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Research Article

A New Identity for Resolvents of Operators

Michael Gil'

Department of Mathematics, Ben-Gurion University of the Negev, P.O. Box 653, 84105 Beer-Sheva, Israel

Correspondence should be addressed to Michael Gil'; gilmi@bezeqint.net

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A new identity for resolvents of operators is suggested. We show that in appropriate situations it is more convenient than the Hilbert identity. In particular, we establish a new invertibility condition for perturbed operators as well as new bounds for the spectrum of perturbed operators. As a particular case we consider perturbations of Hilbert-Schmidt operators.

1. Introduction and the Main Result

Let X be a complex Banach space with a norm $\|\cdot\|$ and the unit operator I. For a linear operator A in X, $\|A\| = \sup_{x\in X}\|Ax\|/\|x\|$, $\sigma(A)$ is the spectrum, A^{-1} is the inverse operator, and $R_\lambda(A) = (A-\lambda I)^{-1}(\lambda \notin \sigma(A))$ is the resolvent.

Everywhere in the following A and $\widetilde{A}$ are bounded operators in X, and $E = \widetilde{A} - A$. Recall the Hilbert identity $R_\lambda(\widetilde{A}) - R_\lambda(A) = -R_\lambda(A)ER_\lambda(\widetilde{A})$ [1]. In particular, it gives the following important result: if a $\lambda \in \mathbb{C}$ is regular for A and

$$\|E\|\,\|R_\lambda(A)\| < 1, \tag{1}$$

then λ is also regular for $\widetilde{A}$. In the present paper we suggest a new identity for resolvents of operators, by which we derive a new invertibility condition for perturbed operators as well as new bounds for the spectrum of perturbed operators. It is shown that in appropriate situations our results improve condition (1). As a particular case we consider perturbations of Hilbert-Schmidt operators.

Put $Z = \widetilde{A}E - EA$. Now we are in a position to formulate and prove our main result.

Theorem 1. *Let a $\lambda \in \mathbb{C}$ be regular for A and $\widetilde{A}$. Then*

$$R_\lambda(\widetilde{A}) - R_\lambda(A) = R_\lambda(\widetilde{A})ZR_\lambda^2(A) - ER_\lambda^2(A). \tag{2}$$

Proof. We have

$$\begin{aligned}
&R_\lambda(\widetilde{A})(\widetilde{A}E - EA)R_\lambda^2(A) - ER_\lambda^2(A)\\
&\quad= \left[R_\lambda(\widetilde{A})(\widetilde{A}E - EA) - E\right]R_\lambda^2(A)\\
&\quad= R_\lambda(\widetilde{A})\left[\widetilde{A}E - EA - (\widetilde{A} - \lambda I)E\right]R_\lambda^2(A)\\
&\quad= R_\lambda(\widetilde{A})(E\lambda - EA)R_\lambda^2(A) = -R_\lambda(\widetilde{A})ER_\lambda(A)\\
&\quad= -R_\lambda(\widetilde{A})(\widetilde{A} - \lambda I - (A - \lambda I))R_\lambda(A)\\
&\quad= -\left(I - R_\lambda(\widetilde{A})(A - \lambda I)\right)R_\lambda(A)\\
&\quad= R_\lambda(\widetilde{A}) - R_\lambda(A),
\end{aligned} \tag{3}$$

as claimed. □

Denote $\eta(A,E,\lambda) = \sup_{0\le t\le 1} t\|(AE - EA + tE^2)R_\lambda^2(A)\|$.

