Research Article

Some Fixed Point Results for Generalized Weak Contraction Mappings in Modular Spaces

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We prove the existence theorem of fixed points for a generalized weak contractive mapping in modular spaces.

1. Introduction

In 1997, Alber and Guerre-Delabriere [1] introduced the concept of weak contraction in Hilbert spaces. Later, Rhoades [2] proved that the result which Alber et al. is also valid in complete metric spaces, the result of Rhoades in the following: A mapping $T : X \rightarrow X$ where (X, d) is a metric space, is said to be *weakly contractive* if

$$d(T(x), T(y)) \leq d(x, y) - \phi(d(x, y)), \quad (1)$$

where $\phi : [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$. In 2008, Dutta and Choudhury [3] introduced a new generalization of contraction in metric spaces and proved the following theorem.

Theorem 1. *Let (X, d) be a complete metric space, and let $T : X \rightarrow X$ be a self-mapping satisfying the following inequality:*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)), \quad (2)$$

where $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ are both continuous and monotone nondecreasing function with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

We note that, if one takes $\psi(t) = t$, then (2) reduces to (1).

Recall that the theory of modular on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [4, 5] and redefined by Musielak and Orlicz [6]. Furthermore, the most complete development of these theories is due to Mazur, Luxemburg, and Turpin [7–9]. In the present time, the theory of modular and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces which in their turn have broad applications [10–14]. In many cases, particularly in applications to integral operators, approximation and fixed point theory, modular-type conditions are much more natural as modular-type assumptions can be more easily verified than their metric or norm counterparts. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated in modular spaces. For instance, fixed point theorems are proved in [15, 16] for nonexpansive mappings. The existences for contraction mapping in modular spaces has been studied in [3, 17–24] and the references therein.

From the above mentioned, we will study the existence of fixed point theorems for mappings satisfying generalized weak contraction mappings in modular spaces.

2. Preliminaries

First, we start with a brief recollection of basic concepts and facts in modular spaces.

Definition 2. Let X be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A functional $\rho : X \rightarrow [0, \infty]$ is called a modular if for arbitrary x and y , elements of X , it satisfies the following conditions:

- (1) $\rho(x) = 0$ if and only if $x = 0$;
- (2) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;
- (3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.
If one replaces (3) by
- (4) $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$, for $\alpha, \beta \geq 0$, $\alpha^s + \beta^s = 1$ with an $s \in (0, 1]$, then the modular ρ is called s -convex modular, and if $s = 1$, ρ is called convex modular.

If ρ is modular in X , then the set

$$X_\rho = \{x \in X : \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0\}, \quad (3)$$

is called a *modular space*. X_ρ is a vector subspace of X .

Definition 3. A modular ρ is said to satisfy the Δ_2 -condition if $\rho(2x_n) \rightarrow 0$, whenever $\rho(x_n) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 4. Let X_ρ be a modular space.

- (1) The sequence $\{x_n\}_{n \in \mathbb{N}}$ in X_ρ is said to be ρ -convergent to $x \in X_\rho$ if $\rho(x_n - x) \rightarrow 0$, as $n \rightarrow \infty$.
- (2) The sequence $\{x_n\}_{n \in \mathbb{N}}$ in X_ρ is said to be ρ -Cauchy if $\rho(x_n - x_m) \rightarrow 0$, as $n, m \rightarrow \infty$.
- (3) A subset C of X_ρ is said to be ρ -closed if the ρ -limit of a ρ -convergent sequence of C always belong to C .
- (4) A subset C of X_ρ is said to be ρ -complete if any ρ -Cauchy sequence in C is ρ -convergent sequence and its ρ -limit is in C .
- (5) A subset C of X_ρ is said to be ρ -bounded if any $\delta_\rho(C) = \sup\{\rho(x - y) : x, y \in C\} < \infty$.

Definition 5. Let X be a nonempty set and $T : X \rightarrow X$. A point $x \in X$ is a fixed point of T if and only if $Tx = x$.

