Research Article

Wellposedness of the Inverse Problem for Dirac Operator

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We study the wellposedness of the inverse problem for Dirac operator. We consider two different problems (unperturbed and perturbed problem) for Dirac operator, and then we prove that if the spectral characteristics of these problems are close to each other, then the difference between their potential functions is sufficiently small.

1. Introduction

Inverse problems are studied for certain special classes of ordinary differential operators. Typically, in inverse eigenvalue problems, one measures the frequencies of a vibrating system and tries to infer some physical properties of the system. An early important result in this direction, which gave vital impetus for the further development of inverse problem theory, was obtained in [1–5].

The Dirac equation is a modern presentation of the relativistic quantum mechanics of electrons, intended to make new mathematical results accessible to a wider audience. It treats in some depth the relativistic invariance of a quantum theory, self-adjointness and spectral theory, qualitative features of relativistic bound and scattering states, and the external field problem in quantum electrodynamics, without neglecting the interpretational difficulties and limitations of the theory.

Inverse problems for Dirac system had been investigated by Moses [6], Prats and Toll [7], Verde [8], Gasymov and Levitan [9], and Panakhov [10, 11]. It is well known [12] that two spectra uniquely determine the matrix-valued potential function. In particular, in [13], eigenfunction expansions for one-dimensional Dirac operators describing the motion of a particle in quantum mechanics are investigated. Recently, Dirac operators have been extensively studied [14–19].

Mizutani showed the wellposedness problem of the Sturm-Liouville operator according to norming constants

and eigenvalues [20]. The purpose of this paper is to give the wellposedness problem for Dirac operator by using Mizutani's method.

Let L denote a matrix operator

$$L \equiv \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}, \quad p_{12}(x) = p_{21}(x), \quad (1)$$

where the $p_{ik}(x)$ ($i, k = 1, 2$) are real functions which are defined and continuous on the interval $[0, \pi]$. Further, let $\varphi(x, \lambda)$ denote a two-component vector-function

$$\varphi(x, \lambda) = \begin{pmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{pmatrix}. \quad (2)$$

Then the equation

$$\left(B \frac{d}{dx} + L - \lambda I \right) \varphi = 0, \quad (3)$$

where λ is a parameter and

$$B \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4)$$

is equivalent to the system of two simultaneous first-order ordinary differential equations

$$\begin{aligned} \frac{d\varphi_2}{dx} + p_{11}(x)\varphi_1 + p_{12}(x)\varphi_2 &= \lambda\varphi_1, \\ -\frac{d\varphi_1}{dx} + p_{21}(x)\varphi_1 + p_{22}(x)\varphi_2 &= \lambda\varphi_2. \end{aligned} \quad (5)$$

For the case in which $p_{12}(x) = p_{21}(x) = 0$, $p_{11}(x) = V(x) + m$, $p_{22}(x) = V(x) - m$, where $V(x)$ is a potential and m is the mass of a particle, the system (5) is known in relativistic quantum theory as a stationary one-dimensional Dirac system [5].

Consider the following eigenvalue problem for Dirac operator:

$$By' + Q(x)y = \lambda y, \quad 0 \leq x \leq \pi, \quad (6)$$

$$y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \quad (7)$$

$$y_1(\pi) = 0, \quad (8)$$

where $Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $p(x)$ and $q(x) \in C^1[0, \pi]$ are real-valued functions, $0 \leq \alpha \leq \pi$, and λ is spectral parameter.

We denote by

$$\varphi(x, \lambda) = \begin{pmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{pmatrix} = \begin{pmatrix} \sin(\lambda x - \alpha) + O\left(\frac{e^{\tau x}}{\lambda}\right) \\ \cos(\lambda x - \alpha) + O\left(\frac{e^{\tau x}}{\lambda}\right) \end{pmatrix} \quad (9)$$

the solution of (6) satisfying the initial conditions

$$\varphi_1(0, \lambda) = -\sin \alpha, \quad \varphi_2(0, \lambda) = \cos \alpha, \quad (10)$$

where $\tau = |\operatorname{Im} \lambda|$.

We show the spectral characteristics of the problems (6)–(8) by $\{\lambda_n, \rho_n\}$ ($n \in \mathbb{Z}$), where $\{\lambda_n\}$ is the spectrum (set of eigenvalues) and $\{\rho_n\}$ is the norming constants of this problem. These spectral characteristics satisfy the following asymptotic expressions, respectively [9]:

$$\begin{aligned} \lambda_n &= n - \frac{\alpha}{\pi} + \frac{\alpha_1}{n} + O\left(\frac{1}{n^2}\right), \\ \rho_n &= \pi + \frac{c_1}{n} + O\left(\frac{1}{n^2}\right), \end{aligned} \quad (11)$$

where α , α_1 , and c_1 are real numbers.

