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Conference Paper

Existence and Multiplicity of Solutions for a Robin Problem Involving the $p(x)$-Laplace Operator

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We study the following nonlinear Robin boundary-value problem $-\Delta_{p(x)}u = \lambda f(x,u)$ in Ω, $|\nabla u|^{p(x)-2}(\partial u/\partial v) + \beta(x)|u|^{p(x)-2}u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\partial u/\partial v$ is the outer unit normal derivative on $\partial\Omega$, $\lambda > 0$ is a real number, p is a continuous function on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$, $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega}\beta(x) > 0$, and $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a continuous function. Using the variational method, under appropriate assumptions on f, we obtain results on existence and multiplicity of solutions.

1. Introduction

The purpose of this paper is to study the existence and multiplicity of solutions for the following Robin problem involving the $p(x)$-Laplacian:

$$\begin{aligned} -\Delta_{p(x)}u &= \lambda f(x,u) \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} + \beta(x)|u|^{p(x)-2}u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\partial u/\partial v$ is the outer unit normal derivative on $\partial\Omega$, $\lambda > 0$ is a real number, p is a continuous function on $\overline{\Omega}$ with $p^- := \inf_{x\in\overline{\Omega}} p(x) > 1$, and $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega}\beta(x) > 0$. The main interest in studying such problems arises from the presence of the $p(x)$-Laplace operator $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, which is a natural extension of the classical p-Laplace operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ obtained in the case when p is a positive constant. However, such generalizations are not trivial since the $p(x)$-Laplace operator possesses a more complicated structure than p-Laplace operator; for example, it is inhomogeneous.

In the recent years increasing attention has been paid to the study of differential and partial differential equations involving variable exponent conditions. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics, and calculus of variations; for information on modelling physical phenomena by equations involving $p(x)$-growth condition we refer to [1–10]. In the past decades a vast amount of literature that deals with the existence for type problems $-\Delta_{p(x)}u = f(x,u)$ with different boundary conditions (Dirichlet, Neumann, Robin, nonlinear, etc.) has appeared. See, for instance, [11–16] and references therein.

In [14], by applying the subsupersolution method and the variational method, under appropriate assumptions on f, the author proves that there exists $\lambda^* > 0$ such that problem (1) has at least two positive solutions if $\lambda \in (0,\lambda^*)$, has at least one positive solution if $\lambda = \lambda^* < +\infty$, and has no positive solution if $\lambda = \lambda^*$. Recently in [11], the authors obtain the existence of at least two nontrivial solutions for problem (1) using a variational approach based on the nonsmooth critical point theory for locally Lipschitz functions.

We make the following assumptions on the function f:

(H0) $f : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exists a constant $C \geq 0$ such that

$$|f(x,s)| \leq C\left(1 + |s|^{\alpha(x)-1}\right) \quad \forall (x,s) \in \Omega\times\mathbb{R}, \tag{2}$$

where $\alpha(x) \in C(\Omega)$ and $1 < \alpha(x) < p^*(x)$ for all $x \in \Omega$.

(H1) There exist $R > 0$, $\mu > p^+ := \sup_{x\in\overline{\Omega}} p(x)$ such that, for all $|s| \geq R$ and $x \in \Omega$,

$$0 < \mu F(x,s) \leq f(x,s)\, s. \tag{3}$$

(H2) One has $f(x,s) = o(|s|^{p^+-1})$ as $s \to 0$ and uniformly for $x \in \Omega$.

(H3) One has $f(x,-s) = -f(x,s)$, $x \in \Omega$, $s \in \mathbb{R}$.

(H4) One has $\sup_{t\in\mathbb{R}} \int_0^t f(x,s)\, ds > 0$, for all $x \in \Omega$.

(H5) There exists $q \in C(\Omega)$ such that $p^+ < q^- := \inf_{x\in\Omega} q(x) \leq q(x) < q^*(x)$ and

$$\limsup_{t\to 0} \sup_{x\in\Omega} \frac{\int_0^t f(x,s)\, ds}{|t|^{q(x)}} < +\infty. \tag{4}$$

The main results of this paper are as follows.

Theorem 1. *If (H0), (H1), and (H2) hold and $\alpha^- > p^+$, then, for any $\lambda \in (0,+\infty)$, (1) has at least a nontrivial weak solution.*

Theorem 2. *If (H0), (H1), and (H3) hold and $\alpha^- > p^+$, then, for any $\lambda \in (0,+\infty)$, (1) has infinite many pairs of weak solutions.*

Theorem 3. *If (H0), (H4), and (H5) hold and $p^- > \alpha^+$, then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, (1) has at least three solutions whose norms are less than ρ.*

To prove our results, we will use a variational method and the theory of variable exponent Sobolev spaces. For the proof of Theorem 1, we will use the Mountain Pass Theorem (see [17, 18]). For the proof of Theorem 2, we will use the Fountain Theorem (see [18, 19]). For the proof of Theorem 3, we will use Ricceri three critical points Theorem (see [20, 21]).

