Research Article

Some Spectral Aspects of the Operator Δ_ν^r over the Sequence Spaces ℓ_p and bv_p ($1 < p < \infty$)

S. Dutta¹ and P. Baliarsingh²

¹ Department of Mathematics, Utkal University, Vanivihar, Bhubaneswar 751004, India

² Department of Mathematics, School of Applied Sciences, KIIT University, Bhubaneswar 751024, India

Correspondence should be addressed to P. Baliarsingh; pb.math10@gmail.com

Received 22 July 2013; Accepted 28 August 2013

Academic Editors: Z. Huang, P. K. Sahoo, and W. Sun

Copyright © 2013 S. Dutta and P. Baliarsingh. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The main idea of the present paper is to compute the spectrum and the fine spectrum of the generalized difference operator Δ_ν^r over the sequence spaces bv_p ($1 < p < \infty$). The operator Δ_ν^r denotes a triangular sequential band matrix (a_{nk}) defined by $\Delta_\nu^r(x) = (\Delta_\nu^r x)_k = \sum_{i=0}^r (-1)^i \binom{r}{i} \nu_{k-i} x_{k-i}$, with $x_k = \nu_k = 0$ for $k < 0$, where $x \in \ell_p$ or bv_p , $r, k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$; the set nonnegative integers and $\nu = (\nu_k)$ is either a constant or strictly decreasing sequence of positive real numbers satisfying certain conditions. Finally, we obtain the spectrum, the point spectrum, the residual spectrum, and the continuous spectrum of the operator Δ_ν^r over the sequence spaces ℓ_p and bv_p . These results are more general and comprehensive than the spectrum of the difference operators Δ , Δ_ν , Δ^2 , Δ_{uv}^2 , and Δ^r and include some other special cases such as the spectrum of the operators $B(r, s)$, $B(r, s, t)$, and Δ_{ab} over the sequence spaces ℓ_p or bv_p ($1 < p < \infty$).

1. Introduction, Preliminaries, and Definitions

In analysis, operator theory is one of the important branch of mathematics which has vast applications in the field applied science and engineering. Operator theory deals with the study related to different properties of operators such as their inverse, spectrum, and fine spectrum. Since the spectrum of a bounded linear operator generalizes the notion of eigen values of the corresponding matrix, therefore, the study of spectrum of an operator takes a prominent position in solving many scientific and engineering problems. Hence, mathematicians and researchers have devoted their works in achieving new ideas and concepts in the concerned field. For instance, the fine spectrum of the Cesàro operator on the sequence space ℓ_p for $1 < p < \infty$ has been studied by Gonzalez [1]. Okutoyi [2] computed the spectrum of the Cesàro operator over the sequence space bv . The fine spectra of the Cesàro operator over the sequence space bv_p have been determined by Akhmedov and Başar [3]. Akhmedov and Başar [4, 5] have studied the fine spectrum of the difference

operator Δ over the sequence spaces ℓ_p and bv_p , where $1 < p < \infty$. Altay and Başar [6] have determined the fine spectrum of the difference operator Δ over the sequence spaces ℓ_p , for $0 < p < 1$. The fine spectrum of the difference operator Δ over the sequence spaces ℓ_1 and bv was investigated by Kayaduman and Furkan [7]. Srivastava and Kumar [8] have examined the fine spectrum of the generalized difference operator Δ_ν over the sequence space c_0 . Recently, the spectrum of the generalized difference operator Δ_ν^r over the sequence spaces c_0 and ℓ_1 has been studied by Dutta and Baliarsingh [9, 10], respectively. The main focus of this paper is to define the difference operator Δ_ν^r and establish its spectral characterization with respect to the Goldberg's classifications.

Let $\nu = (\nu_k)$ be either constant or strictly decreasing sequence of positive real numbers satisfying

$$\lim_{k \rightarrow \infty} \nu_k = L > 0, \quad (1)$$

$$\sup_k \nu_k \leq 2L, \quad \text{where } k \in \mathbb{N}_0.$$

By ω , we denote the space of all sequences of real or complex numbers. Any subspace of ω is called a sequence space, and we write $\ell_\infty, c, c_0, \ell_1, \ell_p$, and bv_p for the spaces of all bounded, convergent, null, absolutely summable, p -summable, and p -bounded variation sequences, respectively. The sequence spaces ℓ_p and bv_p are defined by

$$\begin{aligned}\ell_p &= \left\{ x \in \omega : \sum_{k=1}^{\infty} |x_k|^p < \infty \right\}, \\ bv_p &= \left\{ x \in \omega : \sum_{k=1}^{\infty} |x_k - x_{k-1}|^p < \infty \right\}.\end{aligned}\quad (2)$$

The main goal of this article is to define a generalized difference operator Δ_ν^r as follows.

For a positive integer r , we define the generalized difference operator $\Delta_\nu^r : \ell_p \rightarrow \ell_p$ by $\Delta_\nu^r(x) = (\Delta_\nu^r(x))_k$, where

$$\Delta_\nu^r(x)_k = \sum_{i=0}^r (-1)^i \binom{r}{i} x_{k-i} \nu_{k-i}, \quad (3)$$

with $x_k = \nu_k = 0$ for $k < 0$, $x \in \ell_p$, and $k \in \mathbb{N}_0$. It is clear that the operator Δ_ν^r can be represented by a lower triangular sequential band matrix $\Delta_\nu^r = (a_{nk})$ for $n, k \in \mathbb{N}_0$, and

$$a_{nk} = \begin{cases} \nu_k, & n = k, \\ (-1)^i \binom{r}{i} \nu_{k-i}, & n-1 \leq k \leq n-i, \quad 1 \leq i \leq r, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Equivalently,

$$(a_{nk}) = \begin{pmatrix} \nu_0 & 0 & 0 & 0 & \cdot & \cdot & \cdots \\ -r\nu_0 & \nu_1 & 0 & 0 & \cdot & \cdot & \cdots \\ \frac{r(r-1)}{2!}\nu_0 & -r\nu_1 & \nu_2 & 0 & \cdot & \cdot & \cdots \\ \frac{-r(r-1)(r-2)}{3!}\nu_0 & \frac{r(r-1)}{2!}\nu_1 & -r\nu_2 & \nu_3 & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ (-1)^r\nu_0 & (-1)^{r-1}r\nu_1 & \cdot & \cdot & \nu_r & 0 & \cdots \\ 0 & (-1)^r\nu_1 & \cdot & \cdot & -r\nu_r & \nu_{r+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (5)$$