Now let us consider the second eigenvalue problem

$$By' + \bar{Q}(x)y = \mu y, \quad 0 \leq x \leq \pi, \quad (12)$$

with the same boundary conditions (7), (8), where $\bar{Q}(x) = \begin{pmatrix} \bar{p}(x) & \bar{q}(x) \\ \bar{q}(x) & -\bar{p}(x) \end{pmatrix}$, $\bar{p}(x)$ and $\bar{q}(x) \in C^1[0, \pi]$ are real-valued functions and μ is spectral parameter. We denote the spectral characteristics of the problems (12), (7), and (8) by $\{\mu_n, \sigma_n\}$ ($n \in \mathbb{Z}$), where $\{\mu_n\}$ is the spectrum and $\{\sigma_n\}$ is the norming constants of this problem. These spectral characteristics satisfy the following asymptotic expressions, respectively:

$$\begin{aligned} \mu_n &= n - \frac{\alpha'}{\pi} + \frac{\alpha'_1}{n} + O\left(\frac{1}{n^2}\right), \\ \sigma_n &= \pi + \frac{c'_1}{n} + O\left(\frac{1}{n^2}\right), \end{aligned} \quad (13)$$

where α' , α'_1 , and c'_1 are real constants.

Theorem 1. Let E be a linear topological space and $E_1, E_2 \subset E$, where E_1 and E_2 are linear topological subspaces of E . The transformation operator $X = X_{A,B}$ mapping E_1 to E_2 can be defined as follows [9]:

$$X[\varphi(x, \lambda)] = \tilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \int_0^x K(x, s) \varphi(s, \lambda) ds, \quad (14)$$

where the kernel $K(x, s)$ is a solution of the equation

$$B \frac{\partial K}{\partial x} + \bar{Q}(x) K(x, s) = -\frac{\partial K}{\partial s} B + Q(x) K(x, s) \quad (15)$$

and also satisfies the following conditions:

$$BK(x, x) - K(x, x)B = \bar{Q}(x) - Q(x), \quad (16)$$

$$K_{21}(x, 0) = K_{11}(x, 0) = 0, \quad (17)$$

and $K(x, s) = \begin{pmatrix} K_{11}(x, s) & K_{12}(x, s) \\ K_{21}(x, s) & K_{22}(x, s) \end{pmatrix}$ is continuously differentiable on $[0, \pi]$.

The $K(x, s)$ function in (14) satisfies the following integral equation:

$$\begin{aligned} K(x, s) + F(x, s) + \int_0^x K(x, t) F(t, s) dt &= 0, \\ 0 \leq s \leq x \leq \pi, \end{aligned} \quad (18)$$

where $F(x, s)$ function is as such

$$F(x, s) = \sum_{n=1}^{\infty} \frac{\varphi(x, \mu_n) \varphi^T(s, \mu_n)}{\sigma_n} - \frac{\varphi(x, \lambda_n) \varphi^T(s, \lambda_n)}{\rho_n}, \quad (19)$$

and $F(x, s) = \begin{pmatrix} F_{11}(x, s) & F_{12}(x, s) \\ F_{21}(x, s) & F_{22}(x, s) \end{pmatrix}$ is continuously differentiable on $[0, \pi]$.

Now we present main theorem in this paper.

2. Main Results

Theorem 2. If

$$M \equiv \sum_{n=1}^{\infty} \{|\sigma_n - \alpha_n| + |\mu_n - \lambda_n|\} \quad (20)$$

is sufficiently small, then we have

$$\begin{aligned} \max_{0 \leq x \leq \pi} |\tilde{p}(x) - p(x)| &\leq C' M, \\ \max_{0 \leq x \leq \pi} |\tilde{q}(x) - q(x)| &\leq C'' M, \end{aligned} \quad (21)$$

where $C' > 0$ and $C'' > 0$ are constants.

Proof. Let us solve the Gelfand-Levitan equation (18). We start from $F(s, t)$ and construct the iterated kernels $F^{(n)}(s, t; x)$ ($n = 1, 2, 3, \dots$) as follows:

$$\begin{aligned} F^{(1)}(s, t; x) &= F(s, t), \\ F^{(n+1)}(s, t; x) &= \int_0^x F(s, u) F^{(n)}(u, t; x) du, \end{aligned} \quad (22)$$

for $n \geq 1$.

