Research Article

Unified Metrical Common Fixed Point Theorems in 2-Metric Spaces via an Implicit Relation

Sunny Chauhan,¹ Mohammad Imdad,² and Calogero Vetro³

¹ Department of Mathematics, R.H. Government Postgraduate College, Kashipur 244 713, U.S. Nagar, Uttarakhand, India

² Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India

³ Università degli Studi di Palermo, Dipartimento di Matematica e Informatica, Via Archirafi 34, 90123 Palermo, Italy

Correspondence should be addressed to Sunny Chauhan; sun.gkv@gmail.com

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We prove some common fixed point theorems for two pairs of weakly compatible mappings in 2-metric spaces via an implicit relation. As an application to our main result, we derive Bryant's type generalized fixed point theorem for four finite families of self-mappings which can be utilized to derive common fixed point theorems involving any finite number of mappings. Our results improve and extend a host of previously known results. Moreover, we study the existence of solutions of a nonlinear integral equation.

1. Introduction and Preliminaries

In 1963, Gähler [1] initiated the concept of 2-metric space as a natural generalization of a metric space. The topology induced by 2-metric space is called 2-metric topology which is generated by the set of all open spheres with two centers (see [2, 3]). In this course of development, Iséki [4] studied the fixed point theorems in 2-metric spaces. For more references on the recent development of common fixed point theory in 2-metric spaces, we refer readers to [5–19].

In metric fixed point theory, implicit relations are often utilized to cover several contraction conditions in one go rather than proving a separate theorem for each contraction condition. The first ever attempt to coin an implicit function can be traced back to Popa [20]. Recently, Popa et al. [21] proved some interesting fixed point results for weakly compatible mappings in 2-metric spaces satisfying an implicit relation.

In this paper, utilizing the implicit function due to Popa et al. [21], we prove some common fixed point theorems for two pairs of weakly compatible mappings employing common limit range property. In process, many known results (especially the ones contained in Popa et al. [21, 22])

are enriched and improved. Some related results are also derived. Finally, we study the existence of solutions of a nonlinear integral equation using the presented results.

A 2-metric space is a set X equipped with a real valued function d on X^3 which satisfies the following conditions:

- (M_1) For distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.
- (M_2) $d(x, y, z) = 0$ if at least two of x, y, z are equal.
- (M_3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$, for all $x, y, z \in X$.
- (M_4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$, for all $x, y, z, u \in X$.

The function d is called a 2-metric on the set X whereas the pair (X, d) stands for 2-metric space. Geometrically a 2-metric $d(x, y, z)$ represents the area of a triangle with vertices x, y , and z .

It has been known since Gähler [1] that a 2-metric d is a nonnegative continuous function in any one of its three arguments but it does not need that to be continuous in two arguments. A 2-metric d is said to be continuous if it is continuous in all of its arguments. Throughout this paper d stands for a continuous 2-metric.

Definition 1. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be

- (1) convergent to a point $x \in X$, denoted by $\lim_{n \rightarrow \infty} x_n = x$, if $\lim_{n \rightarrow \infty} d(x_n, x, z) = 0$ for all $z \in X$;
- (2) Cauchy sequence if $\lim_{n, m \rightarrow \infty} d(x_n, x_m, z) = 0$ for all $z \in X$.

A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

In 1986, Jungck [23] introduced the notion of compatible mappings and utilized the same (as a tool) to improve commutativity conditions due to Sessa [24] in common fixed point theorems. However, the study of common fixed points of noncompatible mappings is also equally interesting which has been initiated by Pant [25, 26]. Jungck [27] introduced the notion of weakly compatible mappings in ordinary metric spaces and proved common fixed point theorems under minimal commutativity requirement. In recent years, using this idea several general common fixed point theorems have been proved in metric spaces.

Definition 2. Let $A, S : X \rightarrow X$ be two self-mappings of a 2-metric space (X, d) . Then the pair (A, S) is said to be

- (1) compatible [28] if $\lim_{n \rightarrow \infty} d(ASx_n, SAsx_n, a) = 0$ for all $a \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$, for some $t \in X$;
- (2) noncompatible [22] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$ for some $t \in X$ but $\lim_{n \rightarrow \infty} d(ASx_n, SAsx_n, a)$ for at least one $a \in X$ is either nonzero or nonexistent;
- (3) weakly compatible [29] if they commute at their coincidence points; that is, $ASu = SAu$ whenever $Au = Su$, for some $u \in X$.

For more details on systematic comparisons and illustrations of earlier described notions, we refer readers to [28, 29].

Inspired by the work of Aamri and El Moutawakil [30], Popa et al. [22] studied the notion of property (E.A) in the settings of 2-metric spaces.

Definition 3. A pair (A, S) of self-mappings of a 2-metric space (X, d) is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} d(Ax_n, Sx_n, a) = 0, \quad (1)$$

for all $a \in X$.

In 2005, Liu et al. [31] defined the notion of common property (E.A) for hybrid pairs of mappings which contains the property (E.A).

Definition 4. Two pairs (A, S) and (B, T) of self-mappings of a 2-metric space (X, d) are said to satisfy the common property (E.A), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} \lim_{n \rightarrow \infty} d(Ax_n, t, a) &= \lim_{n \rightarrow \infty} d(Sx_n, t, a) \\ &= \lim_{n \rightarrow \infty} d(By_n, t, a) \\ &= \lim_{n \rightarrow \infty} d(Ty_n, t, a) = 0, \end{aligned} \quad (2)$$

for all $a \in X$ and some $t \in X$.

Notice that the recent results, contained in Popa et al. [22] proved for weakly compatible mappings under the property (E.A), always require the completeness of the underlying subspace for the existence of common fixed point. In 2011, Sintunavarat and Kumam [32] introduced the notion of “common limit range property” which relaxes the requirement on completeness (or closedness) of the underlying subspaces (also see [33–35]). Since then, Imdad et al. [36, 37] extended the notion of common limit range property to two pairs of self-mappings and proved common fixed point theorems in Menger and metric spaces, respectively.