In particular, the operator Δ_ν^r generalizes several operators considered by earlier authors concerning the same literature. For instance,

- (i) for $r = 1$, it generalizes the operator Δ_ν , considered by Srivastava and Kumar [8];

- (ii) for $r = 1$ and $\nu_k = r, \nu_{k-1} = -s$, it generalizes the difference operator $B(r, s)$, considered by Altay and Başar [11];
- (iii) for $r = 1$ and $\nu_k = 1, \nu_{k-1} = 1$, it generalizes the difference operator Δ , considered by Altay and Başar [12];
- (iv) for $r = 1$ and $\nu_k = a_k, \nu_{k-1} = -b_k$, it generalizes the difference operator $\Delta_{a,b}$, considered by Akhmedov and El-Shabrawy [13];
- (v) for $r = 2$ and $\nu_k = u_k, \nu_{k-1} = (1/2)\nu_{k-1}$, it generalizes the difference operator Δ_{uv}^2 , considered by Panigrahi and Srivastava [14];
- (vi) for $r = 2$ and $\nu_k = 1$, it generalizes the 2nd order difference operator Δ^2 , considered by Dutta and Baliarsingh [15];
- (vii) for $r = 2$ and $\nu_k = r, \nu_{k-1} = -(1/2)s, \nu_{k-2} = t$, it generalizes the difference operator $B(r, s, t)$, considered by Furkan et al. [16];

Let X and Y be Banach spaces and $T : X \rightarrow Y$ a bounded linear operator. By $\mathcal{R}(T)$, we denote the range of T ; that is,

$$\mathcal{R}(T) = \{y \in Y : y = Tx; x \in X\}. \quad (6)$$

By $B(X)$, we denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$, then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \{0\}$ be a normed linear space over the complex field and $T : D(T) \rightarrow X$ a linear operator, where $D(T)$ denotes the domain of T . With T , for a complex number λ , we associate an operator $T_\lambda = (T - \lambda I)$, where I is called identity operator on $D(T)$, and if T_λ has an inverse, we denote it by T_λ^{-1} ; that is,

$$T_\lambda^{-1} = (T - \lambda I)^{-1} \quad (7)$$

and is called the *resolvent* operator of T . Many properties of T_λ and T_λ^{-1} depend on λ , and the spectral theory is concerned with those properties. We are interested in the set of all λ 's in the complex plane such that T_λ^{-1} exists, T_λ^{-1} is bounded, and domain of T_λ^{-1} is dense in X . For our investigation, we need some basic concepts in spectral theory which are given as some definitions and lemmas.

Definition 1 (see [17, page 371]). Let X and T be defined as previous. A regular value of T is a complex number λ such that

- (R1) T_λ^{-1} exists;
- (R2) T_λ^{-1} is bounded;
- (R3) T_λ^{-1} is defined on a set which is dense in X .

The *resolvent* set $\rho(T, X)$ of T is the set of all *regular* values of T . Its complement $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T . Furthermore, the spectrum $\rho(T, X)$ is partitioned into three disjoint sets as follows.

- (I) *Point Spectrum* $\sigma_p(T, X)$. It is the set of all $\lambda \in \mathbb{C}$ such that (R1) does not hold. The elements of $\sigma_p(T, X)$ are called eigen values of T .
- (II) *Continuous Spectrum* $\sigma_c(T, X)$. It is the set of all $\lambda \in \mathbb{C}$ such that (R1) holds and satisfies (R3) but does not satisfy (R2).
- (III) *Residual Spectrum* $\sigma_r(T, X)$. It is the set of all $\lambda \in \mathbb{C}$ such that (R1) holds but does not satisfy (R3). The condition (R2) may or may not hold.

Lemma 2 (see [18, page 59]). *A linear operator T has a dense range if and only if the adjoint T^* is one to one.*

Lemma 3 (see [18, page 60]). *The adjoint operator T^* is onto if and only if T has a bounded inverse.*

Let P, Q be two nonempty subsets of the space w of all real or complex sequences and $A = (a_{nk})$ an infinite matrix of complex numbers a_{nk} , where $n, k \in \mathbb{N}_0$. For every $x = (x_k) \in P$ and every positive integer n , we write

$$A_n(x) = \sum_k a_{nk} x_k. \quad (8)$$

The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A . Infinite matrix $A \in (P, Q)$ if and only if $Ax \in Q$ whenever $x \in P$.

Lemma 4 (see [19, page 126]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Lemma 5 (see [20, page 254]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_\infty)$ from ℓ_∞ to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Lemma 6 (see [19, page 174]). *Let $1 < p < \infty$ and $A \in (\ell_\infty, \ell_\infty) \cap (\ell_1, \ell_1)$, then $A \in (\ell_p, \ell_p)$.*

Lemma 7 (see [5]). *Define the spaces d_1 and d_q consisting all sequences $x = (x_k)$ normed by*

$$\begin{aligned} \|x\|_{d_1} &= \sup_{n,k} \left| \sum_{j=k}^n x_j \right|, \\ \|x\|_{d_q} &= \left(\sum_{k=1}^{\infty} \left| \sum_{j=k}^{\infty} x_j \right|^q \right)^{1/q}, \quad (1 < q < \infty). \end{aligned} \quad (9)$$

Then, bv_1^* and bv_p^* are isometrically isomorphic to d_1 and d_q .

The basis of the space bv_p is also constructed and given by the following lemma.

Lemma 8 (see [5]). *Define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}}$ of the elements of the space bv_p , for every fixed $k \in \mathbb{N}$, the set of positive integers, by*

$$b_n^{(k)} = \begin{cases} 0, & (n < k), \\ 1, & (n \geq k). \end{cases} \quad (10)$$

Then, the sequence $b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}}$ is a basis for the space bv_p , and $x \in bv_p$ has a unique representation of the form

$$x = \sum_{k=1}^{\infty} \lambda_k b^{(k)}, \quad \text{where } \lambda_k = x_k - x_{k-1}, \quad \forall k \in \mathbb{N}. \quad (11)$$

2. The Spectrum and Fine Spectrum of Δ_ν^r over ℓ_p ($1 < p < \infty$)

In this section, we compute the point spectrum, the spectrum, the continuous spectrum, the residual spectrum, and the fine spectrum of the operator Δ_ν^r over the sequence space ℓ_p ($1 < p < \infty$).