Definition 6. Let C be a subset of a real numbers $\mathbb{R}$. A mapping $T : C \rightarrow \mathbb{R}$ is called monotone increasing (or monotone nondecreasing) $x \leq y$ if and only if $T(x) \leq T(y)$, for all x and y are elements in C . A mapping $T : C \rightarrow \mathbb{R}$ is called monotone decreasing (or monotone nonincreasing), $x \geq y$ if and only if $T(x) \geq T(y)$ for all x and y are elements in C .

Definition 7. A sequence $\{a_n\}$ of a real number is said to be monotone increasing (or monotone nondecreasing), if it satisfies $a_1 \leq a_2 \leq a_3 \leq \dots$. It is also said to be monotone decreasing (or monotone nonincreasing) if it satisfies $a_1 \geq a_2 \geq a_3 \geq \dots$.

3. A Generalized Weak Contraction in Modular Spaces

In this section, we prove fixed point theorems for mappings satisfying generalized weak contractions in modular spaces.

Proposition 8. Let ρ be a modular space on X . If $a, b \in \mathbb{R}^+$ with $b \geq a$, then $\rho(ax) \leq \rho(bx)$.

Proof. In case $a = b$, we are done. Suppose $b > a$, and then one has $a/b < 1$ and

$$\begin{aligned} \rho(ax) &= \rho\left(\frac{a}{b}bx\right) \\ &= \rho\left(\frac{a}{b}bx + \left(1 - \frac{a}{b}\right)(0)\right) \\ &\leq \rho(bx) + \rho(0) \\ &= \rho(bx). \end{aligned} \quad (4)$$

□

Proposition 9. Let X_ρ be a modular space in which ρ satisfies the Δ_2 -condition and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X_ρ . If $\rho(c(x_n - x_{n-1})) \rightarrow 0$ as $n \rightarrow \infty$, then $\rho(\alpha l(x_n - x_{n-1})) \rightarrow 0$ as $n \rightarrow \infty$, where $c, l, \alpha \in \mathbb{R}^+$ with $c > l$ and $l/c + 1/\alpha = 1$.

Proof. Since $\rho(c(x_n - x_{n-1})) \rightarrow 0$ as $n \rightarrow \infty$, by the Δ_2 -condition, we get

$$\rho(2^m c(x_n - x_{n-1})) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (5)$$

for $m \in \mathbb{N}$. Using (5), Proposition 8, and the sandwich theorem, we conclude that

$$\rho(2Nc(x_n - x_{n-1})) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (6)$$

for $N \in \mathbb{N}$. From the fact that $l/c + 1/\alpha = 1$, we get $\alpha l = (\alpha - 1)c \geq c$, then there exist $N_\alpha \in \mathbb{N}$ such that

$$2(N_\alpha - 1)c \leq (\alpha - 1)c \leq 2(N_\alpha)c. \quad (7)$$

By Proposition 8, we get

$$\begin{aligned} &\rho(2(N_\alpha - 1)c(x_n - x_{n-1})) \\ &\leq \rho((\alpha - 1)c(x_n - x_{n-1})) \\ &\leq \rho(2(N_\alpha)c(x_n - x_{n-1})). \end{aligned} \quad (8)$$

From (6) and (8), we obtain

$$\begin{aligned} &\lim_{n \rightarrow \infty} \rho(\alpha l(x_n - x_{n-1})) \\ &= \lim_{n \rightarrow \infty} \rho((\alpha - 1)c(x_n - x_{n-1})) = 0. \end{aligned} \quad (9)$$

□

Theorem 10. Let X_ρ be a ρ -complete modular space, where ρ satisfies the Δ_2 -condition. Let $c, l \in \mathbb{R}^+$, $c > l$, and $T : X_\rho \rightarrow X_\rho$ be a mapping satisfying the inequality

$$\begin{aligned} &\psi(\rho(c(Tx - Ty))) \\ &\leq \psi(\rho(l(x - y))) - \phi(\rho(l(x - y))), \end{aligned} \quad (10)$$

for all $x, y \in X_\rho$, where $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then, T has a unique fixed point.