Now we define the notion of common limit range property in 2-metric spaces as follows.

Definition 5. A pair (A, S) of self-mappings of a 2-metric space (X, d) is said to satisfy the common limit range property with respect to mapping S , denoted by (CLR_S) , if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} d(Ax_n, t, a) = \lim_{n \rightarrow \infty} d(Sx_n, t, a) = 0, \quad (3)$$

where $t \in S(X)$ and for all $a \in X$.

Thus, one can infer that a pair (A, S) satisfying the property (E.A) along with closedness of the subspace $S(X)$ always enjoys the (CLR_S) property with respect to the mapping S (see [36, Examples 2.16–2.17]).

Definition 6. Two pairs (A, S) and (B, T) of self-mappings of a 2-metric space (X, d) are said to satisfy the common limit range property with respect to mappings S and T , denoted by (CLR_{ST}) , if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} \lim_{n \rightarrow \infty} d(Ax_n, t, a) &= \lim_{n \rightarrow \infty} d(Sx_n, t, a) \\ &= \lim_{n \rightarrow \infty} d(By_n, t, a) \\ &= \lim_{n \rightarrow \infty} d(Ty_n, t, a) = 0, \end{aligned} \quad (4)$$

where $t \in S(X) \cap T(X)$ and for all $a \in X$.

Definition 7 (see [38]). Two families of self-mappings $\{A_i\}_{i=1}^m$ and $\{S_k\}_{k=1}^n$ are said to be pairwise commuting if

- (1) $A_i A_j = A_j A_i$ for all $i, j \in \{1, 2, \dots, m\}$,
- (2) $S_k S_l = S_l S_k$ for all $k, l \in \{1, 2, \dots, n\}$,
- (3) $A_i S_k = S_k A_i$ for all $i \in \{1, 2, \dots, m\}$ and $k \in \{1, 2, \dots, n\}$.

2. Implicit Functions

Let $\mathcal{F}$ be the set of all lower semicontinuous functions $F : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ satisfying the following conditions:

- (F_1) $F(u, 0, 0, u, u, 0) \leq 0$ implies $u = 0$,
- (F_2) $F(u, 0, u, 0, 0, u) \leq 0$ implies $u = 0$,
- (F_3) $F(u, u, 0, 0, u, u) \leq 0$ implies $u = 0$.

The following examples are furnished in Popa et al. [21] establishing the utility of the preceding definition. Another examples can be found in Pathak et al. [39].

Example 8. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2} \right\}, \quad (5)$$

where $k \in (0, 1)$.

Example 9. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - t_1(\alpha t_2 + \beta t_3 + \gamma t_4) - \eta t_5 t_6, \quad (6)$$

where $\alpha > 0$, $\beta, \gamma, \eta \geq 0$, $\alpha + \beta + \gamma < 1$, and $\alpha + \eta < 1$.

Example 10. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - \alpha t_1^2 t_2 - \beta t_1 t_3 t_4 - \gamma t_5^2 t_6 - \eta t_5 t_6^2, \quad (7)$$

where $\alpha > 0$, $\beta, \gamma, \eta \geq 0$, $\alpha + \gamma + \eta < 1$, and $\alpha + \beta < 1$.

Example 11. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - \alpha \frac{t_3^2 t_4^2 + t_5^2 t_6^2}{1 + t_2 + t_3 + t_4}, \quad (8)$$

where $\alpha \in (0, 1)$.

Example 12. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \alpha t_2^2 - \beta \frac{t_5 t_6}{1 + t_3^2 + t_4^2}, \quad (9)$$

where $\alpha > 0$, $\beta \geq 0$ and $\alpha + \beta < 1$.

Example 13. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = \begin{cases} t_1 - a_1 \frac{t_3^2 + t_4^2}{t_3 + t_4} - a_2 t_2 - a_3 (t_5 + t_6), & \text{if } t_3 + t_4 \neq 0, \\ t_1, & \text{if } t_3 + t_4 = 0, \end{cases} \quad (10)$$

where $a_i \geq 0$ with at least one a_i nonzero and $a_1 + a_2 + 2a_3 < 1$.

Example 14. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = \begin{cases} t_1 - \alpha t_2 - \frac{\beta t_3 t_4 + \gamma t_5 t_6}{t_3 + t_4}, & \text{if } t_3 + t_4 \neq 0, \\ t_1, & \text{if } t_3 + t_4 = 0, \end{cases} \quad (11)$$

where $\alpha, \beta, \gamma \geq 0$ such that $1 < 2\alpha + \beta < 2$.

Example 15. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a_1 t_2 - a_2 t_3 - a_3 t_4 - a_4 t_5 - a_5 t_6, \quad (12)$$

where $\sum_{i=1}^5 a_i < 1$.

Example 16. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \left[\beta \max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2} \right\} + (1 - \beta) \times \left[\max \left\{ t_2^2, t_3 t_4, t_5 t_6, \frac{t_3 t_6}{2}, \frac{t_4 t_5}{2} \right\} \right]^{1/2} \right], \quad (13)$$

where $\alpha \in (0, 1)$ and $0 \leq \beta \leq 1$.

Example 17. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \alpha \max \{ t_2^2, t_3^2, t_4^2 \} - \beta \max \left\{ \frac{t_3 t_5}{2}, \frac{t_4 t_6}{2} \right\}, \quad (14)$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma < 1$.

Example 18. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi \left(\max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2} \right\} \right), \quad (15)$$

where $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing upper semicontinuous function with $\phi(0) = 0$ and $\phi(t) < t$ for each $t > 0$.