We put $S(s, t; x) = \sum_{n=1}^{\infty} (-1)^n F^{(n)}(s, t; x)$ and assume that

$$\int_0^\pi \int_0^\pi |F(s, t)|^2 ds dt < 1. \quad (23)$$

Then it is easily seen that

$$K(x, s) = S(x, s; x), \quad \text{for } 0 \leq s \leq x \leq \pi. \quad (24)$$

From (19), we have

$$\begin{aligned} F(x, s) &= \sum_{n=1}^{\infty} \left[\frac{\varphi(x, \mu_n) \varphi^T(s, \mu_n)}{\sigma_n} - \frac{\varphi(x, \lambda_n) \varphi^T(s, \lambda_n)}{\rho_n} \right] \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{\sigma_n} \left[\frac{\varphi_1(x, \mu_n)}{\varphi_2(x, \mu_n)} \right] [\varphi_1(s, \mu_n) \varphi_2(s, \mu_n)] \right. \\ &\quad \left. - \frac{1}{\rho_n} \left[\frac{\varphi_1(x, \lambda_n)}{\varphi_2(x, \lambda_n)} \right] [\varphi_1(s, \lambda_n) \varphi_2(s, \lambda_n)] \right) \\ &= \sum_{n=1}^{\infty} \left\{ \left[\begin{array}{c} \frac{\varphi_1(x, \mu_n) \varphi_1(s, \mu_n)}{\sigma_n} \\ - \frac{\varphi_1(x, \lambda_n) \varphi_1(s, \lambda_n)}{\rho_n} \\ \times \frac{\varphi_1(x, \mu_n) \varphi_2(s, \mu_n)}{\sigma_n} \\ - \frac{\varphi_1(x, \lambda_n) \varphi_2(s, \lambda_n)}{\rho_n} \\ \frac{\varphi_2(x, \mu_n) \varphi_1(s, \mu_n)}{\sigma_n} \\ - \frac{\varphi_2(x, \lambda_n) \varphi_1(s, \lambda_n)}{\rho_n} \\ \times \frac{\varphi_2(x, \mu_n) \varphi_2(s, \mu_n)}{\sigma_n} \\ - \frac{\varphi_2(x, \lambda_n) \varphi_2(s, \lambda_n)}{\rho_n} \end{array} \right] \right\}. \end{aligned} \quad (25)$$

From (25), it is easy to see that

$$F_{11}(x, s) = \sum_{n=1}^{\infty} \left[\frac{\varphi_1(x, \mu_n) \varphi_1(s, \mu_n)}{\sigma_n} - \frac{\varphi_1(x, \lambda_n) \varphi_1(s, \lambda_n)}{\rho_n} \right], \quad (26)$$

$$F_{12}(x, s) = \sum_{n=1}^{\infty} \left[\frac{\varphi_1(x, \mu_n) \varphi_2(s, \mu_n)}{\sigma_n} - \frac{\varphi_1(x, \lambda_n) \varphi_2(s, \lambda_n)}{\rho_n} \right], \quad (27)$$

$$F_{21}(x, s) = \sum_{n=1}^{\infty} \left[\frac{\varphi_2(x, \mu_n) \varphi_1(s, \mu_n)}{\sigma_n} - \frac{\varphi_2(x, \lambda_n) \varphi_1(s, \lambda_n)}{\rho_n} \right], \quad (28)$$

$$F_{22}(x, s) = \sum_{n=1}^{\infty} \left[\frac{\varphi_2(x, \mu_n) \varphi_2(s, \mu_n)}{\sigma_n} - \frac{\varphi_2(x, \lambda_n) \varphi_2(s, \lambda_n)}{\rho_n} \right]. \quad (29)$$

Firstly, we calculate formula of (26). Adding and subtracting $\varphi_1(x, \lambda_n) \varphi_1(s, \lambda_n) / \sigma_n$ to the right side of (26), then we obtain