This paper is organized as follows. First, we will introduce some basic preliminary results and lemmas in Section 2. In Section 3, we will give the proofs of our main results.

2. Preliminaries

For completeness, we first recall some facts on the variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$, where

$$C_+\left(\overline{\Omega}\right) = \left\{p \in C\left(\overline{\Omega}\right), \inf_{x\in\overline{\Omega}} p(x) > 1\right\}. \tag{5}$$

Denote $p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ is a measurable}, \int_\Omega |u|^{p(x)}\, dx < +\infty\right\}, \tag{6}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{\tau > 0; \int_\Omega \left|\frac{u}{\tau}\right|^{p(x)} dx \leq 1\right\}. \tag{7}$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}, \tag{8}$$

with the norm

$$\|u\| = \inf\left\{\tau > 0; \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \left|\frac{u}{\tau}\right|^{p(x)}\right) dx \leq 1\right\}, \qquad \|u\| = |\nabla u|_{p(x)} + |u|_{p(x)}. \tag{9}$$

We refer the reader to [14, 22, 23] for the basic properties of the variable exponent Lebesgue and Sobolev spaces.

Lemma 4 (see [23]). *Both $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are separable, reflexive, and uniformly convex Banach spaces.*

Lemma 5 (see [23]). *Hölder inequality holds, namely,*

$$\int_\Omega |uv|\, dx \leq 2|u|_{p(x)}|v|_{p'(x)} \quad \forall u \in L^{p(x)}(\Omega),\ v \in L^{p'(x)}(\Omega), \tag{10}$$

where $1/p(x) + 1/p'(x) = 1$.

Now, we introduce a norm, which will be used later. For $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_\beta = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} dx + \int_{\partial\Omega} \beta(x) \left|\frac{u(x)}{\lambda}\right|^{p(x)} d\sigma_x \leq 1\right\}. \tag{11}$$

Then, by Theorem 2.1 in [14], $\|u\|_\beta$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|u\|$.

Hereafter, let

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N, \end{cases} \qquad p^\partial(x) = \begin{cases} \dfrac{(N-1)\,p(x)}{N - p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N. \end{cases} \tag{12}$$

Lemma 6 (see [12, 16, 23]). (1) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

(2) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^\partial(x)$ for any $x \in \partial\Omega$, then the trace imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ is compact and continuous.*

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping defined by the following.

Lemma 7 (see [14]). *Denoting*

$$I_\beta(u) = \int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)\,|u|^{p(x)}\,d\sigma_x, \quad \forall u \in W^{1,p(x)}(\Omega), \tag{13}$$

with $\beta^- > 0$, then

(1) $\|u\|_\beta \geq 1 \Rightarrow \|u\|_\beta^{p^-} \leq I_\beta(u) \leq \|u\|_\beta^{p^+}$,

(2) $\|u\|_\beta \leq 1 \Rightarrow \|u\|_\beta^{p^+} \leq I_\beta(u) \leq \|u\|_\beta^{p^-}$,

(3) $\|u\|_\beta \to 0$ *if and only if* $I_\beta(u) \to 0$ *(as* $k \to \infty$*),*

(4) $|u(x)|_{p(x)} \to \infty$ *if and only if* $I_\beta(u) \to \infty$ *(as* $k \to \infty$*).*

Lemma 8 (see [20, 21, 24]). *Let X be a separable and reflexive real Banach space; $\phi : X \to \mathbb{R}$ is a continuous Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\psi : X \to \mathbb{R}$ is a continuous Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(i) $\lim_{\|u\|_X \to \infty}(\phi(u) + \lambda\psi(u)) = \infty$ *for all* $\lambda > 0$,

(ii) *there exist* $r \in \mathbb{R}$ *and* $u_0, u_1 \in X$ *such that* $\phi(u_0) < r < \phi(u_1)$,

(iii)

$$\inf_{u\in\phi^{-1}(-\infty,r]} \psi(u) > \frac{(\phi(u_1) - r)\,\psi(u_0) + (r - \phi(u_0))\,\psi(u_1)}{\phi(u_1) - \phi(u_0)}. \tag{14}$$

Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive constant $\rho > 0$ such that for any $\lambda \in \Lambda$ the equation $\phi'(u) + \lambda\psi'(u) = 0$ has at least three solutions in X whose norms are less than ρ.

