A New Identity for Resolvents of Operators

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A new identity for resolvents of operators is suggested. We show that in appropriate situations it is more convenient than the Hilbert identity. In particular, we establish a new invertibility condition for perturbed operators as well as new bounds for the spectrum of perturbed operators. As a particular case we consider perturbations of Hilbert-Schmidt operators.

1. Introduction and the Main Result

Let $X$ be a complex Banach space with a norm $\| \cdot \|$ and the unit operator $I$. For a linear operator $A$ in $X$, $\|A\| = \sup_{x \in X} \|Ax\|/\|x\|$, $\sigma(A)$ is the spectrum, $A^{-1}$ is the inverse operator, and $R_\lambda(A) = (A - \lambda I)^{-1} (\lambda \notin \sigma(A))$ is the resolvent.

Everywhere in the following $A$ and $\overline{A}$ are bounded operators in $X$, and $E = \overline{A} - A$. Recall the Hilbert identity $R_\lambda(\overline{A}) - R_\lambda(A) = -R_\lambda(A)ER_\lambda(\overline{A})$ [1]. In particular, it gives the following important result: if a $\lambda \in \mathbb{C}$ is regular for $A$ and

$$\|E\| \|R_\lambda(A)\| < 1,$$  \hspace{1cm} (1)

then $\lambda$ is also regular for $\overline{A}$. In the present paper we suggest a new identity for resolvents of operators, by which we derive a new invertibility condition for perturbed operators as well as new bounds for the spectrum of perturbed operators. It is shown that in appropriate situations our results improve condition (1). As a particular case we consider perturbations of Hilbert-Schmidt operators.

Put $Z = \overline{A}E - EA$. Now we are in a position to formulate and prove our main result.

**Theorem 1.** Let a $\lambda \in \mathbb{C}$ be regular for $A$ and $\overline{A}$. Then

$$R_\lambda(\overline{A}) - R_\lambda(A) = R_\lambda(\overline{A})ZR_\lambda^2(A) - ER_\lambda^2(A).$$  \hspace{1cm} (2)

**Proof.** We have

$$R_\lambda(\overline{A})(\overline{A}E - EA)R_\lambda^2(A) - ER_\lambda^2(A)$$

$$= R_\lambda(\overline{A})(\overline{A}E - EA - E)R_\lambda^2(A)$$

$$= R_\lambda(\overline{A})(\overline{A}E - EA - (\overline{A} - \lambda I)E)R_\lambda^2(A)$$

$$= R_\lambda(\overline{A})(E\lambda - EA)R_\lambda^2(A) = -R_\lambda(\overline{A})ER_\lambda(A)$$

$$= -R_\lambda(\overline{A})(\overline{A} - \lambda I - (A - \lambda I))R_\lambda(A)$$

$$= - (I - R_\lambda(\overline{A})(A - \lambda I))R_\lambda(A)$$

$$= R_\lambda(\overline{A}) - R_\lambda(A),$$

as claimed. \hfill \square

Denote $\eta(A, E, \lambda) = \sup_{0 \leq t \leq 1}[\|AE - EA + tE^2\| R_\lambda^2(A)]$.

**Corollary 2.** Let a $\lambda \in \mathbb{C}$ be regular for $A$ and $\eta(A, E, \lambda) < 1$. Then $\lambda$ is regular also for $\overline{A}$.

Indeed, put $A_t = A + tE$ ($t \in [0, 1]$). Since the regular sets of operators are open, $\lambda$ is regular for $A_t$, provided $t$ is small enough. By Theorem 1, we get

$$R_\lambda(A_t) - R_\lambda(A) = R_\lambda(A_t)(t(A + tE)E - tEA)R_\lambda^2(A)$$

$$- tER_\lambda^2(A).$$  \hspace{1cm} (4)
2. Perturbations of Hilbert-Schmidt Operators

In this section $X = H$ is a separable Hilbert space. Let
\[
N_2(A) := \left[ \text{Trace} \left( AA^* \right) \right]^{1/2} < \infty. \tag{11}
\]
That is, $A$ is a Hilbert-Schmidt operator. Introduce the quantity
\[
g(A) := \left[ N_2^2(A) - \sum_{k=1}^{\infty} |\lambda_k(A)|^2 \right]^{1/2}. \tag{12}
\]
The following relations are checked in [3, Section 6.4]:
\[
g^2(A) \leq N_2^2(A) - \text{Trace} A^2, \quad \frac{g^2(A)}{2} \leq \frac{N_2^2(A - A^*)}{2} = 2N_2^2(A_1). \tag{13}
\]
where $A_1 = (A - A^*)/2i$. In our reasonings in the following one can replace $g(A)$ by any of its upper bounds. In particular, one can replace $g(A)$ by $\sqrt{2}N_2(A_1)$.

We need the following result.