Example 19. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi(\max \{ t_2, t_3, t_4, t_5, t_6 \}), \quad (16)$$

where $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is upper semicontinuous and non-decreasing function in each coordinate variable such that $\phi(t, t, \alpha t, \beta t, \gamma t) < t$ for each $t > 0$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma \leq 3$.

Example 20. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \phi(t_2^2, t_3 t_4, t_5 t_6, t_3 t_6, t_4 t_5), \quad (17)$$

where $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is upper semicontinuous and non-decreasing function in each coordinate variable such that $\phi(t, t, \alpha t, \beta t, \gamma t) < t$ for each $t > 0$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma \leq 3$.

Apart from earlier stated definitions, still there are many contractive definitions which meet the requirements (F_1) , (F_2) and (F_3) but due to paucity of the space we have not opted to include more examples.

3. Main Results

We begin with the following observation.

Lemma 21. *Let A, B, S , and T be self-mappings of a 2-metric space (X, d) . Suppose that*

- (1) *the pair (A, S) satisfies the (CLR_S) property (or (B, T) satisfies the (CLR_T) property),*
- (2) *$A(X) \subset T(X)$ (or $B(X) \subset S(X)$),*
- (3) *$T(X)$ (or $S(X)$) is a closed subset of X ,*
- (4) *$\{By_n\}$ converges for every sequence $\{y_n\}$ in X whenever $\{Ty_n\}$ converges (or $\{Ax_n\}$ converges for every sequence $\{x_n\}$ in X whenever $\{Sx_n\}$ converges),*
- (5) *there exists $F \in \mathcal{F}$ such that (for all $x, y \in X$ and $a \in X$)*

$$F(d(Ax, By, a), d(Sx, Ty, a), d(Ax, Sx, a), d(By, Ty, a), d(Sx, By, a), d(Ty, Ax, a)) \leq 0. \quad (18)$$

Then the pairs (A, S) and (B, T) share the (CLR_{ST}) property.

Proof. Since the pair (A, S) satisfies the (CLR_S) property with respect to mapping S , there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} d(Ax_n, t, a) = \lim_{n \rightarrow \infty} d(Sx_n, t, a) = 0, \quad (19)$$

where $t \in S(X)$. As $A(X) \subset T(X)$, for each sequence $\{x_n\}$ there exists a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore, due to closedness of $T(X)$,

$$\lim_{n \rightarrow \infty} d(Ty_n, t, a) = \lim_{n \rightarrow \infty} d(Ax_n, t, a) = 0, \quad (20)$$

where $t \in S(X) \cap T(X)$. Thus in all, we have $Ax_n \rightarrow t$, $Sx_n \rightarrow t$, and $Ty_n \rightarrow t$ as $n \rightarrow \infty$. By (4), the sequence $\{By_n\}$ converges, and in all, we need to show that $By_n \rightarrow t$ as $n \rightarrow \infty$. On using inequality (18) with $x = x_n$, $y = y_n$, we have

$$F(d(Ax_n, By_n, a), d(Sx_n, Ty_n, a), d(Ax_n, Sx_n, a), d(By_n, Ty_n, a), d(Sx_n, By_n, a), d(Ty_n, Ax_n, a)) \leq 0. \quad (21)$$

Let on contrary $By_n \rightarrow t' (\neq t)$ as $n \rightarrow \infty$. Then, taking limit as $n \rightarrow \infty$, we get

$$F(d(t, t', a), d(t, t, a), d(t, t, a), d(t', t, a), d(t, t', a), d(t, t, a)) \leq 0, \quad (22)$$

or

$$F(d(t, t', a), 0, 0, d(t', t, a), d(t, t', a), 0) \leq 0, \quad (23)$$

yielding thereby $d(t, t', a) = 0$ for all $a \in X$ (due to (F_1)). Hence $By_n \rightarrow t$, which shows that the pairs (A, S) and (B, T) share the (CLR_{ST}) property. This concludes the proof. $\square$

Remark 22. In general, the converse of Lemma 21 is not true (see [36, Example 3.5]).

Now, we state and prove our main result for two pairs of weakly compatible mappings satisfying the (CLR_{ST}) property.

Theorem 23. *Let A, B, S , and T be self-mappings of a metric space (X, d) satisfying the inequality (18) (of Lemma 21). If the pairs (A, S) and (B, T) share the (CLR_{ST}) property, then (A, S) and (B, T) have a coincidence point each. Moreover, A, B, S , and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof. Since the pairs (A, S) and (B, T) share the (CLR_{ST}) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} \lim_{n \rightarrow \infty} d(Ax_n, t, a) &= \lim_{n \rightarrow \infty} d(Sx_n, t, a) \\ &= \lim_{n \rightarrow \infty} d(By_n, t, a) \\ &= \lim_{n \rightarrow \infty} d(Ty_n, t, a) \\ &= 0, \end{aligned} \quad (24)$$

where $t \in S(X) \cap T(X)$. Since $t \in S(X)$, there exists a point $u \in X$ such that $Su = t$. We show that $Au = t$. On using inequality (18) with $x = u$ and $y = y_n$, we get

$$F(d(Au, By_n, a), d(Su, Ty_n, a), d(Au, Su, a), d(By_n, Ty_n, a), d(Su, By_n, a), d(Ty_n, Au, a)) \leq 0, \quad (25)$$

which on making $n \rightarrow \infty$, reduces to

$$F(d(Au, t, a), d(t, t, a), d(Au, t, a), d(t, t, a), d(t, t, a), d(t, Au, a)) \leq 0, \quad (26)$$

or

$$F(d(Au, t, a), 0, d(Au, t, a), 0, 0, d(t, Au, a)) \leq 0, \quad (27)$$

implying thereby $d(Au, t, a) = 0$ for all $a \in X$ (due to (F_2)). Hence $Au = Su = t$ which shows that u is a coincidence point of the pair (A, S) .