Theorem 9. *Let $X = W^{1,p(x)}(\Omega)$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function with primitive $F(x,u) = \int_0^u f(x,t)dt$. If the following condition hold:*

$$|f(x,s)| \leq C\left(1 + |s|^{\alpha(x)-1}\right), \quad \forall (x,s) \in \overline{\Omega} \times \mathbb{R}, \tag{15}$$

where $C \geq 0$ is a constant and $\alpha(x) \in C_+(\overline{\Omega})$ such that, for all $x \in \overline{\Omega}$, $\alpha(x) < p^*(x)$, then $\psi(u) = -\int_\Omega F(x,u(x))dx \in C^1(X,\mathbb{R})$ and $D\psi(u,\varphi) = \langle\psi'(u),\varphi\rangle = -\int_\Omega f(x,u(x))\varphi\,dx$; moreover, the operator $\psi' : X \to X^*$ is compact.

Proof. It is easily adapted from Theorem 2.1 in [13]. □

Let $X = W^{1,p(x)}(\Omega)$ and

$$\begin{aligned} \phi(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x, \\ \psi(u) &= -\int_\Omega F(x,u)\,dx, \\ J(u) &= \phi(u) + \lambda\psi(u), \end{aligned} \tag{16}$$

where $F(x,t) = \int_0^t f(x,s)\,ds$.

Obviously $\phi \in C^1(X,\mathbb{R})$ and

$$\begin{aligned} \left(\phi'(u), v\right) &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx \\ &\quad + \int_{\partial\Omega} \beta(x)\,|u|^{p(x)-2}uv\,d\sigma_x, \\ \left(\psi'(u), v\right) &= -\int_\Omega f(x,u)\,v\,dx. \end{aligned} \tag{17}$$

Moreover, we have the following.

Proposition 10 (see [11]). (1) *$\phi' : X \to X^*$ is a continuous, bounded, and strictly monotone operator.*

(2) *$\phi' : X \to X^*$ is a mapping of type $(S)^+$; that is, if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}(\phi'(u_n) - \phi'(u), u_n - u) \leq 0$, then $u_n \to u$ in X.*

(3) *$\phi' : X \to X^*$ is a homeomorphism.*

Definition 11. One says that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of problem (1) if for all $v \in W^{1,p(x)}(\Omega)$

$$\begin{aligned} &\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx + \int_{\partial\Omega} \beta(x)\,|u|^{p(x)-2}uv\,d\sigma_x \\ &\quad = \lambda\int_\Omega f(x,u)\,v\,dx, \end{aligned} \tag{18}$$

where $d\sigma_x$ is the measure on the boundary $\partial\Omega$.

3. Proof of Main Results

To prove Theorem 1, we have to check that the functional J satisfies the following compactness condition (PS).

Definition 12. One says that the C^1-functional $H : X \to \mathbb{R}$ satisfies the Palais-Smale condition ((PS) condition for short) if any sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ for which $(H(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $H'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Lemma 13. *If* (H0), (H1) *hold, then for any $\lambda \in (0,+\infty)$ the functional J satisfies the Palais-Smale condition (PS).*

Proof. Suppose that $(u_n) \subset X$ is a (PS) sequence; that is,

$$\sup|J(u_n)| \leq M, \qquad J'(u_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{19}$$

We claim that (u_n) is bounded in X. Using hypothesis (H1), since $J(u_n)$ is bounded, we have for n large enough

$$\begin{aligned} M+1 &\geq J(u_n) - \frac{1}{\mu}\left\langle J'(u_n), u_n\right\rangle + \frac{1}{\mu}\left\langle J'(u_n), u_n\right\rangle \\ &= \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x \\ &\quad - \lambda\int_\Omega F(x,u_n)\,dx \\ &\quad - \frac{1}{\mu}\left(I_\beta(u_n) - \lambda\int_\Omega f(x,u_n)\,u_n\,dx\right) \\ &\quad + \frac{1}{\mu}\left\langle J'(u_n), u_n\right\rangle \\ &\geq \frac{1}{p^+}I_\beta(u_n) - \lambda\int_\Omega \frac{u_n}{\mu}f(x,u_n)\,dx \\ &\quad - \frac{1}{\mu}\left(I_\beta(u_n) - \lambda\int_\Omega f(x,u_n)\,u_n\,dx\right) \\ &\quad + \frac{1}{\mu}\left\langle J'(u_n), u_n\right\rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\mu}\right)\|u_n\|_\beta^{p^-} - \frac{1}{\mu}\left\|J'(u_n)\right\|_{X^*}\|u_n\| - C_1 \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\mu}\right)C_3\|u_n\|^{p^-} - \frac{C_2}{\mu}\|u_n\| - C_1, \end{aligned} \tag{20}$$