Theorem 9. *The operator $\Delta_\nu^r : \ell_p \rightarrow \ell_p$ is a linear operator and satisfies the following inequalities:*

- (i) if (ν_k) is a constant sequence

$$\left(\sum_{i=0}^r \left| \binom{r}{i} \right|^p \right)^{1/p} \leq \frac{1}{L} \|\Delta_\nu^r\|_{(\ell_p, \ell_p)} \leq \binom{r}{r_{1/2}} (r+1), \quad (12)$$

- (ii) if (ν_k) is a strictly decreasing sequence

$$\left(\sum_{i=0}^r \left| \binom{r}{i} \right|^p \right)^{1/p} \leq \frac{1}{L} \|\Delta_\nu^r\|_{(\ell_p, \ell_p)} \leq 2 \binom{r}{r_{1/2}} (r+1), \quad (13)$$

where

$$r_{1/2} = \begin{cases} \frac{r}{2}, & \text{if } r \text{ is even,} \\ \frac{r-1}{2}, & \text{if } r \text{ is odd.} \end{cases} \quad (14)$$

Proof. (i) Suppose (ν_k) is a constant sequence and $\nu_k = L$ for all $k \in \mathbb{N}_0$. Linearity of the operator Δ_ν^r is trivial, hence

omitted. Let $x = (x_k) \in \ell_p$ for all $k \in \mathbb{N}_0$ and $1 < p < \infty$ such that $\|x\| = 1$. Now, by using Minkowski's inequality we get

$$\begin{aligned}
 \|\Delta_v^r(x)\|_{(\ell_p)} &= L \left(\sum_{k=0}^{\infty} \left| \sum_{i=0}^r (-1)^i \binom{r}{i} x_{k-i} \right|^p \right)^{1/p} \\
 &\leq L \left(\sum_{k=0}^{\infty} \left| \binom{r}{0} x_k \right|^p \right)^{1/p} \\
 &\quad + \left(\sum_{k=0}^{\infty} \left| \binom{r}{1} x_{k-1} \right|^p \right)^{1/p} \\
 &\quad + \cdots + \left(\sum_{k=0}^{\infty} \left| \binom{r}{r} x_{k-r} \right|^p \right)^{1/p} \\
 &\leq \sup_{i \leq r} \binom{r}{i} L \left[\left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} + \left(\sum_{k=0}^{\infty} |x_{k-1}|^p \right)^{1/p} \right. \\
 &\quad \left. + \cdots + \left(\sum_{k=0}^{\infty} |x_{k-r}|^p \right)^{1/p} \right] \\
 &= L \binom{r}{r_{1/2}} (r+1) \|x\|_{\ell_p}.
 \end{aligned} \tag{15}$$

Thus,

$$\frac{1}{L} \|\Delta_v^r\|_{(\ell_p)} \leq \binom{r}{r_{1/2}} (r+1). \tag{16}$$

Suppose we denote $e_1 = (1, 0, 0, \dots)$ as a sequence whose 1st entry is 1 and otherwise 0. Clearly, $e_1 \in \ell_p$ and

$$\|\Delta_v^r(e_1)\|_{(\ell_p)} = L \left(\sum_{i=0}^{\infty} \left| \binom{r}{i} \right|^p \right)^{1/p} \leq \|\Delta_v^r\|_{(\ell_p)} \|e_1\|_{\ell_p}. \tag{17}$$

Thus,

$$\left(\sum_{i=0}^{\infty} \left| \binom{r}{i} \right|^p \right)^{1/p} \leq \frac{1}{L} \|\Delta_v^r\|_{(\ell_p)}. \tag{18}$$

Combining inequations (16) and (18), we complete the proof.

(ii) Suppose (v_k) is a strictly decreasing sequence. Proof of this bit follows from (i) and with the fact that $\sup_k v_k \leq 2L$. $\square$

Theorem 10. The spectrum of Δ_v^r on the sequence space ℓ_p is given by

$$\sigma(\Delta_v^r, \ell_p) = \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 2^r - 1 \right\}. \tag{19}$$

Proof. The proof of this theorem consists of two parts.

Part 1. In the first part, we have to show that

$$\sigma(\Delta_v^r, \ell_p) \subseteq \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 2^r - 1 \right\}. \tag{20}$$

Equivalently, we need to show that if $\alpha \in \mathbb{C}$ with $|1 - (\alpha/L)| > 2^r - 1 \Rightarrow \alpha \notin \sigma(\Delta_v^r, \ell_p)$. Let $\alpha \in \mathbb{C}$ with $|1 - (\alpha/L)| > 2^r - 1$. Now $(\Delta_v^r - \alpha I) = (a_{nk})$ is a triangle and hence has an inverse $(\Delta_v^r - \alpha I)^{-1} = (b_{nk})$, where

$$(b_{nk}) = \begin{pmatrix} \frac{1}{(v_0 - \alpha)} & 0 & 0 & 0 & \cdots \\ \frac{v_0 r}{(v_0 - \alpha)(v_1 - \alpha)} & \frac{1}{(v_1 - \alpha)} & 0 & 0 & \cdots \\ b_{20} & \frac{v_1 r}{(v_1 - \alpha)(v_2 - \alpha)} & \frac{1}{(v_2 - \alpha)} & 0 & \cdots \\ b_{30} & b_{31} & \frac{v_2 r}{(v_2 - \alpha)(v_3 - \alpha)} & \frac{1}{(v_3 - \alpha)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{21}$$

and b_{20} , b_{30} , and b_{31} are as follows:

$$b_{20} = \frac{r^2 v_0 v_1}{(v_0 - \alpha)(v_1 - \alpha)(v_2 - \alpha)} - \frac{r(r-1)v_0}{2!(v_0 - \alpha)(v_2 - \alpha)}. \tag{22}$$

Similarly,

$$b_{31} = \frac{r^2 v_1 v_2}{(v_1 - \alpha)(v_2 - \alpha)(v_3 - \alpha)} - \frac{r(r-1)v_1}{2!(v_1 - \alpha)(v_3 - \alpha)},$$

$$\begin{aligned}
b_{30} = & \frac{r^3 \nu_0 \nu_1 \nu_2}{(\nu_0 - \alpha)(\nu_1 - \alpha)(\nu_2 - \alpha)(\nu_3 - \alpha)} \\
& - \frac{r^2 (r-1) \nu_0 \nu_1}{2! (\nu_0 - \alpha)(\nu_1 - \alpha)(\nu_3 - \alpha)} \\
& - \frac{r^2 (r-1) \nu_0 \nu_2}{2! (\nu_0 - \alpha)(\nu_2 - \alpha)(\nu_3 - \alpha)} \\
& + \frac{r(r-1)(r-2) \nu_0}{3! (\nu_0 - \alpha)(\nu_3 - \alpha)}.
\end{aligned} \tag{23}$$