As $t \in T(X)$, there exists a point $v \in X$ such that $Tv = t$. We assert that $Bv = Tv$. On using inequality (18) with $x = u$, $y = v$, we get

$$F(d(Au, Bv, a), d(Su, Tv, a), d(Au, Su, a), d(Bv, Tv, a), d(Su, Bv, a), d(Tv, Au, a)) \leq 0, \quad (28)$$

which reduces to

$$F(d(t, Bv, a), d(t, t, a), d(t, t, a), d(Bv, t, a), d(t, Bv, a), d(t, t, a)) \leq 0, \quad (29)$$

or

$$F(d(t, Bv, a), 0, 0, d(Bv, t, a), d(t, Bv, a), 0) \leq 0, \quad (30)$$

yielding thereby $d(t, Bv, a) = 0$ for all $a \in X$ (due to (F_1)). Hence $Bv = Tv = t$, which shows that v is a coincidence point of the pair (B, T) .

Since the pair (A, S) is weakly compatible and $Au = Su$, hence $At = ASu = SAu = St$. Now, we assert that t is a common fixed point of the pair (A, S) . On using inequality (18) with $x = t$, $y = v$, we have

$$\begin{aligned} F(d(At, Bv, a), d(St, Tv, a), d(At, St, a), \\ d(Bv, Tv, a), d(St, Bv, a), d(Tv, At, a)) \leq 0, \end{aligned} \quad (31)$$

or

$$F(d(At, t, a), d(At, t, a), 0, 0, d(At, t, a), d(t, At, a)) \leq 0, \quad (32)$$

implying thereby $d(At, t, a) = 0$ for all $a \in X$ (due to (F_3)). Hence $At = t = St$ which shows that t is a common fixed point of the pair (A, S) .

Also the pair (B, T) is weakly compatible and $Bv = Tv$, then $Bt = BTv = TBv = Tt$. On using inequality (18) with $x = u$ and $y = t$, we have

$$\begin{aligned} F(d(Au, Bt, a), d(Su, Tt, a), d(Au, Su, a), \\ d(Bt, Tt, a), d(Su, Bt, a), d(Tt, Au, a)) \leq 0, \end{aligned} \quad (33)$$

or

$$F(d(t, Bt, a), d(t, Bt, a), 0, 0, d(t, Bt, a), d(Bt, t, a)) \leq 0, \quad (34)$$

yielding thereby $d(t, Bt, a) = 0$ for all $a \in X$ (due to (F_3)). Therefore, $Bt = t = Tt$ which shows that t is a common fixed point of the pair (B, T) . Hence t is a common fixed point of both the pairs (A, S) and (B, T) .

To prove the uniqueness, let z be another common fixed point of A, B, S , and T . On using inequality (18) with $x = t$ and $x = z$, we have

$$\begin{aligned} F(d(At, Bz, a), d(St, Tz, a), d(At, St, a), \\ d(Bz, Tz, a), d(St, Bz, a), d(Tz, At, a)) \leq 0, \end{aligned} \quad (35)$$

or

$$\begin{aligned} F(d(t, z, a), d(t, z, a), d(t, t, a), \\ d(z, z, a), d(t, z, a), d(z, t, a)) \leq 0, \end{aligned} \quad (36)$$

or

$$F(d(t, z, a), d(t, z, a), 0, 0, d(t, z, a), d(z, t, a)) \leq 0, \quad (37)$$

implying thereby $d(t, z, a) = 0$ for all $a \in X$ (due to (F_3)). Hence $t = z$. This concludes the proof. $\square$

Remark 24. Theorem 23 improves the corresponding result contained in Popa et al. [21, Theorem 4.1] as completeness (or closedness) of the underlying subspaces is not required.

Now, we present an example which demonstrates the validity of the hypotheses and degree of generality of our main result over comparable ones from the existing literature.

Example 25. Let (X, d) be a metric space wherein $X = [1, 19]$ equipped with usual metric $d(x, y) = |x - y|$. Define a 2-metric on X by $d(x, y, z) = \min\{|x - y|, |y - z|, |z - x|\}$, for all $x, y, z \in X$. Also the self-mappings A, B, S , and T are defined by $A(x) = B(x) = 1$,

$$S(x) = \begin{cases} 1, & \text{if } x = 1, \\ 3, & \text{if } x \in (1, 4], \\ \frac{x+1}{5}, & \text{if } x \in (4, 19), \end{cases} \quad (38)$$

$$T(x) = \begin{cases} 1, & \text{if } x = 1, \\ 15, & \text{if } x \in (1, 4], \\ x - 3, & \text{if } x \in (4, 19). \end{cases}$$

Then we have $S(X) = [1, 4]$ and $T(X) = [1, 16]$. Consider the implicit function $F(t_1, t_2, t_3, t_4, t_5, t_6): (\mathbb{R}_+)^6 \rightarrow \mathbb{R}$ given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2} \right\}, \quad (39)$$

where $k \in (0, 1)$. By a routine calculation, one can verify the following inequality for all $x, y \in X$, $a \in X$, and $k \in (0, 1)$:

$$\begin{aligned} & d(Ax, By, a) \\ & - k \max \left\{ d(Sx, Ty, a), d(Ax, Sx, a), d(By, Ty, a), \right. \\ & \left. \frac{d(Sx, By, a) + d(Ty, Ax, a)}{2} \right\} \leq 0. \end{aligned} \quad (40)$$

Now, if we choose two sequences as $\{x_n\} = \{4 + 1/n\}_{n \in \mathbb{N}}$, $\{y_n\} = \{1\}$ (or $\{x_n\} = \{1\}$, $\{y_n\} = \{4 + 1/n\}_{n \in \mathbb{N}}$), then the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property. In fact

$$\begin{aligned} \lim_{n \rightarrow \infty} d(Ax_n, 1, a) &= \lim_{n \rightarrow \infty} d(Sx_n, 1, a) \\ &= \lim_{n \rightarrow \infty} d(By_n, 1, a) \\ &= \lim_{n \rightarrow \infty} d(Ty_n, 1, a) = 0, \end{aligned} \quad (41)$$

where $1 \in S(X) \cap T(X)$. Thus all the conditions of Theorem 23 are satisfied and 1 is a unique common fixed point of the pairs (A, S) and (B, T) which also remains a point of coincidence as well.