where C_1, C_2, and C_3 are three positive constants. Hence (u_n) is bounded in X since $\mu > p^+$. Without loss of generality, we assume that $u_n \rightharpoonup u$, then $\psi'(u_n) \to \psi'(u)$ since $\psi' : X \to X^*$ is completely continuous. The hypothesis $J'(u_n) \to 0$, that is, $\phi'(u_n) + \lambda\psi'(u_n) \to 0$ and the fact that $\psi'(u_n) \to \psi'(u)$ imply that $\phi'(u_n) \to -\lambda\psi'(u)$. From Proposition 10, ϕ' is a homeomorphism, then $u_n \to u$, and so J satisfies the (PS) condition. The proof is complete. □

Lemma 14. *There exists $r_1, C' > 0$ such that $J(u) \geq C'$ for all $u \in X$ such that $\|u\|_\beta = r_1$.*

Proof. Conditions (H0) and (H2) assure that

$$|F(x,s)| \leq \varepsilon|s|^{p^+} + C(\varepsilon)|s|^{\alpha(x)} \quad \forall (x,s) \in \Omega\times\mathbb{R}. \tag{21}$$

For $\|u\|_\beta$ small enough, we have

$$\begin{aligned} J(u) &\geq \frac{1}{p^+}\|u\|_\beta^{p^+} - \lambda\int_\Omega F(x,u)\,dx \\ &\geq \frac{1}{p^+}\|u\|_\beta^{p^+} - \lambda\int_\Omega\left(\varepsilon|u|^{p^+} + C(\varepsilon)|u|^{\alpha(x)}\right)dx. \end{aligned} \tag{22}$$

Note that $1 < p^+ < \alpha^- \leq \alpha(x) < p^*(x)$, for all $x \in \Omega$; then, by Lemma 6, we have $X \hookrightarrow L^{p^+}(\Omega)$ and $X \hookrightarrow L^{\alpha(x)}(\Omega)$ with a continuous and compact embedding. Furthermore, there exists $C_4, C_5 > 0$ such that

$$\begin{aligned} |u|_{L^{p^+}(\Omega)} &\leq C_4\|u\|_\beta, \\ |u|_{L^{\alpha(x)}(\Omega)} &\leq C_5\|u\|_\beta, \quad \forall u \in X. \end{aligned} \tag{23}$$

Since $\|u\|_\beta$ is small enough, we deduce

$$\int_\Omega |u|^{\alpha(x)}dx \leq \max\left\{|u|_{\alpha(x)}^{\alpha^-}, |u|_{\alpha(x)}^{\alpha^+}\right\} \leq C_6\|u\|_\beta^{\alpha^-}. \tag{24}$$

Replacing in (22), it results that

$$J(u) \geq \frac{1}{p^+}\|u\|_\beta^{p^+} - \lambda\varepsilon C_4^{p^+}\|u\|_\beta^{p^+} - \lambda C(\varepsilon)C_6\|u\|_\beta^{\alpha^-}. \tag{25}$$

Choosing $\varepsilon > 0$ small enough such that $0 < \lambda\varepsilon C_4^{p^+} < 1/2p^+$, then we obtain

$$\begin{aligned} J(u) &\geq \frac{1}{2p^+}\|u\|_\beta^{p^+} - \lambda C(\varepsilon)C_6\|u\|_\beta^{\alpha^-} \\ &\geq \|u\|_\beta^{p^+}\left(\frac{1}{2p^+} - \lambda C(\varepsilon)C_6\|u\|_\beta^{\alpha^- - p^+}\right). \end{aligned} \tag{26}$$

Since $p^+ < \alpha^-$, the function $t \mapsto (1/2p^+ - \lambda C(\varepsilon)C_6 t^{\alpha^- - p^+})$ is strictly positive in a neighborhood of zero. It follows that there exist $r_1 > 0$ and $C' > 0$ such that

$$J(u) \geq C' \quad \forall u \in X : \|u\|_\beta = r_1. \tag{27}$$

The proof is complete. □

Proof of Theorem 1. To apply the Mountain Pass Theorem [17, 18], we need to prove that $J(tu) \to -\infty$ as $t \to +\infty$, for a certain $u \in X$. From condition (H1), we obtain

$$F(x,s) \geq c|s|^\mu \quad \forall (x,s) \in \Omega\times\mathbb{R}. \tag{28}$$

Letting $u \in X$ and $t > 1$, we have

$$\begin{aligned} J(tu) &= \int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\frac{t^{p(x)}\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x \\ &\quad - \lambda\int_\Omega F(x,tu)\,dx \\ &\leq t^{p^+}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\right) \\ &\quad - t^\mu\lambda c\int_\Omega |u|^\mu\,dx. \end{aligned} \tag{29}$$

The fact $\mu > p^+$ implies for any $\lambda \in (0,+\infty)$, $J(tu) \to -\infty$ as $t \to +\infty$.