$$\begin{aligned} F_{11}(x, s) &= \sum_{n=1}^{\infty} \left[\frac{\varphi_1(x, \mu_n) \varphi_1(s, \mu_n)}{\sigma_n} - \frac{\varphi_1(x, \lambda_n) \varphi_1(s, \lambda_n)}{\rho_n} \right] \\ &\quad + \sum_{n=1}^{\infty} \left[\frac{\varphi_1(x, \lambda_n) \varphi_1(s, \lambda_n)}{\sigma_n} \right. \\ &\quad \left. - \frac{\varphi_1(x, \lambda_n) \varphi_1(s, \lambda_n)}{\sigma_n} \right] \\ &= \sum_{n=1}^{\infty} \left[\left(\frac{\rho_n - \sigma_n}{\rho_n \sigma_n} \right) \varphi_1(x, \lambda_n) \varphi_1(s, \lambda_n) \right] \\ &\quad + \frac{1}{\sigma_n} \int_{\lambda_n}^{\mu_n} [\varphi_1(x, \lambda) \varphi_1(s, \lambda)] d\lambda, \end{aligned} \quad (30)$$

where $\cdot = \partial / \partial \lambda$. If the absolute value is taken in the formula of (30) and (9) is used and $s = x$ is written, the following is obtained:

$$\begin{aligned} |F_{11}(x, x)| &\leq \sum_{n=1}^{\infty} \left[\left| \left(\frac{\rho_n - \sigma_n}{\rho_n \sigma_n} \right) c_1 \right| + \left| \frac{1}{\sigma_n} \int_{\lambda_n}^{\mu_n} c_2 d\lambda \right| \right] \\ &\leq \sum_{n=1}^{\infty} \left[|(\rho_n - \sigma_n) c_3| + \left| c_4 \int_{\lambda_n}^{\mu_n} d\lambda \right| \right] \\ &\leq C_1 \sum_{n=1}^{\infty} \{ |\sigma_n - \rho_n| + |\mu_n - \lambda_n| \}, \end{aligned} \quad (31)$$

where c_1, c_2, c_3, c_4 , and C_1 are constants and positive numbers. From (20), we have

$$|F_{11}(x, x)| \leq C_1 M. \quad (32)$$

From (24) and (32) for $n = 1$, we have

$$\begin{aligned} F_{11}^{(2)}(s, t; x) &= \int_0^x F_{11}(s, u) F_{11}^{(1)}(u, t; x) du \\ |F_{11}^{(2)}(s, t; x)| &\leq \frac{(C_1 M \pi)^2}{\pi}, \end{aligned} \quad (33)$$

similarly for $n = 2$

$$\begin{aligned} F_{11}^{(3)}(s, t; x) &= \int_0^x F_{11}(s, u) F_{11}^{(2)}(u, t; x) du \\ |F_{11}^{(3)}(s, t; x)| &\leq \frac{(C_1 M \pi)^3}{\pi}, \\ &\vdots \end{aligned} \quad (34)$$

consequently,

$$\begin{aligned} F_{11}^{(n)}(s, t; x) &= \int_0^x F_{11}(s, u) F_{11}^{(n-1)}(u, t; x) du \\ |F_{11}^{(n)}(s, t; x)| &\leq \frac{(C_1 M \pi)^n}{\pi}. \end{aligned} \quad (35)$$

From (24), we get the following:

$$K_{11}(x, s) \leq \sum_{n=1}^{\infty} (-1)^n F_{11}^{(n)}(s, t; x). \quad (36)$$

From (35) and (36), we have

$$|K_{11}(x, x)| \leq \sum_{n=1}^{\infty} \left| (-1)^n \frac{(C_1 M \pi)^n}{\pi} \right| = \sum_{n=1}^{\infty} \frac{(C_1 M \pi)^n}{\pi}. \quad (37)$$

If $C_1 M \pi$ is small, for example, $C_1 M \pi < 1/2$, then we have

$$\begin{aligned} |K_{11}(x, x)| &\leq \sum_{n=1}^{\infty} \frac{(C_1 M \pi)^n}{\pi} \\ &= C_1 M + \pi(C_1 M)^2 + \pi^2(C_1 M)^3 + \cdots \\ &\leq C_1 M \left(1 + \frac{1}{2} + \left(\frac{1}{2} \right)^2 + \cdots \right) \\ &\leq 2C_1 M. \end{aligned} \quad (38)$$

If the operations for $F_{11}(x, s)$ are repeated, we get the following:

$$\begin{aligned} |F_{12}(x, s)| &\leq \sum_{n=1}^{\infty} \left[\left| \left(\frac{\rho_n - \sigma_n}{\rho_n \sigma_n} \right) c_5 \right| + \left| \frac{1}{\sigma_n} \int_{\lambda_n}^{\mu_n} c_6 d\lambda \right| \right], \\ &\leq \sum_{n=1}^{\infty} \left[|(\rho_n - \sigma_n) c_7| + \left| c_8 \int_{\lambda_n}^{\mu_n} d\lambda \right| \right], \\ &\leq C_2 \sum_{n=1}^{\infty} [|\sigma_n - \rho_n| + |\mu_n - \lambda_n|], \end{aligned} \quad (39)$$

where c_5, c_6, c_7, c_8 , and C_2 are constants and positive numbers.