Notice that in the context of this example, $S(X)$ and $T(X)$ are not complete (or closed) subsets of X ; therefore, Theorem 4.1 of Popa et al. [21] cannot be used in the context of this example which establishes the genuineness of our extension.

Theorem 26. Let A, B, S , and T be self-mappings of a 2-metric space (X, d) satisfying all the hypotheses of Lemma 21. Then A, B, S , and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. In view of Lemma 21, the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property so that there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} \lim_{n \rightarrow \infty} d(Ax_n, t, a) &= \lim_{n \rightarrow \infty} d(Sx_n, t, a) \\ &= \lim_{n \rightarrow \infty} d(Ty_n, t, a) \\ &= \lim_{n \rightarrow \infty} d(By_n, t, a) = 0, \end{aligned} \quad (42)$$

where $t \in S(X) \cap T(X)$. The rest of the proof can be completed on the lines of the proof of Theorem 23; therefore, we omit the details. $\square$

The following example demonstrates the utility of Theorem 26 over Theorem 23.

Example 27. In the setting of Example 25, replace the self-mappings S and T by the following, besides retaining the rest:

$$\begin{aligned} S(x) &= \begin{cases} 1, & \text{if } x = 1, \\ 4, & \text{if } x \in (1, 4], \\ \frac{x+1}{5}, & \text{if } x \in (4, 19), \end{cases} \\ T(x) &= \begin{cases} 1, & \text{if } x = 1, \\ 16, & \text{if } x \in (1, 4], \\ x-3, & \text{if } x \in (4, 19). \end{cases} \end{aligned} \quad (43)$$

Then, like earlier example, the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property. Consider the implicit function $F(t_1, t_2, t_3, t_4, t_5, t_6): (\mathbb{R}_+)^6 \rightarrow \mathbb{R}$ given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2} \right\}, \quad (44)$$

which yields the inequality

$$\begin{aligned} d(Ax, By, a) &- k \max \left\{ d(Sx, Ty, a), d(Ax, Sx, a), d(By, Ty, a), \right. \\ &\quad \left. \frac{d(Sx, By, a) + d(Ty, Ax, a)}{2} \right\} \leq 0, \end{aligned} \quad (45)$$

for all $x, y \in X, a \in X$ and $k \in (0, 1)$. Clearly, inequality (45) holds true. Also, $S(X) = [1, 4]$, $T(X) = [1, 16]$ and the pairs (A, S) and (B, T) commute at 1 which is also their common coincidence point as well. Thus all the conditions of Theorem 26 are satisfied and 1 is a unique common fixed point of the involved mappings A, B, S , and T .

Here, it can be pointed out that Theorem 23 is not applicable to this example as both $S(X), T(X)$ are complete subspaces of X ; this demonstrates the situational utility of Theorem 26 over Theorem 23.

Corollary 28. The conclusions of Lemma 21, Theorems 23 and 26 remain true if inequality (18) is replaced by one of the following contractive conditions, for all $x, y \in X, a \in X$

$$\begin{aligned} d(Ax, By, a) &\leq k \max \left\{ d(Sx, Ty, a), d(Ax, Sx, a), d(By, Ty, a), \right. \\ &\quad \left. \frac{1}{2} [d(Sx, By, a) + d(Ty, Ax, a)] \right\}, \end{aligned} \quad (46)$$

where $k \in (0, 1)$;

$$\begin{aligned} d^2(Ax, By, a) &\leq d(Ax, By, a) \\ &\quad \times [\alpha d(Sx, Ty, a) + \beta d(Ax, Sx, a) + \gamma d(By, Ty, a)] \\ &\quad + \eta d(Sx, By, a) d(Ty, Ax, a), \end{aligned} \quad (47)$$

where $\alpha > 0, \beta, \gamma, \eta \geq 0, \alpha + \beta + \gamma < 1$ and $\alpha + \eta < 1$;

$$\begin{aligned} d^3(Ax, By, a) &\leq \alpha d^2(Ax, By, a) d(Sx, Ty, a) \\ &\quad + \beta (Ax, By, a) d(Ax, Sx, a) d(By, Ty, a) \\ &\quad + \gamma d^2(Sx, By, a) d(Ty, Ax, a) \\ &\quad + \eta d(Sx, By, a) d^2(Ty, Ax, a), \end{aligned} \quad (48)$$

where $\alpha > 0, \beta, \gamma, \eta \geq 0, \alpha + \beta < 1$ and $\alpha + \gamma + \eta < 1$;

$$\begin{aligned} d^3(Ax, By, a) &\leq \alpha \left(d^2(Ax, Sx, a) d^2(By, Ty, a) \right. \\ &\quad + d^2(Sx, By, a) d^2(Ty, Ax, a)) \\ &\quad \times (1 + d(Sx, Ty, a) \\ &\quad \left. + d(Sx, Ax, a) + d(Ty, By, a) \right)^{-1}, \end{aligned} \quad (49)$$

where $\alpha \in (0, 1)$;

$$\begin{aligned} d^2(Ax, By, a) &\leq \alpha d^2(Sx, Ty, a) \\ &\quad + \beta \frac{d(Sx, By, a) d(Ty, Ax, a)}{1 + d^2(Ax, Sx, a) + d^2(By, Ty, a)}, \end{aligned} \quad (50)$$

where $\alpha > 0$, $\beta \geq 0$ and $\alpha + \beta < 1$;