It follows that there exists $e \in X$ such that $\|e\|_\beta > r_1$ and $J(e) < 0$. According to the Mountain Pass Theorem, J admits a critical value $\tau \geq C'$ which is characterized by

$$\tau = \inf_{h\in\Gamma}\sup_{t\in[0,1]} J(h(t)), \tag{30}$$

where

$$\Gamma = \{h \in C([0,1], X) : h(0) = 0, h(1) = e\}. \tag{31}$$

This completes the proof. □

Since X is a separable and reflexive Banach space [22, 25], there exist $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that

$$\begin{gathered} f_n(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases} \\ X = \overline{\operatorname{span}}\{e_n : n = 1, 2, \ldots\}, \\ X^* = \overline{\operatorname{span}}^{W^*}\{f_n : n = 1, 2, \ldots\}. \end{gathered} \tag{32}$$

For $k = 1, 2, \ldots$ denote

$$\begin{gathered} X_n = \operatorname{span}\{e_n\}, \qquad Y_n = \bigoplus_{j=1}^{n} X_j, \\ Z_n = \overline{\bigoplus_{j=n}^{\infty} X_j}. \end{gathered} \tag{33}$$

Lemma 15. *For $\alpha(x) \in C_+(\overline{\Omega})$, $\alpha(x) < p^*(x)$, and $x \in \overline{\Omega}$, let*

$$\gamma_k = \sup\left\{|u|_{L^{\alpha(x)}(\Omega)} : \|u\|_\beta = 1, u \in Z_k\right\}. \tag{34}$$

Then $\lim_{k\to\infty}\gamma_k = 0$.

Proof. It is clear that $0 < \gamma_{k+1} < \gamma_k$, so γ_k converges to $\gamma \geq 0$. Let $u_k \in Z_k$ such that

$$\|u_k\|_\beta = 1, \qquad 0 \leq \gamma_k - |u_k|_{L^{\alpha(x)}(\Omega)} < \frac{1}{k}. \tag{35}$$

Then, there exists a subsequence, noted also by (u_k), such that $u_k \rightharpoonup u$ in X and

$$\langle f_i, u\rangle = \lim_{k\to+\infty}\langle f_i, u_k\rangle = 0, \quad \forall i = 1, 2, \ldots. \tag{36}$$

Thus $u = 0$ and so $u_k \rightharpoonup 0$ in X. According to Lemma 6, there is a compact embedding of X into $L^{\alpha(x)}(\Omega)$, which assure that $u_k \to 0$ in $L^{\alpha(x)}(\Omega)$. Hence we get $\gamma_k \to 0$ as $k \to +\infty$. □

Proof of Theorem 2. We will use the Fountain theorem [18, 19]. Obviously, J is an even functional and satisfies the (PS) condition according to (H1) and (H2). We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that

(A1) $b_k := \inf\{J(u) : u \in Z_k, \|u\|_\beta = r_k\} \to +\infty$ as $k \to +\infty$,

(A2) $a_k := \max\{J(u) : u \in Y_k, \|u\|_\beta = \rho_k\} \leq 0$ as $k \to +\infty$.

(A1) For $u \in Z_k$ such that $\|u\|_\beta = r_k > 1$, by condition (H0), we have

$$\begin{aligned} J(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x \\ &\quad - \lambda\int_\Omega F(x,u)\,dx \\ &\geq \frac{1}{p^+}\|u\|_\beta^{p^-} - \lambda\int_\Omega C\left(1 + |u|^{\alpha(x)}\right)dx \\ &\geq \frac{1}{p^+}\|u\|_\beta^{p^-} - \lambda C\int_\Omega |u|^{\alpha(x)}dx - C_7. \end{aligned} \tag{37}$$