Proof. Let $x_0 \in X_\rho$, and we construct the sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_n = Tx_{n-1}$, $n = 1, 2, 3, \dots$. First, we prove that the sequence $\{\rho(c(Tx_n - Tx_{n-1}))\}$ converges to 0. Indeed

$$\begin{aligned} \psi(\rho(c(x_n - x_{n+1}))) &\leq \psi(\rho(l(x_{n-1} - x_n))) \\ &\quad - \phi(\rho(l(x_{n-1} - x_n))) \quad (11) \\ &\leq \psi(\rho(l(x_{n-1} - x_n))). \end{aligned}$$

By monotone nondecreasing of ψ and Proposition 8, we have

$$\begin{aligned} \rho(c(x_n - x_{n+1})) &\leq \rho(l(x_{n-1} - x_n)) \\ &\leq \rho(c(x_{n-1} - x_n)). \quad (12) \end{aligned}$$

This means that the sequence $\{\rho(c(x_n - x_{n-1}))\}$ is monotone decreasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} \rho(c(x_n - x_{n-1})) = r. \quad (13)$$

If $r > 0$, taking $n \rightarrow \infty$ in the inequality (11), we get

$$\begin{aligned} \psi(r) &\leq \psi(r) - \phi(r) \\ &< \psi(r), \quad (14) \end{aligned}$$

which is a contradiction, thus $r = 0$. So we have

$$\rho(c(x_n - x_{n-1})) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (15)$$

Next, we prove that the sequence $\{cx_n\}_{n \in \mathbb{N}}$ is a ρ -Cauchy. Suppose that $\{cx_n\}_{n \in \mathbb{N}}$ is not ρ -Cauchy, then there exist $\varepsilon > 0$ and subsequence $\{x_{m_k}\}, \{x_{n_k}\}$ with $m_k > n_k \geq k$ such that

$$\rho(c(x_{m_k} - x_{n_k})) \geq \varepsilon, \quad \rho(c(x_{m_{k-1}} - x_{n_k})) < \varepsilon. \quad (16)$$

Now, let $\alpha \in \mathbb{R}^+$ such that $l/c + 1/\alpha = 1$, then we get

$$\begin{aligned} \psi(\rho(c(x_{m_k} - x_{n_k}))) &\leq \psi(\rho(l(x_{m_{k-1}} - x_{n_{k-1}}))) \\ &\quad - \phi(\rho(l(x_{m_{k-1}} - x_{n_{k-1}}))) \quad (17) \\ &\leq \psi(\rho(l(x_{m_{k-1}} - x_{n_{k-1}}))) \end{aligned}$$

which implies that

$$\rho(c(x_{m_k} - x_{n_k})) \leq \rho(l(x_{m_{k-1}} - x_{n_{k-1}})). \quad (18)$$

We have

$$\begin{aligned} \rho(l(x_{m_{k-1}} - x_{n_{k-1}})) &= \rho(l(x_{m_{k-1}} - x_{n_k} + x_{n_k} - x_{n_{k-1}})) \\ &= \rho\left(\frac{l}{c}(x_{m_{k-1}} - x_{n_k})\right. \\ &\quad \left. + \frac{1}{\alpha}l(x_{n_k} - x_{n_{k-1}})\right) \\ &\leq \rho(c(x_{m_{k-1}} - x_{n_k})) \\ &\quad + \rho(\alpha l(x_{n_k} - x_{n_{k-1}})) \\ &< \varepsilon + \rho(\alpha l(x_{n_k} - x_{n_{k-1}})). \quad (19) \end{aligned}$$

By (16), (18), and (19), we get

$$\begin{aligned} \varepsilon &\leq \rho(c(x_{m_k} - x_{n_k})) \\ &\leq \rho(l(x_{m_{k-1}} - x_{n_{k-1}})) \\ &< \varepsilon + \rho(\alpha l(x_{n_k} - x_{n_{k-1}})). \end{aligned} \quad (20)$$