In fact, for $k \in \mathbb{N}_0$, one can calculate

$$\begin{aligned}
b_{kk} &= \frac{1}{\nu_k - \alpha}, \\
b_{k+1,k} &= \frac{r \nu_k}{(\nu_k - \alpha)(\nu_{k+1} - \alpha)}, \\
b_{k+2,k} &= \frac{r^2 \nu_k \nu_{k+1}}{(\nu_k - \alpha)(\nu_{k+1} - \alpha)(\nu_{k+2} - \alpha)} \\
& - \frac{r(r-1) \nu_k}{2! (\nu_k - \alpha)(\nu_{k+2} - \alpha)}, \\
b_{k+3,k} &= \frac{r^3 \nu_k \nu_{k+1} \nu_{k+2}}{(\nu_k - \alpha)(\nu_{k+1} - \alpha)(\nu_{k+2} - \alpha)(\nu_{k+3} - \alpha)} \\
& - \frac{r^2 (r-1) \nu_k \nu_{k+1}}{2! (\nu_k - \alpha)(\nu_{k+1} - \alpha)(\nu_{k+3} - \alpha)} \\
& - \frac{r^2 (r-1) \nu_k \nu_{k+1}}{2! (\nu_k - \alpha)(\nu_{k+2} - \alpha)(\nu_{k+3} - \alpha)} \\
& + \frac{r(r-1)(r-2) \nu_k}{3! (\nu_k - \alpha)(\nu_{k+3} - \alpha)}, \\
& \vdots
\end{aligned} \tag{24}$$

and so on.

Now, let

$$\begin{aligned}
S_k &= \sum_{n=0}^{\infty} |b_{nk}| = |b_{kk}| + |b_{k+1,k}| + |b_{k+2,k}| + \cdots \\
&= \left| \frac{1}{\nu_k - \alpha} \right| + \left| \frac{r \nu_k}{(\nu_k - \alpha)(\nu_{k+1} - \alpha)} \right| \\
&+ \left| \frac{r^2 \nu_k \nu_{k+1}}{(\nu_k - \alpha)(\nu_{k+1} - \alpha)(\nu_{k+2} - \alpha)} \right| \\
&- \frac{r(r-1) \nu_k}{2! (\nu_k - \alpha)(\nu_{k+2} - \alpha)} \Big|
\end{aligned}$$

$$\begin{aligned}
&+ \left| \frac{r^3 \nu_k \nu_{k+1} \nu_{k+2}}{(\nu_k - \alpha)(\nu_{k+1} - \alpha)(\nu_{k+2} - \alpha)(\nu_{k+3} - \alpha)} \right| \\
&- \frac{r^2 (r-1) \nu_k \nu_{k+1}}{2! (\nu_k - \alpha)(\nu_{k+1} - \alpha)(\nu_{k+3} - \alpha)} \\
&- \frac{r^2 (r-1) \nu_k \nu_{k+1}}{2! (\nu_k - \alpha)(\nu_{k+2} - \alpha)(\nu_{k+3} - \alpha)} \\
&+ \frac{r(r-1)(r-2) \nu_k}{3! (\nu_k - \alpha)(\nu_{k+3} - \alpha)} \Big| + \cdots.
\end{aligned} \tag{25}$$

Now,

$$\begin{aligned}
&\lim_k S_k \\
&= \lim_k \left| \frac{1}{(\nu_k - \alpha)} \right| + \lim_k \left| \frac{r \nu_k}{(\nu_k - \alpha)(\nu_{k+1} - \alpha)} \right| + \cdots \\
&\leq \frac{1}{|Lr|} \left\{ \left| \frac{Lr}{L - \alpha} \right| + \left| \frac{Lr}{L - \alpha} \right|^2 + \left| \frac{Lr}{L - \alpha} \right|^3 \right. \\
&\quad + \frac{(r-1)}{2!} \left| \frac{Lr}{L - \alpha} \right|^2 + \frac{|Lr|^4}{|L - \alpha|^4} + 2 \cdot \frac{(r-1)}{2!} \left| \frac{Lr}{L - \alpha} \right|^3 \\
&\quad \left. + \frac{(r-1)(r-2)}{3!} \left| \frac{Lr}{L - \alpha} \right|^2 + \cdots \right\} \\
&= \frac{1}{|Lr|} \left\{ |\beta_r| + |\beta_r|^2 + |\beta_r|^3 + \frac{(r-1)}{2!} |\beta_r|^2 \right. \\
&\quad + |\beta_r|^4 + 2 \cdot \frac{(r-1)}{2!} |\beta_r|^3 \\
&\quad \left. + \frac{(r-1)(r-2)}{3!} |\beta_r|^2 + \cdots \right\} \\
&= \frac{1}{|Lr|} \left\{ |\beta_r| + |\beta_r|^2 \left(\sum_{i \geq 1} n_{2(i)} \right) \right. \\
&\quad \left. + |\beta_r|^3 \left(\sum_{i \geq 1} n_{3(i)} \right) + |\beta_r|^4 \left(\sum_{i \geq 1} n_{4(i)} \right) + \cdots \right\} \\
&= \frac{1}{|Lr|} \left\{ |\beta_r| + \left(\frac{2^r - 1}{r} \right) |\beta_r|^2 \right. \\
&\quad \left. + \left(\frac{2^r - 1}{r} \right)^2 |\beta_r|^3 + \left(\frac{2^r - 1}{r} \right)^3 |\beta_r|^4 + \cdots \right\} \\
&\leq \frac{1}{|L - \alpha|} \left\{ 1 + \left| \frac{L(2^r - 1)}{L - \alpha} \right| + \left| \frac{L(2^r - 1)}{L - \alpha} \right|^2 \right. \\
&\quad \left. + \left| \frac{L(2^r - 1)}{L - \alpha} \right|^3 + \left| \frac{L(2^r - 1)}{L - \alpha} \right|^4 + \cdots \right\} \\
&= \frac{1}{|L - \alpha| - |L(2^r - 1)|} < \infty,
\end{aligned} \tag{26}$$

where $\beta_r = Lr/(L - \alpha)$ and $n_{k(i)}$ denote the coefficients of $|\beta_r|^k$ for $k \geq 2$. As per the assumption, $|L(2^r - 1)/(L - \alpha)| < 1$;

hence, $\lim_k S_k < \infty$. Now, (S_k) is a sequence of positive real numbers and is convergent; this implies the boundedness of (S_k) and hence $(\Delta_v^r - \alpha I)^{-1} \in B(\ell_1)$. In a similar way it can be shown that $(\Delta_v^r - \alpha I)^{-1} \in B(\ell_\infty)$. By using Lemma 6, we obtain that $(\Delta_v^r - \alpha I)^{-1} \in B(\ell_p)$. Hence, we have

$$\sigma(\Delta_v^r, \ell_p) \subseteq \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 2^r - 1 \right\}. \quad (27)$$