If $|u|_{\alpha(x)} \leq 1$ then $\int_\Omega |u|^{\alpha(x)}dx \leq |u|_{\alpha(x)}^{\alpha^-} \leq 1$. And if $|u|_{\alpha(x)} > 1$ then $\int_\Omega |u|^{\alpha(x)}dx \leq |u|_{\alpha(x)}^{\alpha^+} \leq (\gamma_k\|u\|_\beta)^{\alpha^+}$. So we obtain that

$$\begin{aligned} J(u) &\geq \begin{cases} \dfrac{1}{p^+}\|u\|_\beta^{p^-} - C_8(\lambda) - C_7 & \text{if } |u|_{\alpha(x)} \leq 1 \\ \dfrac{1}{p^+}\|u\|_\beta^{p^-} - C_8(\lambda)\left(\gamma_k\|u\|_\beta\right)^{\alpha^+} - C_7 & \text{if } |u|_{\alpha(x)} > 1 \end{cases} \\ &\geq \frac{1}{p^+}\|u\|_\beta^{p^-} - C_8(\lambda)\left(\gamma_k\|u\|_\beta\right)^{\alpha^+} - C_9 \\ &= r_k^{p^-}\left(\frac{1}{p^+} - C_8(\lambda)\gamma_k^{\alpha^+}r_k^{\alpha^+-p^-}\right) - C_9. \end{aligned} \tag{38}$$

If we take $C_8(\lambda)\gamma_k^{\alpha^+}r_k^{\alpha^+-p^-} = 1/\alpha^+$, that is, $r_k = (C_8(\lambda)\alpha^+\gamma_k^{\alpha^+})^{1/(p^--\alpha^+)}$, we obtain

$$J(u) \geq r_k^{p^-}\left(\frac{1}{p^+} - \frac{1}{\alpha^+}\right) - C_9. \tag{39}$$

Since $\gamma_k \to 0$ and $p^- \leq p^+ < \alpha^- \leq \alpha^+$, we have $r_k \to +\infty$ as $k \to +\infty$. Consequently,

$$J(u) \longrightarrow +\infty \quad \text{as } \|u\|_\beta \longrightarrow +\infty,\ u \in Z_k. \tag{40}$$

So (A1) holds.

(A2) Condition (H1) implies that there exist positive constants C_{10}, C_{11} such that

$$F(x,s) \geq C_{10}|s|^\mu - C_{11}, \quad \forall (x,s) \in \Omega\times\mathbb{R}. \tag{41}$$

Let $u \in Y_k$ be such that $\|u\|_\beta = \rho_k > r_k > 1$. Then

$$\begin{aligned} J(u) &\leq \frac{1}{p^-}\|u\|_\beta^{p^+} - \lambda\int_\Omega\left(C_{10}|u|^\mu - C_{11}\right)dx \\ &\leq \frac{1}{p^-}\|u\|_\beta^{p^+} - \lambda C_{10}\int_\Omega |u|^\mu\,dx + C_{12}. \end{aligned} \tag{42}$$

Note that the space Y_k has finite dimension; then all norms are equivalents and we obtain

$$J(u) \leq \frac{1}{p^-}\|u\|_\beta^{p^+} - \lambda C_{13}\|u\|_\beta^\mu + C_{12}. \tag{43}$$

Finally,

$$J(u) \longrightarrow -\infty \quad \text{as } \|u\|_\beta \longrightarrow +\infty,\ u \in Y_k \tag{44}$$

since $\mu > p^+$. The assertion (A2) is then satisfied and the proof of Theorem 2 is complete. □

Proof of Theorem 3. We will use Ricceri three critical points Theorem (see [20, 21]). Notice that ϕ is a continuous convex functional, so it is weakly lower semicontinuous and its inverse derivative is continuous. From Theorem 9 the precondition of Lemma 8 is satisfied. Now we only need to verify that conditions (i), (ii), and (iii) in Lemma 8 are fulfilled.

Firstly for $u \in X$ such that $\|u\|_\beta \geq 1$, we have

$$\begin{aligned}\psi(u) &= -\int_\Omega F(x,u)\,dx = -\int_\Omega \left(\int_0^{u(x)} f(x,t)\,dt\right)dx \\ &\leq C\int_\Omega \left(|u(x)| + \frac{1}{\alpha(x)}|u|^{\alpha(x)}\right)dx \\ &\leq C\int_\Omega |u(x)|\,dx + \frac{C}{\alpha^-}\int_\Omega |u(x)|^{\alpha(x)}\,dx.\end{aligned} \tag{45}$$