**Theorem 6.** Let $A$ be a Hilbert-Schmidt operator. Then
\[
\| R_A(A) \| \leq \lim_{k \to \infty} \frac{g^k(A)}{\rho^{k+1}(A, \lambda)} \sqrt{k!} \left( \lambda \notin \sigma(A) \right), \tag{14}
\]
where $\rho(A, \lambda) = \inf_{s \in \sigma(A)} |s - \lambda|$, the distance between $\lambda$ and the spectrum of $A$.

For the proof see [3, Theorem 6.4.1]. Now Corollary 3 implies the following.

**Corollary 7.** If $\lambda$ is regular for $A$, condition (11) holds and
\[
\zeta(A, E) \sum_{k=0}^{\infty} \frac{g^k(A)}{\rho^{k+1}(A, \lambda) \sqrt{k!}} < 1, \tag{15}
\]
then $\lambda$ is regular for $A$.

For any $\mu \in \sigma(A)$, due to Corollary 7, we have
\[
\zeta(A, E) \sum_{k=0}^{\infty} \frac{g^k(A)}{\rho^{k+1}(A, \mu) \sqrt{k!}} \geq 1. \tag{16}
\]
Hence it follows that $\rho(A, \mu) \leq x_0$, where $x_0$ is the unique positive root of
\[
\zeta(A, E) \sum_{k=0}^{\infty} \frac{g^k(A)}{A^{k+1} \sqrt{k!}} = 1. \tag{17}
\]
But $sv_A(\overline{A}) = \sup_{\mu \in \sigma(A)} \rho(A, \mu)$. We thus arrive at our next result.

**Theorem 8.** Let $A$ be a Hilbert-Schmidt operator and $\overline{A}$ be an arbitrary bounded operator in $H$. Then $sv_A(\overline{A}) \leq x_0$, where $x_0$ is the unique positive root of (17).
3. Estimates for $x_0$ and $z(\bar{g})$

Denote

$$\gamma(b,c) := \frac{b\sqrt{2}}{\ln^{1/2} \left[ 1/2 + \sqrt{1/4 + b^2/c^2} \right]}.$$  \hspace{1cm} (21)

Note that $\gamma(b,c) \rightarrow 0$ as $b \rightarrow 0$ and $c > 0$. Similarly, $\gamma(b,c) \rightarrow 0$ as $c \rightarrow 0$ and $b > 0$.

**Lemma 10.** The following inequalities are true:

$$x_0 \leq \gamma(g(A), \zeta(A, E)), \quad (22)$$

$$z(\bar{g}) \leq \gamma(\bar{g}, \zeta(A, E)). \quad (23)$$

**Proof.** Substituting $x = g(A)y$ into (17), with the notation $q = \zeta(A, E)/g(A)$, we get

$$1 = \sum_{k=0}^{\infty} \frac{1}{y^{k+1} \sqrt{k!}}.$$  \hspace{1cm} (24)

By the Schwarz inequality

$$\left( \sum_{k=0}^{\infty} \frac{1}{y^{k+1} \sqrt{k!}} \right)^2 \leq \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{2^k}{y^{2k}k!} = 2e^{2y^2}.$$  \hspace{1cm} (25)

Let $y_0 = x_0/g(A)$ be the unique positive root of (24). Then

$$1 \leq \frac{q\sqrt{2}}{y_0} e^{1/y_0^2} \quad \text{or} \quad 1 \leq \frac{2q^2}{y_0^2} e^{2/y_0^2},$$  \hspace{1cm} (26)

and therefore, $y_0 \leq \bar{y}$, where $\bar{y}$ is the unique positive root of

$$1 = \frac{2q^2}{y^2} e^{2/y^2}.$$  \hspace{1cm} (27)

We need the following simple result proved in [10, Lemma 1.6.5].

**Lemma 11.** The unique positive root $z_0$ of the equation

$$ze^z = a \quad (a = \text{const} > 0) \quad (28)$$

satisfies the estimate

$$z_0 \geq \ln \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + a} \right]. \quad (29)$$

If, in addition, the condition $a \geq e$ holds, then $z_0 \geq \ln a - \ln \ln a$.

Put in (27) $z = 2/y^2$. Then we obtain (28) with $a = 1/q^2$. Now (29) implies

$$\bar{y} \leq \frac{\sqrt{2}}{\ln^{1/2} \left[ 1/2 + \sqrt{1/4 + (1/q^2)^2} \right]}.$$

Since $\bar{y} \geq y_0 = x_0/g(A)$ we get inequality (22). Similarly, inequality (23) can be proved.

Now Theorem 8 and Corollary 9 imply the following.

**Corollary 12.** Let $A$ be a Hilbert-Schmidt operator and $\overline{A}$ an arbitrary bounded operator in $H$. Then $sv_A(\overline{A}) \leq \gamma(g(A), \zeta(A, E))$. If both $A$ and $\overline{A}$ are Hilbert-Schmidt operators, then $hd(A, \overline{A}) \leq \zeta(\bar{g}, \zeta(A, E))$.

**References**


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