Corollary 2. *Let a $\lambda \in \mathbb{C}$ be regular for A and $\eta(A,E,\lambda) < 1$. Then λ is regular also for $\widetilde{A}$.*

Indeed, put $A_t = A + tE$ $(t \in [0,1])$. Since the regular sets of operators are open, λ is regular for A_t, provided t is small enough. By Theorem 1, we get

$$\begin{aligned}
R_\lambda(A_t) - R_\lambda(A) = {}& R_\lambda(A_t)(t(A+tE)E - tEA)R_\lambda^2(A)\\
&- tER_\lambda^2(A).
\end{aligned} \tag{4}$$

Hence,

$$\begin{aligned}\|R_\lambda(A_t)\| &- \left\|R_\lambda(A) - tER_\lambda^2(A)\right\| \\ &\le \|R_\lambda(A_t)\| \left\|\left[t(EA - AE) + t^2E^2\right] R_\lambda^2(A)\right\| \\ &\le \|R_\lambda(A_t)\| \eta(A, E, \lambda).\end{aligned} \tag{5}$$

Thus,

$$\|R_\lambda(A_t)\| \le \frac{\left\|R_\lambda(A) - tER_\lambda^2(A)\right\|}{1 - \eta(A, E, \lambda)}. \tag{6}$$

Taking $t = 1$, we obtain the required result.

Furthermore, we have $\eta(A, E, \lambda) \le \zeta^2(A, E)\|R_\lambda^2(A)\|$, where

$$\zeta(A, E) := \sqrt{\|AE - EA\| + \left\|E^2\right\|}. \tag{7}$$

Now Corollary 2 implies.

Corollary 3. *If a $\lambda \notin \sigma(A)$ and $\zeta(A, E)\|R_\lambda(A)\| < 1$, then $\lambda \notin \sigma(\widetilde{A})$.*

Example 4. Let $A = cI$ and $\widetilde{A} = cI + V$ with a nonzero number c and a nilpotent operator V in X, such that $V^2 = 0$.

It is clear that $E = V$ and A is invertible. We have $AE - EA = 0$ and $\zeta(A, E) = 0$. Therefore Corollary 3 gives us the sharp result: $\widetilde{A}$ is invertible for all nonzero c.

At the same time (1) gives the invertibility condition $\|V\| < |c|$.

Example 5. Let $X = Y_1 + Y_2$ be a direct sum of two spaces Y_1 and Y_2. Besides $Y_1 = Y_2 = Y$ is a Banach space with a norm $\|\cdot\|_Y$. The norm in X is introduced by $\|x\| = \|y_1\|_Y + \|y_2\|_Y$, with an $x = y_1 + y_2$ ($y_1 \in Y_1, y_2 \in Y_2$). Let us consider the operator matrices

$$A = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} B & C \\ 0 & B \end{pmatrix}, \tag{8}$$

where C and B are commuting operators in Y. Let $\lambda \notin \sigma(A)$. Again $E = \widetilde{A} - A$. It is simple to check that $E^2 = 0$, $\|E\| = \|C\|_Y$, and $AE - EA = 0$. Corollary 3 implies $\lambda \notin \sigma(\widetilde{A})$. At the same time, due to (1), we can assert that $\lambda \notin \sigma(\widetilde{A})$ only if $\|C\|_Y\|R_\lambda(A)\| < 1$.

Furthermore, following the notions of the matrix perturbation theory, cf. [2], we will say that *the spectral variation of $\widetilde{A}$ with respect to A is*

$$sv_A\left(\widetilde{A}\right) := \sup_{\mu\in\sigma\left(\widetilde{A}\right)} \inf_{s\in\sigma(A)} |\mu - s|, \tag{9}$$

and the Hausdorff distance between the spectra of A and $\widetilde{A}$ is

$$\mathrm{hd}\left(A, \widetilde{A}\right) := \max\left\{sv_A\left(\widetilde{A}\right), sv_{\widetilde{A}}(A)\right\}. \tag{10}$$

2. Perturbations of Hilbert-Schmidt Operators

In this section $X = H$ is a separable Hilbert space. Let

$$N_2(A) := \left[\mathrm{Trace}\left(AA^*\right)\right]^{1/2} < \infty. \tag{11}$$