Using the Hölder inequality and the Sobolev embedding theorem, we have for some positive constants C_{14} and C_{15}

$$\begin{aligned}C\int_\Omega |u(x)|\,dx &\leq 2C|1|_{\alpha'(x)}|u|_{\alpha(x)} \leq C_{14}\ \|u\|_\beta, \\ \frac{C}{\alpha^-}\int_\Omega |u|^{\alpha(x)}\,dx &\leq \frac{C}{\alpha^-}\max\left\{|u|^{\alpha^+}_{\alpha(x)}, |u|^{\alpha^-}_{\alpha(x)}\right\} \leq C_{15}\|u\|^{\alpha^+}_\beta,\end{aligned} \tag{46}$$

where $1/\alpha(x)+1/\alpha'(x) = 1$. Combining all together we obtain

$$|\psi(u)| \leq C_{14}\|u\|_\beta + C_{15}\|u\|^{\alpha^+}_\beta. \tag{47}$$

On the other hand,

$$\begin{aligned}\phi(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x \\ &\geq \frac{1}{p^+}\|u\|^{p^-}_\beta.\end{aligned} \tag{48}$$

Then for any $\lambda > 0$,

$$\phi(u) + \lambda\psi(u) \geq \frac{1}{p^+}\|u\|^{p^-}_\beta - \lambda C_{14}\|u\|_\beta - \lambda C_{15}\|u\|^{\alpha^+}_\beta. \tag{49}$$

From $p^- > \alpha^+$ we obtain

$$\lim_{\|u\|_\beta\to\infty}(\phi(u) + \lambda\psi(u)) = \infty, \tag{50}$$

then (i) of Lemma 8 is verified.

Secondly, letting $u_0 = 0$, then obviously we have $\phi(u_0) = \psi(u_0) = 0$.

We claim that there exist $r > 0$ and $u_1 \in X$ such that $\phi(u_0) > r$ and

$$\inf_{u\in\phi^{-1}((-\infty,r])}\psi(u) > r\frac{\psi(u_1)}{\phi(u_1)}. \tag{51}$$

From (H5), there exits $\eta \in [0,1]$ and $C_{16} > 0$ such that

$$\begin{aligned}&F(x,t) \leq C_{16}|t|^{q(x)} \leq C_{16}|t|^{q^-}, \\ &\forall t \in [-\eta,\eta] \text{ uniformly for } x \in \Omega.\end{aligned} \tag{52}$$

In view of (H0), if we put

$$C_{17} := \max\left\{C_{16}, \sup_{\eta\leq|t|<1}\frac{A\left(1+|t|^{\alpha^-}\right)}{|t|^{q^-}}, \sup_{|t|\geq 1}\frac{A\left(1+|t|^{\alpha^+}\right)}{|t|^{q^-}}\right\}, \tag{53}$$

where A is a positive constant, then we have

$$F(x,t) \leq C_{17}|t|^{q^-}, \quad \forall t \in \mathbb{R} \text{ uniformly for } x \in \Omega. \tag{54}$$

Fix r such that $0 < r < 1$. If $(1/p^+)\|u\|^{p^+}_\beta \leq r < 1$, then, by the Sobolev embedding theorem ($X \hookrightarrow L^{q^-}(\Omega)$ is continuous), we have

$$\int_\Omega F(x,u)\,dx \leq C_{17}\int_\Omega |u|^{q^-}dx \leq C_{18}\|u\|^{q^-}_\beta \leq C_{19}r^{q^-/p^+}, \tag{55}$$

where C_{18} and C_{19} are two positive constants.

Since $q^- > p^+$, then we have

$$\lim_{r\to 0^+}\frac{1}{r}\sup_{(1/p^+)\|u\|^{p^+}_\beta\leq r}\int_\Omega F(x,u)\,dx = 0. \tag{56}$$

By (H4), we can choose a constant $b \in X \setminus \{0\}$ such that $\int_\Omega F(x,b)dx > 0$.

Fix r_0 such that $r_0 < (1/p^+)\min\{\|b\|^{p^+}_\beta, \|b\|^{p^-}_\beta, 1\}$.

If $\|b\|_\beta > 1$, then we have

$$\frac{1}{p^+}\|b\|^{p^-}_\beta \leq \phi(b) \leq \frac{1}{p^-}\|b\|^{p^+}_\beta. \tag{57}$$

From (56) and (57), we deduce that, when $0 < r < r_0$, then $\phi(b) > r$ and

$$\sup_{(1/p^+)\|u\|^{p^+}_\beta\leq r}\int_\Omega F(x,u)\,dx \leq \frac{r}{2}\frac{\int_\Omega F(x,u)\,dx}{(1/p^-)\|b\|^{p^+}_\beta}. \tag{58}$$

Thus

$$\sup_{(1/p^+)\|u\|^{p^+}_\beta\leq r}\int_\Omega F(x,u)\,dx \leq r\frac{\int_\Omega F(x,u)\,dx}{(1/p^-)\|b\|^{p^+}_\beta}. \tag{59}$$