Using (15) and Proposition 9, we have

$$\lim_{k \rightarrow \infty} \rho(\alpha l(x_{n_k} - x_{n_{k-1}})) = 0. \quad (21)$$

From (20) and (21), we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \rho(c(x_{m_k} - x_{n_k})) \\ = \lim_{k \rightarrow \infty} \rho(l(x_{m_{k-1}} - x_{n_{k-1}})) = \varepsilon. \quad (22) \end{aligned}$$

Letting $k \rightarrow \infty$ in (17), by property of ψ and (22), we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon) \quad (23)$$

which is a contradiction. Therefore, $\{cx_n\}_{n \in \mathbb{N}}$ is ρ -Cauchy. Since X_ρ is ρ -complete there exists a point $u \in X_\rho$ such that $\rho(c(x_n - u)) \rightarrow 0$ as $n \rightarrow \infty$. Consequently, $\rho(l(x_n - u)) \rightarrow 0$ as $n \rightarrow \infty$. Next, we prove that u is a unique fixed point of T . Putting $x = x_{n-1}$ and $y = u$ in (10), we obtain

$$\begin{aligned} \psi(\rho(c(x_n - Tu))) &\leq \psi(\rho(l(x_{n-1} - u))) \\ &\quad - \phi(\rho(l(x_{n-1} - u))). \quad (24) \end{aligned}$$

Taking $n \rightarrow \infty$ in the inequality (24), we have

$$\psi(\rho(c(u - Tu))) \leq \psi(0) - \phi(0) = 0, \quad (25)$$

which implies that $\rho(c(Tu - u)) = 0$ and $Tu = u$. Suppose that there exists $v \in X_\rho$ such that $Tv = v$ and $v \neq u$, and then we have

$$\begin{aligned} \psi(\rho(c(u - v))) &= \psi(\rho(c(Tu - Tv))) \\ &\leq \psi(\rho(l(u - v))) - \phi(\rho(l(u - v))) \\ &< \psi(\rho(l(u - v))) \\ &\leq \psi(\rho(c(u - v))), \quad (26) \end{aligned}$$

which is a contradiction. Hence $u = v$ and the proof is complete. $\square$

Corollary 11. Let X_ρ be a ρ -complete modular space, where ρ satisfies the Δ_2 -condition. Let $c, l \in \mathbb{R}^+$, $c > l$, and $T : X_\rho \rightarrow X_\rho$ be a mapping satisfying the inequality

$$\begin{aligned} \rho(c(Tx - Ty)) &\leq \rho(l(x - y)) \\ &\quad - \phi(\rho(l(x - y))), \quad (27) \end{aligned}$$

for all $x, y \in X_\rho$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is continuous and monotone nondecreasing function with $\phi(t) = 0$ if and only if $t = 0$. Then, T has a unique fixed point.

Proof. Take $\psi(t) = t$, and then we obtain the Corollary 11. $\square$

Theorem 12. Let X_ρ be a ρ -complete modular space, where ρ satisfies the Δ_2 -condition and let $T : X_\rho \rightarrow X_\rho$ be a mapping satisfying the inequality

$$\psi(\rho((Tx - Ty))) \leq \psi(m(x, y)) - \phi(m(x, y)) \quad (28)$$

for all $x, y \in X_\rho$, where

$$m(x, y) = \max \left\{ \rho(x - y), \rho(x - Tx), \rho(y - Ty), \left(\rho\left(\frac{1}{2}(x - Ty)\right) + \rho\left(\frac{1}{2}(y - Tx)\right) \right) (2)^{-1} \right\} \quad (29)$$

and $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then, T has a unique fixed point.