That is, A is a Hilbert-Schmidt operator. Introduce the quantity

$$g(A) := \left[N_2^2(A) - \sum_{k=1}^{\infty} |\lambda_k(A)|^2\right]^{1/2}. \tag{12}$$

The following relations are checked in [3, Section 6.4]:

$$\begin{aligned} g^2(A) &\le N_2^2(A) - \left|\mathrm{Trace}\, A^2\right|, \\ g^2(A) &\le \frac{N_2^2(A - A^*)}{2} = 2N_2^2(A_I), \end{aligned} \tag{13}$$

where $A_I = (A - A^*)/2i$. In our reasonings in the following one can replace $g(A)$ by any of its upper bounds. In particular, one can replace $g(A)$ by $\sqrt{2}N_2(A_I)$.

We need the following result.

Theorem 6. *Let A be a Hilbert-Schmidt operator. Then*

$$\|R_\lambda(A)\| \le \sum_{k=0}^{\infty} \frac{g^k(A)}{\rho^{k+1}(A, \lambda)\sqrt{k!}} \quad (\lambda \notin \sigma(A)), \tag{14}$$

where $\rho(A, \lambda) = \inf_{s\in\sigma(A)}|s - \lambda|$, the distance between λ and the spectrum of A.

For the proof see [3, Theorem 6.4.1]. Now Corollary 3 implies the following.

Corollary 7. *If λ is regular for A, condition* (11) *holds and*

$$\zeta(A, E) \sum_{k=0}^{\infty} \frac{g^k(A)}{\rho^{k+1}(A, \lambda)\sqrt{k!}} < 1, \tag{15}$$

then λ is regular for $\widetilde{A}$.

For any $\mu \in \sigma(A)$, due to Corollary 7, we have

$$\zeta(A, E) \sum_{k=0}^{\infty} \frac{g^k(A)}{\rho^{k+1}(A, \mu)\sqrt{k!}} \ge 1. \tag{16}$$

Hence it follows that $\rho(A, \mu) \le x_0$, where x_0 is the unique positive root of

$$\zeta(A, E) \sum_{k=0}^{\infty} \frac{g^k(A)}{x^{k+1}\sqrt{k!}} = 1. \tag{17}$$

But $sv_A(\widetilde{A}) = \sup_{\mu\in\sigma(\widetilde{A})}\rho(A, \mu)$. We thus arrive at our next result.

Theorem 8. *Let A be a Hilbert-Schmidt operator and $\widetilde{A}$ be an arbitrary bounded operator in H. Then $sv_A(\widetilde{A}) \le x_0$, where x_0 is the unique positive root of* (17).

In Section 3 we obtain an estimate for x_0.

If A is normal, then $g(A) = 0$, and consequently $x_0 = \zeta(A, E)$.

Assume that both A and $\widetilde{A}$ are Hilbert-Schmidt operators. Set

$$\widehat{g} = \max\left\{g(A), g\left(\widetilde{A}\right)\right\}. \tag{18}$$

Now Theorem 8 implies the following.

Corollary 9. *Let both A and $\widetilde{A}$ be Hilbert-Schmidt operators. Then $hd(A, \widetilde{A}) \le z(\widehat{g})$, where $z(\widehat{g})$ is the unique positive root of the equation*

$$\zeta(A, E) \sum_{k=0}^{\infty} \frac{\widehat{g}^k}{x^{k+1}\sqrt{k!}} = 1. \tag{19}$$

In the following, we suggest an estimate also for $z(\widehat{g})$.

Note that in [3, Theorem 8.5.1], the inequality $sv_A(\widetilde{A}) \le \widehat{x}(\|E\|, A)$ is proved, where $\widehat{x}(\|E\|, A)$ is the unique positive root

$$\frac{\|E\|}{x} \exp\left[\frac{1}{2} + \frac{g^2(A)}{x^2}\right] = 1. \tag{20}$$

If $E \ne 0$ this inequality gives us a nonzero result. At the same time, if $Z = 0$ (as in the above given examples), then Theorem 8 and Corollary 9 give us the sharp result $sv_A(\widetilde{A}) = \mathrm{hd}(A, \widetilde{A}) = 0$.