$$\begin{aligned} d(Ax, By, a) &\leq a_1 \frac{d^2(Ax, Sx, a) + d^2(By, Ty, a)}{d(Ax, Sx, a) + d(By, Ty, a)} \\ &\quad + a_2 d(Sx, Ty, a) \\ &\quad + a_3 (d(Sx, By, a) + d(Ty, Ax, a)), \end{aligned} \quad (51)$$

where $a_i \geq 0$ with at least one a_i nonzero and $a_1 + a_2 + 2a_3 < 1$;

$$\begin{aligned} d(Ax, By, a) &\leq \alpha d(Sx, Ty, a) \\ &\quad + ((\beta d(Ax, Sx, a) d(By, Ty, a) \\ &\quad + \gamma d(Sx, By, a) d(Ty, Ax, a)) \\ &\quad \times (d(Ax, Sx, a) + d(By, Ty, a))^{-1}), \end{aligned} \quad (52)$$

where $\alpha, \beta, \gamma \geq 0$ such that $1 < 2\alpha + \beta < 2$;

$$\begin{aligned} d(Ax, By, a) &\leq a_1 d(Sx, Ty, a) \\ &\quad + a_2 d(Ax, Sx, a) + a_3 d(By, Ty, a) \\ &\quad + a_4 d(Sx, By, a) + a_5 d(Ty, Ax, a), \end{aligned} \quad (53)$$

where $\sum_{i=1}^5 a_i < 1$;

$$\begin{aligned} d(Ax, By, a) &\leq \alpha \left[\beta \max \left\{ d(Sx, Ty, a), d(Ax, Sx, a), d(By, Ty, a), \right. \right. \\ &\quad \left. \left. \frac{1}{2} [d(Sx, By, a) + d(Ty, Ax, a)] \right\} \right. \\ &\quad + (1 - \beta) \\ &\quad \times \left[\max \left\{ d^2(Sx, Ty, a), \right. \right. \\ &\quad d(Ax, Sx, a) d(By, Ty, a), \\ &\quad d(Sx, By, a) d(Ty, Ax, a), \\ &\quad \frac{1}{2} d(Ax, Sx, a) d(Ty, Ax, a), \\ &\quad \left. \left. \frac{1}{2} d(By, Ty, a) d(Sx, By, a) \right\} \right]^{1/2} \Big], \end{aligned} \quad (54)$$

where $\alpha \in (0, 1)$ and $0 \leq \beta \leq 1$;

$$\begin{aligned} d^2(Ax, By, a) &\leq \alpha \max \{d^2(Sx, Ty, a), d^2(Ax, Sx, a), d^2(By, Ty, a)\} \\ &\quad + \beta \max \left\{ \frac{1}{2} d(Ax, Sx, a) d(Sx, By, a), \right. \\ &\quad \left. \frac{1}{2} d(By, Ty, a) d(Ty, Ax, a) \right\} \\ &\quad + \gamma d(Sx, By, a) d(Ty, Ax, a), \end{aligned} \quad (55)$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma < 1$;

$$\begin{aligned} d(Ax, By, a) &\leq \phi \left(\max \left\{ d(Sx, Ty, a), d(Ax, Sx, a), d(By, Ty, a), \right. \right. \\ &\quad \left. \left. \frac{1}{2} [d(Sx, By, a) + d(Ty, Ax, a)] \right\} \right), \end{aligned} \quad (56)$$

where $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing upper semicontinuous function with $\phi(0) = 0$ and $\phi(t) < t$ for each $t > 0$;

$$\begin{aligned} d(Ax, By, a) &\leq \phi \left(\max \{d(Sx, Ty, a), d(Ax, Sx, a), d(By, Ty, a), \right. \\ &\quad \left. d(Sx, By, a), d(Ty, Ax, a)\} \right), \end{aligned} \quad (57)$$

where $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an upper semicontinuous and non-decreasing function in each coordinate variable such that $\phi(t, t, \alpha t, \beta t, \gamma t) < t$ for each $t > 0$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma \leq 3$;

$$\begin{aligned} d^2(Ax, By, a) &\leq \phi \left(d^2(Sx, Ty, a), \right. \\ &\quad d(Ax, Sx, a) d(By, Ty, a), \\ &\quad d(Sx, By, a) d(Ty, Ax, a), \\ &\quad d(Ax, Sx, a) d(Ty, Ax, a), \\ &\quad \left. d(By, Ty, a) d(Sx, By, a) \right), \end{aligned} \quad (58)$$

where $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an upper semicontinuous and non-decreasing function in each coordinate variable such that $\phi(t, t, \alpha t, \beta t, \gamma t) < t$ for each $t > 0$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma \leq 3$.

Proof. The proof of each of inequalities (46)–(58) easily follows from Theorem 23 in view of Examples 8–20. $\square$

Remark 29. Corollary 28 improves and generalizes a multitude of well-known results especially those contained in [8–12, 15, 17, 21, 22, 28, 40–44] and others whereas some of these

present 2-metric space version of certain existing results of literature (e.g., Chugh and Kumar [45], Ali and Imdad [46], Jeong and Rhoades [47], Hardy and Rogers [48], Lal et al. [49], and others) besides yielding some results which are seeming new to the literature.

By choosing A , B , S , and T suitably, we can deduce corollaries involving two as well as three self-mappings. For the sake of naturality, we only derive the following corollary involving a pair of self-mappings.

Corollary 30. *Let A and S be self-mappings of a 2-metric space (X, d) . Suppose that*

- (1) *the pair (A, S) satisfies the (CLR_S) property,*
- (2) *there exists $F \in \mathcal{F}$ such that*

$$F(d(Ax, Ay, a), d(Sx, Sy, a), d(Ax, Sx, a), d(Ay, Sy, a), d(Sx, Ay, a), d(Sy, Ax, a)) \leq 0, \quad (59)$$

for all $x, y \in X$ and $a \in X$.