Proof. First, we prove that the sequence $\{\rho(c(T^n x - T^{n-1} x))\}$ converges to 0. Since,

$$\begin{aligned} \psi(\rho(T^n x - T^{n-1} x)) &\leq \psi(m(T^{n-1} x, T^{n-2} x)) \\ &\quad - \phi(m(T^{n-1} x, T^{n-2} x)). \quad (30) \\ &\leq \psi(m(T^{n-1} x, T^{n-2} x)). \end{aligned}$$

By monotone nondecreasing of ψ , we have

$$\rho(T^n x - T^{n-1} x) \leq m(T^{n-1} x, T^{n-2} x). \quad (31)$$

From the definition of $m(x, y)$, we get

$$\begin{aligned} m(T^{n-1} x, T^{n-2} x) &= \max \left\{ \rho(T^{n-1} x - T^{n-2} x), \right. \\ &\quad \left. \rho(T^n x - T^{n-1} x), \frac{\rho((1/2)(T^n x - T^{n-2} x))}{2} \right\} \\ &= \max \left\{ \rho(T^{n-1} x - T^{n-2} x), \rho(T^n x - T^{n-1} x), \right. \\ &\quad \left. \frac{\rho(T^n x - T^{n-1} x) + \rho(T^{n-1} x - T^{n-2} x)}{2} \right\} \\ &= \max \left\{ \rho(T^{n-1} x - T^{n-2} x), \rho(T^n x - T^{n-1} x) \right\}. \quad (32) \end{aligned}$$

If $\rho(T^n x - T^{n-1} x) > \rho(T^{n-1} x - T^{n-2} x) \geq 0$, then $m(T^{n-1} x, T^{n-2} x) = \rho(T^n x - T^{n-1} x)$. Furthermore it is implied that

$$\begin{aligned} \psi(\rho(T^n x - T^{n-1} x)) &\leq \psi(m(T^{n-1} x, T^{n-2} x)) \\ &\quad - \phi(m(T^{n-1} x, T^{n-2} x)) \\ &\leq \psi(\rho(T^n x - T^{n-1} x)) \\ &\quad - \phi(\rho(T^n x - T^{n-1} x)) \\ &< \psi(\rho(T^n x - T^{n-1} x)) \end{aligned} \quad (33)$$

which is a contradiction, and, hence,

$$\begin{aligned} \rho(T^n x - T^{n-1} x) &\leq m(T^{n-1} x - T^{n-2} x) \\ &= \rho(T^{n-1} x - T^{n-2} x). \end{aligned} \quad (34)$$

So, we have that the sequence $\{\rho(T^n x - T^{n-1} x)\}$ is monotone decreasing and bounded below. Hence there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} \rho(T^n x - T^{n-1} x) = r. \quad (35)$$

If $r > 0$, taking $n \rightarrow \infty$ in the inequality (30), we get

$$\begin{aligned} \psi(r) &\leq \psi(r) - \phi(r) \\ &< \psi(r) \end{aligned} \quad (36)$$

which is a contradiction, and thus $r = 0$. So, we have

$$\lim_{n \rightarrow \infty} \rho(T^n x - T^{n-1} x) = 0. \quad (37)$$

Next, we prove that the sequence $\{T^n(x)\}_{n \in \mathbb{N}}$ is ρ -Cauchy. Suppose $\{T^n(x)\}_{n \in \mathbb{N}}$ is not ρ -Cauchy, and there exist $\varepsilon > 0$ and sequence of integers $\{m_k\}, \{n_k\}$ with $\{m_k\} > \{n_k\} \geq k$ such that

$$\rho((T^{m_k} x - T^{n_k} x)) \geq \varepsilon, \quad \rho(2(T^{m_k-1} x - T^{n_k} x)) < \varepsilon. \quad (38)$$