From (20), we have

$$|F_{12}(x, s)| \leq C_2 M. \quad (40)$$

Similarly from (24) and (40) for $n = 1$, we have

$$\begin{aligned} F_{12}^{(2)}(s, t; x) &= \int_0^x F_{12}(s, u) F_{12}^{(1)}(u, t; x) du \\ |F_{12}^{(2)}(s, t; x)| &\leq \frac{(C_2 M \pi)^2}{\pi}, \end{aligned} \quad (41)$$

for $n = 2$,

$$\begin{aligned} F_{12}^{(3)}(s, t; x) &= \int_0^x F_{12}(s, u) F_{12}^{(2)}(u, t; x) du \\ |F_{12}^{(3)}(s, t; x)| &\leq \frac{(C_2 M \pi)^3}{\pi}, \end{aligned} \quad (42)$$

consequently,

$$\begin{aligned} &\vdots \\ F_{12}^{(n)}(s, t; x) &= \int_0^x F_{12}(s, u) F_{12}^{(n-1)}(u, t; x) du, \end{aligned} \quad (43)$$

$$|F_{12}^{(n)}(s, t; x)| \leq \frac{(C_2 M \pi)^n}{\pi}.$$

From (24), we get the following:

$$K_{12}(x, s) = \sum_{n=1}^{\infty} (-1)^n F_{12}^{(n)}(s, t; x). \quad (44)$$

If we repeat the above operations, we have

$$|K_{12}(x, x)| \leq \sum_{n=1}^{\infty} \left| (-1)^n \frac{(C_2 M \pi)^n}{\pi} \right| = \sum_{n=1}^{\infty} \frac{(C_2 M \pi)^n}{\pi}. \quad (45)$$

If $C_2 M \pi$ is small, for example, $C_2 M \pi < 1/2$, then we have

$$\begin{aligned} |K_{12}(x, x)| &\leq \sum_{n=1}^{\infty} \frac{(C_2 M \pi)^n}{\pi} \\ &= C_2 M + \pi(C_2 M)^2 + \pi^2(C_2 M)^3 + \cdots \\ &\leq C_2 M \left(1 + \frac{1}{2} + \left(\frac{1}{2} \right)^2 + \cdots \right) \\ &\leq 2C_2 M. \end{aligned} \quad (46)$$

When the similar operations are repeated, the following is obtained:

$$\begin{aligned} |K_{21}(x, x)| &\leq 2C_3 M, \\ |K_{22}(x, x)| &\leq 2C_4 M, \end{aligned} \quad (47)$$

where $C_3 > 0$ and $C_4 > 0$ are constants.

On the other side, from (16) we get the following:

$$\begin{aligned}
 BK(x, x) - K(x, x)B &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \\
 &\quad - \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} K_{12} + K_{21} & K_{22} - K_{11} \\ K_{22} - K_{11} & -K_{12} - K_{21} \end{pmatrix} \\
 &= \begin{pmatrix} K_{12} + K_{21} & K_{22} - K_{11} \\ K_{22} - K_{11} & -K_{12} - K_{21} \end{pmatrix} \\
 &= \begin{pmatrix} \tilde{p}(x) - p(x) & \tilde{q}(x) - q(x) \\ \tilde{q}(x) - q(x) & p(x) - \tilde{p}(x) \end{pmatrix}.
 \end{aligned} \tag{48}$$

From here, we have

$$\begin{aligned}
 \tilde{p}(x) - p(x) &= K_{12} + K_{21}, \\
 \tilde{q}(x) - q(x) &= K_{22} - K_{11}.
 \end{aligned} \tag{49}$$

Consequently from (38), (46), (47), and (49), we obtained the desired estimate

$$\begin{aligned}
 \max_{0 \leq x \leq \pi} |\tilde{p}(x) - p(x)| &\leq C' M, \\
 \max_{0 \leq x \leq \pi} |\tilde{q}(x) - q(x)| &\leq C'' M,
 \end{aligned} \tag{50}$$

where $C' > 0$ and $C'' > 0$ are constants. $\square$

3. Conclusion

The more norming constants and spectrums which are taken as spectral characteristics of Dirac operators are close to each other, the more difference of potential functions is sufficiently small.

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