Then (A, S) has a coincidence point. Moreover, if the pair (A, S) is weakly compatible then the pair has a unique common fixed point in X .

As an application of Theorem 23, we state a Bryant's [50] type generalized common fixed point theorem involving four finite families of self-mappings.

Theorem 31. *Let $\{A_i\}_{i=1}^m$, $\{B_j\}_{j=1}^n$, $\{S_k\}_{k=1}^p$, and $\{T_l\}_{l=1}^q$ be four finite families of self-mappings of a 2-metric space (X, d) with $A = A_1 A_2 \cdots A_m$, $B = B_1 B_2 \cdots B_n$, $S = S_1 S_2 \cdots S_p$, and $T = T_1 T_2 \cdots T_q$ satisfying condition (18). Suppose that the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property; then (A, S) and (B, T) have a point of coincidence each.*

Moreover $\{A_i\}_{i=1}^m$, $\{B_j\}_{j=1}^n$, $\{S_k\}_{k=1}^p$, and $\{T_l\}_{l=1}^q$ have a unique common fixed point if the families $(\{A_i\}, \{S_k\})$ and $(\{B_j\}, \{T_l\})$ commute pairwise wherein $i \in \{1, 2, \dots, m\}$, $k \in \{1, 2, \dots, p\}$, $j \in \{1, 2, \dots, n\}$, and $l \in \{1, 2, \dots, q\}$.

Proof. The proof of this theorem can be completed on the lines of the corresponding theorem of Imdad et al. [38]. $\square$

Remark 32. Note that

- (1) a result similar to Theorem 31 can be outlined in respect of Theorem 23;
- (2) Theorem 31 improves and extends the corresponding results contained in Popa et al. [22].

Now, we indicate that Theorem 31 can be utilized to derive common fixed point theorems for any finite number of mappings. As a sample, we derive the following theorem for five mappings by setting one family of two members while the rest are three, of single members.

Corollary 33. *Let A , B , R , S , and T be self-mappings of a 2-metric space (X, d) . Suppose that*

- (1) *the pairs (A, SR) and (B, T) share the $(CLR_{(SR)(T)})$ property,*
- (2) *there exists $F \in \mathcal{F}$ such that*

$$F(d(Ax, By, a), d(SRx, Ty, a), d(Ax, SRx, a), d(By, Ty, a), d(SRx, By, a), d(Ty, Ax, a)) \leq 0, \quad (60)$$

for all $x, y \in X$ and $a \in X$.

Then (A, SR) and (B, T) have a coincidence point each. Moreover, A , B , R , S , and T have a unique common fixed point provided both pairs (A, SR) and (B, T) commute pairwise; that is, $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$.

Similarly, one can derive a common fixed point theorem for six mappings by setting two families of two members while the rest two are of single members.

Corollary 34. *Let A , B , H , R , S , and T be self-mappings of a 2-metric space (X, d) . Suppose that*

- (1) *the pairs (A, SR) and (B, TH) share the $(CLR_{(SR)(TH)})$ property,*
- (2) *there exists $F \in \mathcal{F}$ such that*

$$F(d(Ax, By, a), d(SRx, THy, a), d(Ax, SRx, a), d(By, THy, a), d(SRx, By, a), d(THy, Ax, a)) \leq 0, \quad (61)$$

for all $x, y \in X$ and $a \in X$.

Then (A, SR) and (B, TH) have a coincidence point each. Moreover, A , B , H , R , S , and T have a unique common fixed point provided both of the pairs (A, SR) and (B, TH) commute pairwise; that is, $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$, $BH = HB$, and $TH = HT$.

By setting $A_1 = A_2 = \cdots = A_m = A$, $B_1 = B_2 = \cdots = B_n = B$, $S_1 = S_2 = \cdots = S_p = S$, and $T_1 = T_2 = \cdots = T_q = T$ in Theorem 31, we deduce the following.

Corollary 35. *Let A , B , S , and T be self-mappings of a 2-metric space (X, d) . Suppose that*

- (1) *the pairs (A^m, S^p) and (B^n, T^q) share the $(CLR_{S^p T^q})$ property,*
- (2) *there exists $F \in \mathcal{F}$ such that*

$$F(d(A^m x, B^n y, a), d(S^p x, T^q y, a), d(A^m x, S^p x, a), d(B^n y, T^q y, a), d(S^p x, B^n y, a), d(T^q y, A^m x, a)) \leq 0, \quad (62)$$

for all $x, y \in X$, $a \in X$ and where m, n, p, q are fixed positive integers.

Then A , B , S , and T have a unique common fixed point provided $AS = SA$ and $BT = TB$.

Remark 36. Corollary 35 is a slight but partial generalization of Theorem 23 as the commutativity requirements (i.e., $AS = SA$ and $BT = TB$) in this corollary are relatively stronger as compared to weak compatibility in Theorem 23.

Remark 37. Results similar to Corollary 35 can be derived in respect of Theorem 23 and Corollary 28.

4. Application

Inspired by Pathak et al. [51], we study the existence of solutions of a nonlinear integral equation using the results proved in Section 3. Let $X = C([0, \infty), \mathbb{R})$ be the set of continuous real valued functions defined on $[0, \infty)$ endowed with the metric given by

$$\rho(x, y) = \sup_{t \in [0, \infty)} |x(t) - y(t)| \quad \forall x, y \in X. \quad (63)$$

Clearly, (X, ρ) is a complete metric space. Now, consider the integral equation:

$$\begin{aligned} x(t) = & u_1(t) - u_2(t) + \alpha \int_0^t h(t, s) f(s, x(s)) ds \\ & + \beta \int_0^\infty k(t, s) g(s, x(s)) ds, \end{aligned} \quad (64)$$

for all $t \in [0, \infty)$, where $u_1(t), u_2(t) \in X$ with $u_1(t) \geq u_2(t)$ are known, $h(t, s), k(t, s), g(s, x(s)), f(s, x(s)) \in C([0, \infty) \times [0, \infty), \mathbb{R})$, $\alpha, \beta \in \mathbb{R}$.