Since,

$$\begin{aligned} \psi(\rho((T^{m_k} x - T^{n_k} x))) &\leq \psi(m(T^{m_k-1} x, T^{n_k-1} x)) \\ &\quad - \phi(m(T^{m_k-1} x, T^{n_k-1} x)) \quad (39) \\ &\leq \psi(m(T^{m_k-1} x, T^{n_k-1} x)) \end{aligned}$$

which implies that

$$\rho((T^{m_k} x - T^{n_k} x)) \leq m(T^{m_k-1} x, T^{n_k-1} x). \quad (40)$$

On the other hand,

$$\begin{aligned}
& m(T^{m_k-1}x, T^{n_k-1}x) \\
&= \max \left\{ \rho(T^{m_k-1}x - T^{n_k-1}x), \right. \\
&\quad \rho(T^{m_k}x - T^{m_k-1}x), \\
&\quad \rho(T^{n_k}x - T^{n_k-1}x), \\
&\quad \left(\rho\left(\frac{1}{2}(T^{m_k}x - T^{n_k-1}x)\right) \right. \\
&\quad \left. + \rho\left(\frac{1}{2}(T^{m_k-1}x - T^{n_k}x)\right) \right) (2)^{-1} \Big\}, \quad (41) \\
&\rho((T^{m_k-1}x - T^{n_k-1}x)) \\
&= \rho((T^{m_k-1}x - T^{n_k}x + T^{n_k}x - T^{n_k-1}x)) \\
&\leq \rho(2(T^{m_k-1}x - T^{n_k}x)) \\
&\quad + \rho(2(T^{n_k}x - T^{n_k-1}x)) \\
&< \varepsilon + \rho(2(T^{n_k}x - T^{n_k-1}x)).
\end{aligned}$$

For the last term in $m(T^{m_k-1}x, T^{n_k-1}x)$, by Proposition 8, we have

$$\begin{aligned}
& \left(\rho\left(\frac{1}{2}(T^{m_k}x - T^{n_k-1}x)\right) + \rho\left(\frac{1}{2}(T^{m_k-1}x - T^{n_k}x)\right) \right) (2)^{-1} \\
&= \left(\rho\left(\frac{1}{2}(T^{m_k-1}x - T^{n_k}x)\right) \right. \\
&\quad + \rho\left(\frac{1}{2}(T^{m_k}x - T^{m_k-1}x + T^{m_k-1}x - T^{n_k}x) \right. \\
&\quad \left. \left. + \frac{1}{2}(T^{n_k}x - T^{n_k-1}x)\right) \right) (2)^{-1} \\
&\leq \left(\rho\left(\frac{1}{2}(T^{m_k-1}x - T^{n_k}x)\right) \right. \\
&\quad + \rho(T^{m_k}x - T^{m_k-1}x + T^{m_k-1}x - T^{n_k}x) \\
&\quad \left. + \rho(T^{n_k}x - T^{n_k-1}x) \right) (2)^{-1} \\
&\leq \left(\rho\left(\frac{1}{2}(T^{m_k-1}x - T^{n_k}x)\right) + \rho(2(T^{m_k}x - T^{m_k-1}x)) \right. \\
&\quad \left. + \rho(2(T^{m_k-1}x - T^{n_k}x)) + \rho(T^{n_k}x - T^{n_k-1}x) \right) (2)^{-1} \\
&< \varepsilon + \frac{\rho(T^{n_k}x - T^{n_k-1}x) + \rho(2(T^{m_k}x - T^{m_k-1}x))}{2}. \quad (42)
\end{aligned}$$

It follow from (41) and (42) that

$$\begin{aligned}
& m(T^{m_k-1}x, T^{n_k-1}x) \\
&= \max \left\{ \rho(T^{m_k-1}x - T^{n_k-1}x), \right. \\
&\quad \rho(T^{m_k}x - T^{m_k-1}x), \rho(T^{n_k}x - T^{n_k-1}x), \\
&\quad \left(\rho\left(\frac{1}{2}(T^{m_k}x - T^{n_k-1}x)\right) \right. \\
&\quad \left. + \rho\left(\frac{1}{2}(T^{m_k-1}x - T^{n_k}x)\right) \right) (2)^{-1} \Big\} \\
&< \max \left\{ \varepsilon + \rho(2(T^{n_k}x - T^{n_k-1}x)), \right. \\
&\quad \rho(T^{m_k}x - T^{m_k-1}x), \rho(T^{n_k}x - T^{n_k-1}x), \varepsilon \\
&\quad \left. + \frac{\rho(T^{n_k}x - T^{n_k-1}x) + \rho(2(T^{m_k}x - T^{m_k-1}x))}{2} \right\}. \quad (43)
\end{aligned}$$