Part 2. In this part, we need to show that $\{\alpha \in \mathbb{C} : |1 - (\alpha/L)| \leq 2^r - 1\} \subseteq \sigma(\Delta_v^r, \ell_p)$. Consider $\alpha \neq \nu_k, L$ for all $k \in \mathbb{N}_0$ and $|1 - (\alpha/L)| < 2^r - 1$. Clearly $(\Delta_v^r - \alpha I)$ is a triangle and hence $(\Delta_v^r - \alpha I)^{-1}$ exists, but (S_k) is unbounded:

$$\Rightarrow (\Delta_v^r - \alpha I)^{-1} \notin B(\ell_p) \quad \text{with} \quad \left| 1 - \frac{\alpha}{L} \right| < 2^r - 1. \quad (28)$$

Again $\alpha \in \mathbb{C}$ and $\alpha \neq \nu_k, L$ for all $k \in \mathbb{N}_0$ with $|1 - (\alpha/L)| = 2^r - 1$ which implies $\lim_k S_k = \infty$. Thus, (S_k) is unbounded:

$$\Rightarrow (\Delta_v^r - \alpha I)^{-1} \notin B(\ell_p) \quad \text{with} \quad \left| 1 - \frac{\alpha}{L} \right| = 2^r - 1. \quad (29)$$

Finally, we prove the result under the assumptions $\alpha = L$ and $\alpha = \nu_k$. Then, we have the following cases.

Case 1. If (ν_k) is a constant sequence and $\nu_k = L$, then we have

$$(\Delta_v^r - \alpha I) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ -r\nu_0 & 0 & 0 & 0 & \cdots \\ \frac{r(r-1)}{2!}\nu_0 & -r\nu_1 & 0 & 0 & \cdots \\ \frac{-r(r-1)(r-2)}{3!}\nu_0 & \frac{r(r-1)}{2!}\nu_1 & -r\nu_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (30)$$

which is not invertible. Thus, the condition (R1) fails.

Case 2. If (ν_k) is a strictly decreasing sequence and $\alpha = \nu_k$, then

$$(\Delta_v^r - \alpha I)x = \begin{pmatrix} (\nu_0 - \nu_k)x_0 \\ -r\nu_0 x_0 + (\nu_1 - \nu_k)x_1 \\ \frac{r(r-1)}{2!}\nu_0 x_0 - r\nu_1 x_1 + (\nu_2 - \nu_k)x_2 \\ \vdots \end{pmatrix}. \quad (31)$$

As (ν_k) is strictly decreasing sequence, then for fixed k , $(\Delta_v^r - \alpha I)x = \mathbf{0} \Rightarrow x_0 = 0, x_1 = 0, x_2 = 0 \cdots x_{k-1} = 0, x_{n+1} = (r\nu_k/(\nu_{n+1} - \nu_k))x_n$ for all $n \geq k$. This shows that $(\Delta_v^r - \alpha I)$ is not one to one for $\alpha = \nu_k$, thus, we have $\nu_k \in \sigma(\Delta_v^r, \ell_p)$ for $k \in \mathbb{N}_0$. Again, for $\alpha = L$, then (S_k) is unbounded. So, the condition (R2) fails. Thus,

$$\left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 2^r - 1 \right\} \subseteq \sigma(\Delta_v^r, \ell_p). \quad (32)$$

Combining (27) and (32), we conclude the proof. $\square$

Theorem 11. The point spectrum of the operator Δ_v^r over ℓ_p is given by

$$\sigma_p(\Delta_v^r, \ell_p) = \emptyset. \quad (33)$$

Proof Consider $\Delta_v^r x = \alpha x$ for $x \neq \theta$ in ℓ_p , which gives the system of linear equations

$$\begin{aligned} \nu_0 x_0 &= \alpha x_0, \\ -r\nu_0 x_0 + \nu_1 x_1 &= \alpha x_1, \\ \frac{r(r-1)}{2!}\nu_0 x_0 - r\nu_1 x_1 + \nu_2 x_2 &= \alpha x_2, \\ \frac{-r(r-1)(r-2)}{3!}\nu_0 x_0 + \frac{r(r-1)}{2!}\nu_1 x_1 - r\nu_2 x_2 + \nu_3 x_3 &= \alpha x_3, \\ &\vdots \end{aligned} \quad (34)$$

As per the definition of (ν_k) , we have the following cases.

Case 1. Suppose (ν_k) is a constant sequence. That is, $\nu_k = L$ for all $k \in \mathbb{N}_0$. On solving system of (34), we obtain that $x = \theta$, which contradicts to our assumption. Thus, $\sigma_p(\Delta_v^r, \ell_p) = \emptyset$.

Case 2. Suppose (ν_k) is a strictly decreasing sequence and consider $\Delta_v^r x = \alpha x$ for $x \neq \mathbf{0} = (0, 0, 0, \dots)$ in ℓ_p . On solving system of (34), we obtain that $\alpha = \nu_0$ for $x_0 \neq 0$, $x_1 = (r\nu_0/(\nu_1 - \nu_0))x_0$, $x_2 = [r^2\nu_1\nu_0/(\nu_1 - \nu_0)(\nu_2 - \nu_0) - r(r-1)\nu_0/2!(\nu_2 - \nu_0)]x_0$ and so on. Proceeding this way, it can be proved that for $x_0 \neq 0$, $(\Delta_v^r - \alpha I)x = \mathbf{0}$ has a nonzero solution $x = (x_k)$, but $x \notin \ell_p$, and if $x_0 = 0$, then $x_k = 0$ for each $k > 0$, which is a contradiction. Therefore, $\sigma_p(\Delta_v^r, \ell_p) = \emptyset$. However, for a better explanation, we take a counter example.

