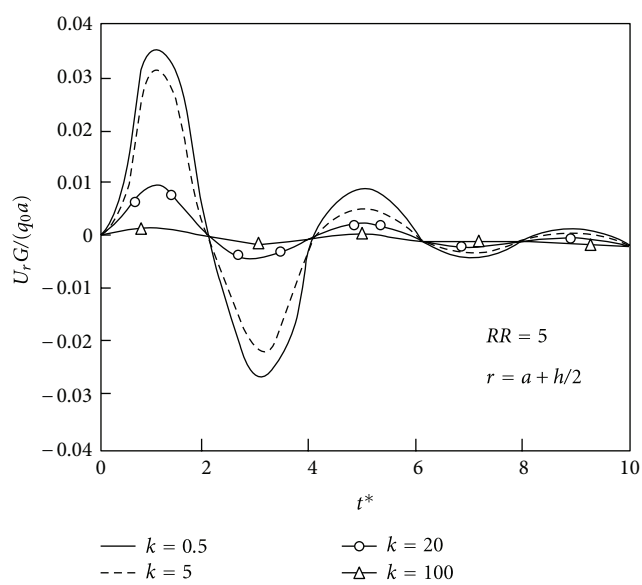

Figure 3: Curve of radial displacement versus parameter k (fluid pressure).

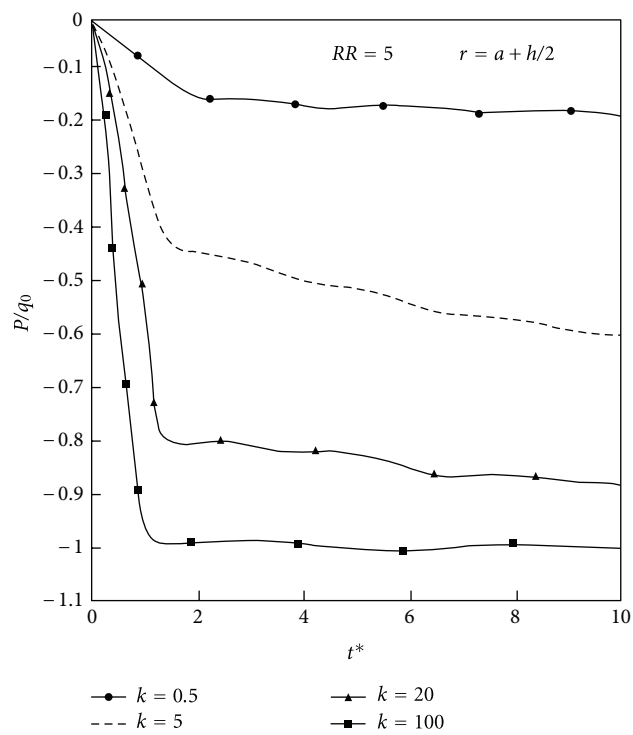

Figure 4: Curve of pore pressure versus parameter k (fluid pressure).

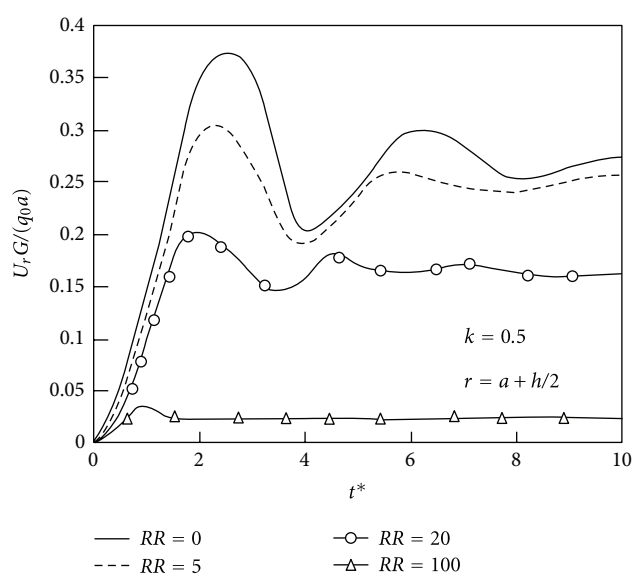

Figure 5: Curve of radial displacement versus relative rigidity.