Now, we formulate our result.

Theorem 38. Assume that the following hypotheses hold:

- (a) $\int_0^\infty \sup_{s \in [0, \infty)} |h(t, s)| dt \leq 1/(|\alpha| + 1)$ and $\int_0^\infty \sup_{s \in [0, \infty)} |k(t, s)| dt \leq 1/(|\beta| + 1)$;
- (b) for all $s \in [0, \infty)$ there exists $K_1 > 0$ such that $|f(s, x(s)) - f(s, y(s))| \leq K_1 |x(s) - y(s)|$, for all $x, y \in X$;
- (c) for all $s \in [0, \infty)$, there exists $K_2 > 0$ such that $|g(s, x(s)) - g(s, y(s))| \leq K_2 |x(s) - y(s)|$, for all $x, y \in X$;
- (d) $K_1 + K_2 < 1$;
- (e) $|\alpha \int_0^t h(t, s) f(s, x(s)) ds - u_2(t) - z(t)| \leq (K_1/(1 - K_2)) |x(t) - z(t) - u_1(t) - \beta \int_0^\infty k(t, s) g(s, x(s)) ds|$, for all $x, z \in X$;
- (f) $\beta \int_0^\infty k(t, s) g(s, \alpha \int_0^s h(s, r) f(r, x(r)) dr + u_1(s) - u_2(s)) ds = 0$, for all $x \in X$;
- (g) there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = l \in A(X)$.

Then, the integral equation (64) has a unique solution in X .

Proof. We consider the operators $A, S : X \rightarrow X$ defined by

$$A(x)(t) = -u_2(t) + \alpha \int_0^t h(t, s) f(s, x(s)) ds, \quad (65)$$

$$S(x)(t) = x(t) - u_1(t) - \beta \int_0^\infty k(t, s) g(s, x(s)) ds.$$

It is easy to show that x is a solution to (64) if and only if x is a coincidence point of A and S . To establish the existence of such a point, we will use our Corollary 30. Then, we have to check that all the hypotheses of Corollary 30 are satisfied.

Firstly, suppose that d is the 2-metric on X given by $d(x, y, z) = \min\{\rho(x, y), \rho(y, z), \rho(z, x)\}$ for all $x, y, z \in X$. Since (X, ρ) is a complete metric space, then also (X, d) is complete. Next, we show that $A(X) \subseteq S(X)$. In fact, by using hypothesis (f), for all $x \in X$ we get

$$\begin{aligned} & S(A(x)(t) + u_1(t)) \\ &= A(x)(t) + u_1(t) - u_1(t) \\ &\quad - \beta \int_0^\infty k(t, s) g(s, A(x)(s) + u_1(s)) ds \\ &= A(x)(t) \\ &\quad - \beta \int_0^\infty k(t, s) \\ &\quad \times g\left(s, -u_2(s) \right. \\ &\quad \left. + \alpha \int_0^s h(s, r) f(r, x(r)) dr + u_1(s)\right) ds \\ &= A(x)(t). \end{aligned} \quad (66)$$

Therefore, from hypothesis (g) and containment of ranges, we deduce that the pair (A, S) satisfies the (CLR_S) property. Now, by using hypotheses (a) and (b), for all $x, y \in X$ and $t \in [0, \infty)$, we have

$$\begin{aligned} & |A(x)(t) - A(y)(t)| \\ &\leq |\alpha| \int_0^t |h(t, s)| |f(t, x(s)) - f(t, y(s))| ds \\ &\leq |\alpha| K_1 \int_0^t |h(t, s)| |x(s) - y(s)| ds \\ &\leq K_1 \rho(x, y). \end{aligned} \quad (67)$$

Similarly, we get

$$\begin{aligned}
 & |S(x)(t) - S(y)(t)| \\
 &= \left| x(t) - y(t) \right. \\
 &\quad \left. - \beta \int_0^\infty k(t, s) [g(s, x(s)) - g(s, y(s))] ds \right| \\
 &\geq |x(t) - y(t)| \\
 &\quad - |\beta| \int_0^\infty |k(t, s)| |g(s, x(s)) - g(s, y(s))| ds \\
 &\geq |x(t) - y(t)| \\
 &\quad - |\beta| K_2 \int_0^\infty |k(t, s)| |x(s) - y(s)| ds \\
 &\geq |x(t) - y(t)| - K_2 \sup_{t \in [0, \infty)} |x(t) - y(t)|.
 \end{aligned} \tag{68}$$

This implies easily

$$\rho(S(x), S(y)) \geq (1 - K_2) \rho(x, y). \tag{69}$$

Therefore, combining opportunely (67) and (69), we obtain

$$\rho(A(x), A(y)) \leq \frac{K_1}{1 - K_2} \rho(S(x), S(y)), \tag{70}$$

for all $x, y \in X$. Moreover, using hypothesis (e) and after routine calculations (omitted) we obtain

$$\rho(A(x), z) \leq \frac{K_1}{1 - K_2} \rho(S(x), z), \tag{71}$$

for all $x, z \in X$. Then, from (70), (71), and hypothesis (d), we conclude that condition (53) is satisfied with $A = B, S = T, a_1 = K_1/(1 - K_2) < 1$ and $a_2 = a_3 = a_4 = a_5 = 0$.

Now, applying Corollary 30, where inequality (59) is replaced by inequality (53), we obtain the existence of a solution to (64).

On the lines of the proof of Theorem 4.1 in [51], one can show that hypothesis (f) also implies that the pair (A, S) is weakly compatible. Thus, from Corollary 30 we obtain the uniqueness of the solution to (64). $\square$

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