Research Article

A Regularity Criterion for Compressible Nematic Liquid Crystal Flows

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We prove a blow-up criterion for local strong solutions to a simplified hydrodynamic flow modeling the compressible, nematic liquid crystal materials in a bounded domain.

1. Introduction

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary \( \partial \Omega \). We consider the following simplified version of Ericksen-Leslie system modeling the hydrodynamic flow of compressible nematic liquid crystals:

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p(\rho) - \mu \Delta u &= -(\lambda + \mu) \text{div} u - \Delta d \cdot \nabla d, \\
\partial_t d + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d, \\
|d| &= 1 \quad \text{in} \quad \Omega \times (0, \infty), \\
u &= 0, \quad d = d_0(x) \quad \text{on} \quad \partial \Omega \times (0, \infty), \\
(\rho, u, d)(x, 0) &= (\rho_0, u_0, d_0)(x), \\
|d_0| &= 1, \quad x \in \Omega \subset \mathbb{R}^3.
\end{align*}
\]

Here \( \rho \) is the density of the fluid, \( u \) is the fluid velocity, \( d \) represents the macroscopic average of the nematic liquid crystal orientation field, and \( p(\rho) := a \rho^\gamma \) is the pressure with positive constants \( a > 0 \) and \( \gamma > 1 \). Two real constants \( \mu \) and \( \lambda \) are the shear viscosity and the bulk viscosity coefficients of the fluid, respectively, which are assumed to satisfy the following physical condition:

\[
\mu > 0, \quad 3\lambda + 2\mu \geq 0.
\]

Equations (1) and (2) are the well-known compressible Navier-Stokes system with the external force \( -\Delta d \cdot \nabla d \). Equation (3) is the well-known heat flow of harmonic map when \( u = 0 \).

Recently, Huang et al. [1] prove the following local-in-time well-posedness.

**Proposition 1.** Let \( \rho_0 \in W^{1,q} \) for some \( q \in (3,6] \) and \( \rho_0 \geq 0 \) in \( \Omega \), \( u_0 \in H^2 \), \( d_0 \in H^3 \) and \( |d_0| = 1 \) in \( \Omega \). If, in addition, the compatibility condition

\[
-\mu \Delta u_0 - (\lambda + \mu) \text{div} u_0 - \nabla p(\rho_0) - \Delta d_0 \cdot \nabla d_0 = \sqrt{\rho_0} g
\]

for some \( g \in L^2(\Omega) \) (7)

holds, then there exist \( T_0 > 0 \) and a unique strong solution \( (\rho, u, d) \) to the problem (1)–(5).

Based on the above proposition, Huang et al. [2] prove the regularity criterion

\[
\int_0^T \left( \| \nabla (u) \|_{L^\infty} + \| \nabla d \|_{L^\infty}^2 \right) dt < \infty
\]

(8)
to the problem (1)–(3), (5) with the boundary condition

\[
u = 0 = \frac{\partial d}{\partial y} \quad \text{on} \quad \partial \Omega \times (0, \infty)
\]

or

\[
u \cdot y = \text{curl} u \times y = \frac{\partial d}{\partial y} = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty).
\]

(10)
Here,

$$D(u) := \frac{1}{2} \left( \nabla u + \nabla u \right),$$  \hspace{1cm} (11)

where $\nu$ is the unit outward normal vector to $\partial \Omega$.

When $\Omega = \mathbb{R}^3$, Huang and Wang [3] show the following regularity criterion:

$$\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^2(0,T;L^2)} + \|\nabla d\|_{L^2(0,T;L^2)} < \infty,$$  \hspace{1cm} (12)

with $r_i$ and $s_i$ satisfying

$$\frac{2}{s_i} + \frac{3}{r_i} = 1, \quad 3 < r_i \leq \infty, \quad i = 1, 2.$$  \hspace{1cm} (13)

When the term $|\nabla d|^2$ is replaced by $d - |d|^2 d$, the problem (1)–(5) has been studied by L. M. Liu and X. G. Liu [4]; they proved the following regularity criterion:

$$\int_0^T \left( \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^\infty} \right) dt < \infty.$$  \hspace{1cm} (14)

The aim of this paper is to study the regularity criterion of local strong solutions to the problem (1)–(5). We will prove

**Theorem 2.** Let the assumptions in Proposition 1 hold true. If (12) holds true with $0 < T < \infty$, then the solution $(\rho, u, d)$ can be extended beyond $T > 0$.

**Remark 3.** Theorem 2 is also true for the boundary condition (9). But it is an open problem to prove (12) when the homogeneous Dirichlet boundary condition $u = 0$ is replaced by

$$u \cdot \nu = 0, \quad \text{curl} u \times \nu = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty).$$  \hspace{1cm} (15)

2. **Proof of Theorem 2**

Since $(\rho, u, d)$ is the local strong solution, we only need to prove a priori estimates.