Example 12. Suppose $\nu_k = (k+2) - [(k+2)(k+1)]^{1/2}$ for $k \in \mathbb{N}_0$. Clearly, (ν_k) is a strictly decreasing sequence with

$$\begin{aligned} \nu_0 &= 2 - \sqrt{2} \approx 0.585786437, \\ \nu_1 &= 3 - \sqrt{6} \approx 0.550510257, \\ \nu_2 &= 4 - \sqrt{12} \approx 0.535898384 \text{ and so on.} \end{aligned} \quad (35)$$

Also,

$$\lim_{k \rightarrow \infty} \nu_k = L = \lim_{k \rightarrow \infty} \frac{k+2}{k(k+2) + \sqrt{(k+1)(k+2)}} = \frac{1}{2}. \quad (36)$$

In fact, $\sup_k \nu_k = \nu_0 = 2 - \sqrt{2} \leq 2L = 2 \cdot (1/2) = 1$; hence, (ν_k) satisfies the conditions (1). Our claim is to show that $\sigma_p(\Delta_v^r, \ell_p) = \emptyset$. Supposing for the contrary that we take $\sigma_p(\Delta_v^r, \ell_p) \neq \emptyset$, equivalently, let $\nu_0 \in \sigma_p(\Delta_v^r, \ell_p)$. From the system of (34), we obtain that for $r = 1$

$$\begin{aligned} -\nu_k x_k + \nu_{k+1} x_{k+1} &= \nu_0 x_{k+1}, \\ \Rightarrow \left| \frac{x_{k+1}}{x_k} \right| &= \left| \frac{\nu_k}{\nu_{k+1} - \nu_0} \right| \\ \Rightarrow \lim_{k \rightarrow \infty} \left| \frac{x_{k+1}}{x_k} \right| &= \left| \frac{1/2}{\frac{1}{2} - (2 + \sqrt{2})} \right| = \left| \frac{1}{2\sqrt{2} - 3} \right| \approx 5.8284 > 1. \end{aligned} \quad (37)$$

Therefore, $x \notin \ell_p$, and hence $\nu_0 \notin \sigma_p(\Delta_\nu^r, \ell_p)$ for $r = 1$. Again for $r = 2$, the system of (34) reduces to

$$\begin{aligned} \nu_0 x_0 &= \nu_0 x_0, \\ -2\nu_0 x_0 + \nu_1 x_1 &= \nu_0 x_1, \\ \nu_0 x_0 - 2\nu_1 x_1 + \nu_2 x_2 &= \nu_0 x_2, \\ &\vdots \\ \nu_{k-2} x_{k-2} - 2\nu_{k-1} x_{k-1} + \nu_k x_k &= \nu_0 x_k, \\ &\vdots \end{aligned} \quad (38)$$

From the system of (38), it is clear that

$$\begin{aligned} \left| \frac{x_1}{x_0} \right| &= \left| \frac{2\nu_0}{\nu_1 - \nu_0} \right| \approx 33.21144358, \\ \left| \frac{x_2}{x_1} \right| &= \left| \frac{3\nu_1 + \nu_0}{2(\nu_2 - \nu_0)} \right| \approx 22.42337679, \\ \left| \frac{x_3}{x_2} \right| &= \left| \frac{2\nu_2 - \nu_1 x_1/x_2}{\nu_3 - \nu_0} \right| \approx 18.92786968 \text{ and so on.} \end{aligned} \quad (39)$$

Indeed, for each $0 < j < k$ the ratio $x_j/x_{j-1} < 0$ and $|x_j/x_{j-1}| > 1$. Supposing that $|x_k/x_{k-1}| > 1$ as $k \rightarrow \infty$, now by the inductive hypothesis we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{x_{k+1}}{x_k} \right| &= \lim_{k \rightarrow \infty} \left| \frac{2\nu_k - \nu_{k-1} x_{k-1}/x_k}{\nu_{k+1} - \nu_0} \right| \\ &= \left| \frac{1 - 0.5x_{k-1}/x_k}{1/2 - 2 + \sqrt{2}} \right| \geq \left| \frac{1 + 0.5}{1/2 - 2 + \sqrt{2}} \right| \\ &\approx 17.48528137 > 1. \end{aligned} \quad (40)$$

By using inductive principle, it is trivial to prove that $|x_{k+1}/x_k| > 1$ for all $k \in \mathbb{N}_0$, and therefore, $x \notin \ell_p$. Proceeding this way, it can be shown that $\nu_1 \notin \sigma_p(\Delta_\nu^r, \ell_p)$ and similarly for all $k \in \mathbb{N}_0$, $\nu_k \notin \sigma_p(\Delta_\nu^r, \ell_p)$. Hence, $\sigma_p(\Delta_\nu^r, \ell_p) = \emptyset$. $\square$

Theorem 13. The point spectrum of the dual operator $(\Delta_\nu^r)^*$ of Δ_ν^r over $\ell_p^* \cong \ell_q$ is given by

$$\sigma_p((\Delta_\nu^r)^*, \ell_q) = \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq r \right\}. \quad (41)$$

Proof. Suppose $(\Delta_\nu^r)^* f = \alpha f$ for $\mathbf{0} \neq f \in \ell_q$, where $1 < p < \infty$, $1/p + 1/q = 1$ and

$$\begin{aligned} (\Delta_\nu^r)^* &= \begin{pmatrix} \nu_0 & -r\nu_0 & \frac{r(r-1)}{2!}\nu_0 & \frac{-r(r-1)(r-2)}{3!}\nu_0 & \cdots \\ 0 & \nu_1 & -r\nu_1 & \frac{r(r-1)}{2!}\nu_1 & \cdots \\ 0 & 0 & \nu_2 & -r\nu_2 & \cdots \\ 0 & 0 & 0 & \nu_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \\ f &= \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix}. \end{aligned} \quad (42)$$

Now consider the system of linear equations $(\Delta_\nu^r)^* f = \alpha f$; that is,

$$\begin{aligned} \nu_0 f_0 - r\nu_0 f_1 + \frac{r(r-1)}{2!}\nu_0 f_2 + \cdots &= \alpha f_0, \\ \nu_1 f_1 - r\nu_1 f_2 + \frac{r(r-1)}{2!}\nu_1 f_3 + \cdots &= \alpha f_1, \\ \nu_2 f_2 - r\nu_2 f_3 + \frac{r(r-1)}{2!}\nu_2 f_4 + \cdots &= \alpha f_2, \\ &\vdots \end{aligned} \quad (43)$$