Theorem 6 supplements the recent perturbation results for operators see the interesting papers [4–9] and references given therein.

3. Estimates for x_0 and $z(\widehat{g})$

Denote

$$\gamma(b, c) := \frac{b\sqrt{2}}{\ln^{1/2}\left[1/2 + \sqrt{1/4 + b^2/c^2}\right]}. \tag{21}$$

Note that $\gamma(b, c) \to 0$ as $b \to 0$ and $c > 0$. Similarly, $\gamma(b, c) \to 0$ as $c \to 0$ and $b > 0$.

Lemma 10. *The following inequalities are true:*

$$x_0 \le \gamma\left(g(A), \zeta(A, E)\right), \tag{22}$$

$$z\left(\widehat{g}\right) \le \gamma\left(\widehat{g}, \zeta(A, E)\right). \tag{23}$$

Proof. Substituting $x = g(A)y$ into (17), with the notation $q = \zeta(A, E)/g(A)$, we get

$$1 = q \sum_{k=0}^{\infty} \frac{1}{y^{k+1}\sqrt{k!}}. \tag{24}$$

By the Schwarz inequality

$$\begin{aligned}\left(\sum_{k=0}^{\infty} \frac{1}{y^k\sqrt{k!}}\right)^2 &= \left(\sum_{k=0}^{\infty} \frac{(\sqrt{2})^k}{(\sqrt{2})^k y^k \sqrt{k!}}\right)^2 \\ &\le \sum_{j=0}^{\infty} 2^{-j} \sum_{k=0}^{\infty} \frac{2^k}{y^{2k} k!} \\ &= 2e^{2/y^2}.\end{aligned} \tag{25}$$

Let $y_0 = x_0/g(A)$ be the unique positive root of (24). Then

$$1 \le \frac{q\sqrt{2}}{y_0} e^{1/y_0^2} \quad \text{or} \quad 1 \le \frac{2q^2}{y_0^2} e^{2/y_0^2} \tag{26}$$

and therefore, $y_0 \le \widehat{y}$, where $\widehat{y}$ is the unique positive root of

$$1 = \frac{2q^2}{y^2} e^{2/y^2}. \tag{27}$$

We need the following simple result proved in [10, Lemma 1.6.5]. □

Lemma 11. *The unique positive root z_0 of the equation*

$$ze^z = a \quad (a = const > 0) \tag{28}$$

satisfies the estimate

$$z_0 \ge \ln\left[\frac{1}{2} + \sqrt{\frac{1}{4} + a}\right]. \tag{29}$$

If, in addition, the condition $a \ge e$ holds, then $z_0 \ge \ln a - \ln\ln a$.

Put in (27) $z = 2/y^2$. *Then we obtain* (28) *with* $a = 1/q^2$. *Now* (29) *implies*

$$\widehat{y} \le \frac{\sqrt{2}}{\ln^{1/2}\left[1/2 + \sqrt{1/4 + (1/q^2)}\right]}. \tag{30}$$

Since $\widehat{y} \ge y_0 = x_0/g(A)$ we get inequality (22). *Similarly, inequality* (23) *can be proved.*

Now Theorem 8 and Corollary 9 imply the following.

Corollary 12. *Let A be a Hilbert-Schmidt operator and $\widetilde{A}$ an arbitrary bounded operator in H. Then $sv_A(\widetilde{A}) \le \gamma(g(A), \zeta(A, E))$. If both A and $\widetilde{A}$ are Hilbert-Schmidt operators, then $hd(A, \widetilde{A}) \le \gamma(\widehat{g}, \zeta(A, E))$.*

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