First, testing (2) and (3) by $u$ and $\Delta d + |\nabla d|^2 d$, respectively, and adding the resulting equations together, we see that

$$\frac{d}{dt} \left( \int \left( \frac{1}{2} \rho |u|^2 + \frac{1}{2} |\nabla d|^2 + \frac{a \rho^2}{\gamma - 1} \right) dx \right) + \int \left( \rho |\nabla u|^2 + (\lambda + \mu) (\text{div} u)^2 + |\Delta d + |\nabla d|^2 d|^2 \right) dx = 0,$$  \hspace{1cm} (16)

which gives

$$\int \left( \rho |u|^2 + |\nabla d|^2 \right) dx + \int_0^T \left( \|\nabla u\|^2 + |\Delta d + |\nabla d|^2 d|^2 \right) dx dt \leq C.$$  \hspace{1cm} (17)

We decompose the velocity $u$ into two parts: $u = v + w$, where $v(t) \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfies

$$\mu \Delta v + (\lambda + \mu) \nabla \text{div} v = \nabla p(\rho),$$  \hspace{1cm} (18)

and thus $w(t) \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfies

$$\mu \Delta w + (\lambda + \mu) \nabla \text{div} w = \rho \dot{u} + \Delta d \cdot \nabla d,$$  \hspace{1cm} (19)

where we used $\dot{u} := \partial_u u + \nabla u$ to denote the material derivative of $u$. Then, together with the standard $W^{1,p}$ theory and $H^2$ theory for elliptic systems, we obtain

$$\|\nabla v\|_{L^2} \leq C \|\rho\|_{L^\infty},$$  \hspace{1cm} (20)

$$\|\nabla w\|_{L^2} + \|\nabla w\|_{L^2} \leq C \rho \dot{u} + C \Delta d \nabla d.$$  \hspace{1cm} (21)

for any $0 < \epsilon < 1$, where we have used the Hölder inequality

$$\|\nabla u\|_{L^2}^{1/(r_1 - 2)} \leq C \|\nabla u\|_{L^2}^{1/r_1} \|\nabla u\|_{L^2}^{3/r_1},$$  \hspace{1cm} (22)
and the Gagliardo-Nirenberg inequality

\[ \| \nabla^2 d \|_{L^2}^{1/(r_2-2)} \leq C \| \nabla d \|_{L^2}^{1/(r_2-2)} \| d \|_{L^2}^{r_2/(r_2-2)}. \]

By the $H^3$ theory of the elliptic equations, it follows from (3) that

\[ \| d \|_{H^3} \leq C (1 + \| \Delta d \|_{L^2}). \]

which yields

\[ \| d \|_{H^1} \leq C (1 + \| \nabla d \|_{L^2} + \| u \|_{L^1}^{1/2} \| \nabla d \|_{L^2}^{1/2} + \| \nabla d \|_{L^2}^{1/2} \| \nabla u \|_{L^2}). \]

Using (27) and (20), we have

\[ - \int \partial_t \rho \div u \, dx = - \int \partial_t \rho \div v \, dx - \int \partial_t \rho \div w \, dx \]

\[ = \int \nabla \rho \div w \, dx + \int \partial_t \rho \div w \, dx \]

\[ + (y-1) \int p \, \div u \, dx \]

\[ \leq - \frac{d}{dt} \int \left( \frac{\mu}{2} | \nabla u |^2 + \frac{\lambda + \mu}{2} (\div u)^2 \right) \, dx \]

\[ - \int p \, \div u \, dx - \frac{d}{dt} \int M(d) : \nabla u \, dx \]

\[ + \int \rho | \dot{u} |^2 \, dx \]

\[ \leq \| \rho \dot{u} \|_{L^2} \| u \|_{L^2} + C \| \nabla u \|_{L^2} \| \nabla \partial_t d \|_{L^2}. \]

Inserting (28) into (26) and using (20), we have

\[ \frac{1}{2} \frac{d}{dt} \int \left( \mu | \nabla u |^2 + (\lambda + \mu) (\div u)^2 \right) \, dx \]

\[ + \frac{d}{dt} \int \left( \frac{\mu}{2} | \nabla v |^2 + \frac{\lambda + \mu}{2} (\div v)^2 \right) \, dx \]

\[ - \frac{d}{dt} \int p \, \div u \, dx - \frac{d}{dt} \int M(d) : \nabla u \, dx \]

\[ + \int \rho | \dot{u} |^2 \, dx \]

\[ \leq \rho \dot{u} \|_{L^2} \| u \|_{L^2} + C \| \nabla u \|_{L^2} \| \nabla \partial_t d \|_{L^2}. \]

Combining (21), (25), and (29), taking $\epsilon$ small enough, and using the Gronwall inequality, we conclude that

\[ \| u \|_{L^2(0, T; H^1)} + \| d \|_{L^2(0, T; H^3)} + \| d \|_{L^2(0, T; H^4)} \]

\[ + C \| \dot{u} \|_{L^2(0, T; L^2)} \leq C. \]
Now by the same calculations as those in [3, 5], we prove
that
\[
\|\rho\|_{L^\infty(0,T;W^{1,q})} + \|\partial_t \rho\|_{L^\infty(0,T;L^q)} \leq C,
\]
\[
\|\sqrt{\rho} u\|_{L^\infty(0,T;L^2)} + \|\partial_t u\|_{L^2(0,T;H^1)} \leq C,
\]
\[
\|u\|_{L^\infty(0,T;H^2)} + \|u\|_{L^2(0,T;W^{2,q})} \leq C,
\]
\[
\|d\|_{L^\infty(0,T;H^3)} \leq C,
\]
\[
\|\partial_t d\|_{L^\infty(0,T;H^1)} \leq C.
\]
(31)

This completes the proof.

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