By (37), (38), (40), (43), and the Δ_2 -condition of ρ , we have

$$\lim_{k \rightarrow \infty} \rho((T^{m_k}x - T^{n_k}x)) = \lim_{k \rightarrow \infty} m(T^{m_k-1}x, T^{n_k-1}x) = \varepsilon. \quad (44)$$

Taking $k \rightarrow \infty$ in (39), by (44) and the continuity of ψ , we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon) \quad (45)$$

which is a contradiction. Hence, $\{T^n(x)\}_{n \in \mathbb{N}}$ is ρ -Cauchy. Since X_ρ is ρ -complete, there exists a point $u \in X_\rho$ such that $\rho(T^n x - u) \rightarrow 0$ as $n \rightarrow \infty$. Next, we prove that u is a unique fixed point of T . Suppose that $Tu \neq u$, then $\rho(u - Tu) > 0$.

Since,

$$\begin{aligned}
& \psi(\rho(T^n x - Tu)) \leq \psi(m(T^{n-1}x, u)) \\
&\quad - \phi(m(T^{n-1}x, u)), \quad (46)
\end{aligned}$$

$$\begin{aligned}
& m(T^{n-1}x, u) \\
&= \max \left\{ \rho(T^{n-1}x - u), \rho(T^{n-1}x - T^n x), \right. \\
&\quad \rho(u - Tu), \left(\rho\left(\frac{1}{2}(T^{n-1}x - Tu)\right) \right. \\
&\quad \left. + \rho\left(\frac{1}{2}(u - T^n x)\right) \right) (2)^{-1} \Big\} \quad (47) \\
&\rightarrow \max \left\{ 0, 0, \rho(u - Tu), \frac{\rho((1/2)(u - Tu))}{2} \right\} \\
&= \rho(u - Tu)
\end{aligned}$$

as $n \rightarrow \infty$.

Taking $n \rightarrow \infty$ in (46), by using (47), we get

$$\begin{aligned}\psi(\rho(u - Tu)) &\leq \psi(\rho(u - Tu)) - \phi(\rho(u - Tu)) \\ &< \psi(\rho(u - Tu))\end{aligned}\quad (48)$$

which is a contradiction. Hence, $\rho(u - Tu) = 0$ and $Tu = u$. If there exists point $v \in X_\rho$ such that $Tv = v$ and $u \neq v$, then using an argument similar to the above we get

$$\begin{aligned}\psi(\rho(u - v)) &= \psi(\rho(Tu - Tv)) \\ &\leq \psi(m(u, v)) - \phi(m(u, v)) \\ &\leq \psi(\rho(u - v)) - \phi(\rho(u - v)) \\ &< \psi(\rho(u - v))\end{aligned}\quad (49)$$

which is a contradiction. Hence, $u = v$ and the proof is complete. $\square$

Corollary 13. Let X_ρ be a ρ -complete modular space, where ρ satisfies the Δ_2 -condition, and let $T : X_\rho \rightarrow X_\rho$ be a mapping satisfying the inequality

$$\rho((Tx - Ty)) \leq m(x, y) - \phi(m(x, y)) \quad (50)$$

for all $x, y \in X_\rho$, where $m(x, y) = \max\{\rho(x - y), \rho(x - Tx), \rho(y - Ty), (\rho((1/2)(x - Ty)) + \rho((1/2)(y - Tx)))/2\}$ and $\phi : [0, \infty) \rightarrow [0, \infty)$ is continuous and monotone nondecreasing function with $\phi(t) = 0$ if and only if $t = 0$. Then, T has a unique fixed point.

Proof. Taking $\psi(t) = t$, we obtain the Corollary 13. $\square$

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