Since $r < r_0$, we get

$$\phi^{-1}((-\infty,r]) \subseteq \left\{u \in X : \frac{1}{p^+}\|u\|^{p^+}_\beta \leq r\right\}. \tag{60}$$

Then

$$\sup_{\phi(u)\leq r} -\psi(u) < -r\frac{\psi(u_1)}{\phi(u_1)}, \tag{61}$$

with $u_1 = b$ which implies that

$$\inf_{u\in\phi^{-1}((-\infty,r])}\psi(u) > r\frac{\psi(u_1)}{\phi(u_1)}. \tag{62}$$

So we can find $r > 0$, $u_1 = b$, and $\phi(b) > r$ satisfying (ii) and (iii) of Lemma 8.

If $\|b\|_\beta \leq 1$, we have

$$\frac{1}{p^+}\|b\|^{p^+}_\beta \leq \phi(b) \leq \frac{1}{p^-}\|b\|^{p^-}_\beta. \tag{63}$$

From (56) and (63), we deduce that, when $0 < r < r_0$, then $\phi(b) > r$ and

$$\sup_{(1/p^+)\|u\|_\beta^{p^+} \le r} \int_\Omega F(x,u)\,dx \le \frac{r}{2} \frac{\int_\Omega F(x,u)\,dx}{(1/p^-)\,\|b\|_\beta^{p^-}}. \tag{64}$$

Thus

$$\sup_{(1/p^+)\|u\|_\beta^{p^+} \le r} \int_\Omega F(x,u)\,dx \le r \frac{\int_\Omega F(x,u)\,dx}{(1/p^-)\,\|b\|_\beta^{p^-}}. \tag{65}$$

Since $r < r_0$, we get

$$\phi^{-1}((-\infty, r]) \subseteq \left\{ u \in X : \frac{1}{p^+}\|u\|_\beta^{p^+} \le r \right\}. \tag{66}$$

Then

$$\sup_{\phi(u)\le r} -\psi(u) < -r\frac{\psi(u_1)}{\phi(u_1)}, \tag{67}$$

with $u_1 = b$.

Therefore

$$\inf_{u\in\phi^{-1}((-\infty,r])} \psi(u) > r\frac{\psi(u_1)}{\phi(u_1)}. \tag{68}$$

Then we obtain

$$\inf_{u\in\phi^{-1}((-\infty,r])} \psi(u) > \frac{(\phi(u_1) - r)\,\psi(u_0) + (r - \phi(u_0))\,\psi(u_1)}{\phi(u_1) - \phi(u_0)}. \tag{69}$$

This means that condition (iii) in Lemma 8 is verified. Now since all the assumptions of Lemma 8 are verified, there exist an open interval $\Lambda \subset (0,\infty)$ and a positive constant $\rho > 0$ such that for any $\lambda \in \Lambda$ the equation $\phi'(u) + \lambda\psi'(u) = 0$ has at least three solutions in X whose norms are less than ρ. □

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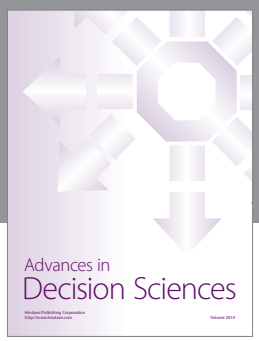

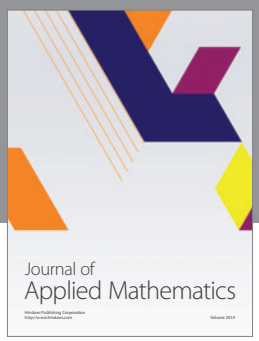

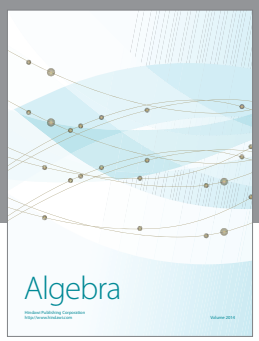

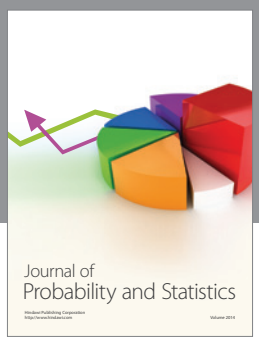

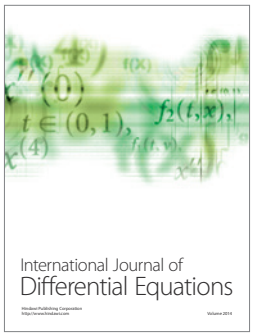

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