On solving the previous system of equations, for each $k \in \mathbb{N}_0$, we observe that

$$\begin{aligned} |f_k| &= \left| \frac{r\nu_k}{\nu_k - \alpha} \left(f_{k+1} - \frac{r-1}{2!}f_{k+2} + \frac{(r-1)(r-2)}{3!}f_{k+3} \right. \right. \\ &\quad \left. \left. + \cdots + (-1)^r f_{k+r} \right) \right|, \end{aligned} \quad (44)$$

since $|r\nu_k/(\nu_k - \alpha)| \geq |rL/(L - \alpha)|$ for both the constant and strictly decreasing sequences (ν_k) satisfying (1). From (44), it is clear that for each $k \in \mathbb{N}_0$, $f = (0, 0, \dots, f_k, f_{k+1}, 0, 0, \dots)$ is an eigen vector that corresponds to the eigen value α satisfying $|rL/(L - \alpha)| \geq 1$. This follows from the fact that

$$|f_k| \geq \left| \frac{rL}{1 - \alpha} \right| |f_{k+1}| \geq |f_{k+1}|. \quad (45)$$

Therefore,

$$\lim_{k \rightarrow \infty} \left| \frac{f_{k+1}}{f_k} \right| < 1 \quad (46)$$

which implies that

$$\sum_k \left| \frac{f_{k+1}}{f_k} \right|^q < \infty. \quad (47)$$

As a consequence $x \in \ell_q$.

Conversely, it is trivial to show that if $x \in \ell_q$, then $|1 - (\alpha/L)| \leq r$. Suppose for the contrary, we take $x \in \ell_q$ and $|1 - (\alpha/L)| > r$. From (44), it can be shown that $\lim_{k \rightarrow \infty} |f_{k+1}/f_k| > 1$ for $|1 - (\alpha/L)| > r$. This implies that $x \notin \ell_q$, which is a contradiction, and this step completes the proof. $\square$

Theorem 14. The residual spectrum of the operator Δ_ν^r over ℓ_p is given by

$$\sigma_r(\Delta_\nu^r, \ell_p) = \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq r \right\}. \quad (48)$$

Proof. For $|1 - (\alpha/L)| < r$, the operator $\Delta_\nu^r - \alpha I$ has an inverse. By Theorem 13, the operator $(\Delta_\nu^r)^* - \alpha I$ is not one to one for $\alpha \in \mathbb{C}$ with $|1 - (\alpha/L)| \leq r$. By using Lemma 2, we have $R(\Delta_\nu^r - \alpha I) \neq \ell_p$. Hence,

$$\sigma_r(\Delta_\nu^r, \ell_p) = \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq r \right\}. \quad (49)$$

$\square$

Theorem 15. The continuous spectrum of the operator Δ_ν^r over ℓ_p is given by

$$\sigma_c(\Delta_\nu^r, \ell_p) = \left\{ \alpha \in \mathbb{C} : r < \left| 1 - \frac{\alpha}{L} \right| \leq 2^r - 1 \right\}. \quad (50)$$

Proof. Since $\sigma_p(\Delta_\nu^r, \ell_p)$, $\sigma_r(\Delta_\nu^r, \ell_p)$, and $\sigma_c(\Delta_\nu^r, \ell_p)$ are disjoint subsets of $\sigma(\Delta_\nu^r, \ell_p)$, and their union contributes to $\sigma(\Delta_\nu^r, \ell_p)$. Therefore, from Theorems 9, 10, 11, and 14, we obtain that

$$\sigma_c(\Delta_\nu^r, \ell_p) = \left\{ \alpha \in \mathbb{C} : r < \left| 1 - \frac{\alpha}{L} \right| \leq 2^r - 1 \right\}. \quad (51)$$

$\square$

3. The Spectrum of Δ_ν^r over bv_p ($1 < p < \infty$)

In this section, we determine the spectrum of the operator Δ_ν^r over the sequence space bv_p ($1 < p < \infty$). Now, we give some theorems starting with the result concerning the boundedness and linearity of the operator Δ_ν^r .

Theorem 16. Consider the following:

$$\Delta_\nu^r \in B(bv_p). \quad (52)$$

Proof. Supposing $x = (x_k) \in bv_p$ and by using Minkowski's inequality we have

$$\begin{aligned} \|\Delta_\nu^r(x)\|_{(bv_p)} &= \left(\sum_{k=0}^{\infty} \left| \sum_{i=0}^r (-1)^i \binom{r}{i} (\nu_{k-i} x_{k-i} - \nu_{k-i-1} x_{k-i-1}) \right|^p \right)^{1/p} \\ &\leq \left(\sum_{k=0}^{\infty} \left| \binom{r}{0} (\nu_k x_k - \nu_{k-1} x_{k-1}) \right|^p \right)^{1/p} \\ &\quad + \left(\sum_{k=0}^{\infty} \left| \binom{r}{1} (\nu_{k-1} x_{k-1} - \nu_{k-2} x_{k-2}) \right|^p \right)^{1/p} + \cdots \\ &\quad + \left(\sum_{k=0}^{\infty} \left| \binom{r}{r} (\nu_{k-r} x_{k-r} - \nu_{k-r-1} x_{k-r-1}) \right|^p \right)^{1/p} \\ &\leq \binom{r}{r_{1/2}} (r+1) \nu_0 \|x\|_{(bv_p)}. \end{aligned} \quad (53)$$

Thus,

$$\|\Delta_\nu^r\|_{(bv_p)} \leq \binom{r}{r_{1/2}} (r+1) \nu_0. \quad (54)$$

This step concludes the proof. $\square$

Theorem 17. The spectrum of the operator Δ_ν^r over bv_p is given by

$$\sigma(\Delta_\nu^r, bv_p) = \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 2^r - 1 \right\}. \quad (55)$$

Proof. Firstly, we need to prove that $(\Delta_\nu^r - \alpha I)^{-1}$ exists and is in $B(bv_p)$ for $|1 - (\alpha/L)| > 2^r - 1$, and after that the operator $\Delta_\nu^r - \alpha I$ is not invertible for $|1 - (\alpha/L)| \leq 2^r - 1$.