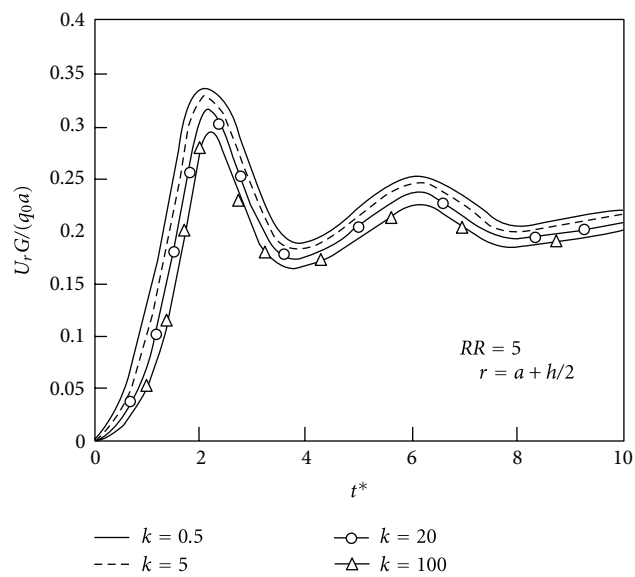

Figure 6: Curve of radial displacement versus parameter k.

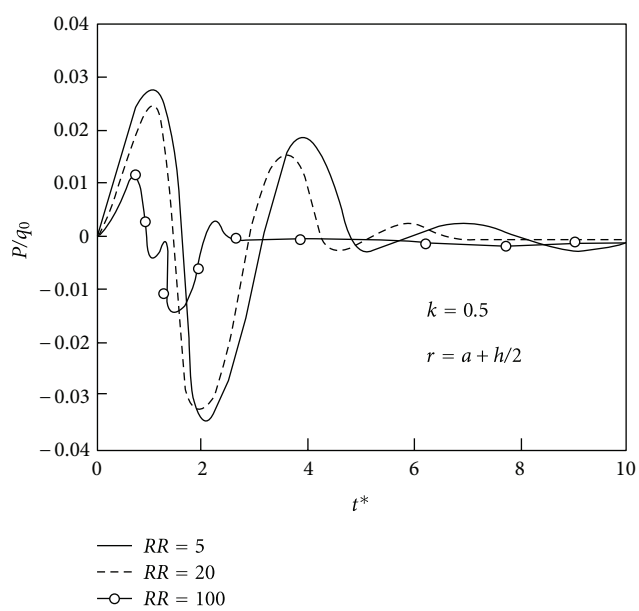

Figure 7: Curve of pore pressure versus relative rigidity.

decreases with the increase of relative rigidity. The influence of permeability parameter k on radial displacement is indicated in Figure 6. It can be seen that the influence of parameter k on radial displacement induced by axisymmetric radial surface load is not remarkable.

The histories of dimensionless pore pressure are shown in Figure 7 at the interface of shell and soil for the parameter $k = 0.5$. Pore pressure is zero at $t^* = 0$ and increases rapidly with time in the interval $0 < t^* \leq T^*$ and reaches to its peak value nearly at $t^* = T^*$. Thereafter, it decreases with time and reaches to its maximum suction values. With increasing time the values of suction decreases and pore pressure is noted once again. On the other hand, the pore pressure decreases with the increase of parameter k (Figure 8). As a result, both the relative rigidity and parameter k have great influence on the pore pressure under the condition of axisymmetric radial surface load.

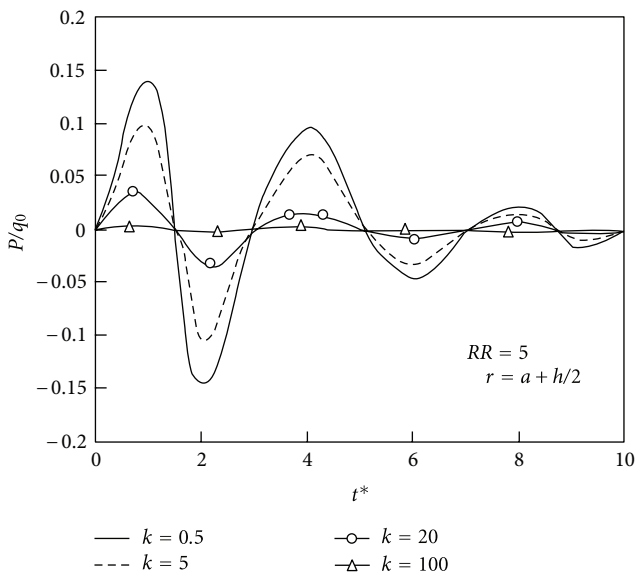

Figure 8: Curve of pore pressure versus parameter k.

5. Conclusions

An extensive parameters study conducted to investigate the influence of therelative rigidity of shell and soil and permeability parameter k, showed that permeability parameter k depends on the relative permeability of the liner and soil as well as the geometry of the liner. Relative rigidity and parameter k have significant influences on the transient response of spherical cavity with a shell embedded in viscoelastic saturated soil. The solutions under permeable and impermeable boundary conditions are only two extreme cases. Thus partially sealed boundary condition and the relative rigidity of shell and soil in the designing and computation of spherical shell in viscoelastic saturated medium are remarkable.

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