Let $|1 - (\alpha/L)| > 2^r - 1$. Since $\Delta_\nu^r - \alpha I$ is a triangle, hence $(\Delta_\nu^r - \alpha I)^{-1}$ exists. Let $y = (y_k) \in bv_p$; this implies that $y_k - y_{k-1} \in \ell_p$. Solving the system equation $\Delta_\nu^r - \alpha I = y$, we obtain that

$$\begin{aligned} x_0 &= \frac{y_0}{\nu_0 - \alpha}, \\ x_1 &= \frac{y_1}{\nu_1 - \alpha} + \frac{r\nu_0 y_0}{(\nu_1 - \alpha)(\nu_0 - \alpha)}, \\ x_2 &= \frac{y_2}{\nu_2 - \alpha} + \frac{r\nu_1 y_1}{(\nu_2 - \alpha)(\nu_1 - \alpha)} \\ &\quad + \left\{ \frac{r^2 \nu_1 \nu_0}{(\nu_2 - \alpha)(\nu_1 - \alpha)(\nu_0 - \alpha)} \right. \\ &\quad \left. - \binom{r}{2} \frac{\nu_0}{(\nu_2 - \alpha)(\nu_0 - \alpha)} \right\} y_0, \\ &\quad \vdots \end{aligned} \quad (56)$$

By using notation b_{nk} , we can write for each $k \in \mathbb{N}_0$

$$x_k = \sum_{j=0}^k b_{k-j,k} y_{k-j}. \quad (57)$$

Now,

$$\begin{aligned} x_k - x_{k-1} &= b_{kk} y_k + (b_{k,k-1} - b_{k-1,k-1}) y_{k-1} \\ &\quad + (b_{k,k-2} - b_{k-1,k-2}) y_{k-2} + \cdots \\ &\quad + (b_{k,0} - b_{k-1,0}) y_0 \\ &= (B - B^*) y, \end{aligned} \quad (58)$$

where $B = (b_{nk})$ and

$$B^* = \begin{pmatrix} 0 & 0 \\ (b_{nk}) & 0 \end{pmatrix}. \quad (59)$$

By Theorem 10, it is clear that $B, B^* \in (\ell_p, \ell_p)$ and $y \in \ell_p$. Therefore, $(x_k - x_{k-1}) \in \ell_p$ equivalently, $x = (x_k) \in bv_p$. Thus,

$$\sigma(\Delta_v^r, bv_p) \subset \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 2^r - 1 \right\}. \quad (60)$$

Again for $\alpha \in \{\alpha \in \mathbb{C} : |1 - (\alpha/L)| \leq 2^r - 1\}$ and $\alpha \neq L$ or v_k , $\Delta_v^r - \alpha I$ is a triangle, and hence $(\Delta_v^r - \alpha I)^{-1}$ exists. Suppose $y = (1, 1, 1, \dots) \in bv_p$; then by Theorem 10, the series $\sum_k |x_k - x_{k-1}|$ is divergent. Finally, for $\alpha = L$ or v_k , $(\Delta_v^r - \alpha I)^{-1} \notin B(\ell_p)$, this shows that

$$\left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 2^r - 1 \right\} \subset \sigma(\Delta_v^r, bv_p). \quad (61)$$

Combining (60) and (61), we complete the proof. $\square$

Since the spectrum and fine spectrum of the operator Δ_v^r over the sequence space bv_p are similar to that over the space ℓ_p , we avoid the repetition of similar statements as discussed in the last section and give the results via following theorems without proof.

Theorem 18. Consider the following:

- (i) $\sigma(\Delta_v^r, bv_p) = \{\alpha \in \mathbb{C} : |1 - (\alpha/L)| \leq 2^r - 1\}$.
- (ii) $\sigma_p(\Delta_v^r, bv_p) = \emptyset$.
- (iii) $\sigma_r(\Delta_v^r, bv_p) = \{\alpha \in \mathbb{C} : |1 - (\alpha/L)| \leq r\}$.
- (iv) $\sigma_c(\Delta_v^r, bv_p) = \{\alpha \in \mathbb{C} : r < |1 - (\alpha/L)| \leq 2^r - 1\}$.

4. Conclusion

In the present article, we give a unifying approach to most of the difference operators defined earlier in the literature and determine the spectrum and fine spectrum of the generalized r th difference operators Δ_v^r over the sequence spaces ℓ_p and bv_p ($1 < p < \infty$). Therefore, the results and theorems presented in this article are more general and comprehensive than the works done by earlier authors. Now, by choosing r and the sequence (v_k) suitably, our work generalizes various other known results studied by several authors (see [4–6, 16]).

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this article.

References

- [1] M. Gonzalez, “The fine spectrum of the Cesàro operator in ℓ_p ($1 < p < \infty$),” *Archiv der Mathematik*, vol. 44, no. 4, pp. 355–358, 1985.
- [2] J. T. Okutoyi, “On the spectrum of C_1 as an operator on bv ,” *Communications de la Faculté des Sciences de l’Université d’Ankara A*, vol. 141, pp. 197–207, 1992.
- [3] A. M. Akhmedov and F. Başar, “The fine spectra of Cesàro operator C_1 over the sequence space bv_p ,” *Mathematical Journal of Okayama University*, vol. 50, pp. 135–147, 2008.
- [4] A. M. Akhmedov and F. Başar, “The fine spectra of the difference operator Δ over the sequence space ℓ_p , ($1 \leq p < \infty$),” *Demonstratio Mathematica*, vol. 39, no. 3, pp. 586–595, 2006.
- [5] A. M. Akhmedov and F. Başar, “The fine spectra of the difference operator Δ over the sequence space bv_p , ($1 \leq p < \infty$),” *Acta Mathematica Sinica (English Series)*, vol. 23, no. 10, pp. 1757–1768, 2007.
- [6] B. Altay and F. Başar, “The fine spectrum and the matrix domain of the difference operator Δ on the sequence space ℓ_p , ($0 < p < 1$),” *Communications in Mathematical Analysis*, vol. 2, pp. 1–11, 2007.
- [7] K. Kayaduman and H. Furkan, “The fine spectra of the difference operator Δ over the sequence spaces ℓ_1 and bv ,” *International Mathematical Forum*, vol. 1, no. 24, pp. 1153–1160, 2006.
- [8] P. D. Srivastava and S. Kumar, “On the fine spectrum of the generalized difference operator Δ_v over the sequence space c_0 ,” *Communications in Mathematical Analysis*, vol. 6, no. 1, pp. 8–21, 2009.
- [9] S. Dutta and P. Baliarsingh, “On a spectral classification of the operator Δ_v^r over the sequence space c_0 ,” *Proceedings of the National Academy of Sciences, India A*. In press.
- [10] S. Dutta and P. Baliarsingh, “On the fine spectra of the generalized r th difference operator Δ_v^r on the sequence space ℓ_1 ,” *Applied Mathematics and Computation*, vol. 219, pp. 1776–1784, 2012.
- [11] B. Altay and F. Başar, “On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces c_0 and c ,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 18, pp. 3005–3013, 2005.
- [12] B. Altay and F. Başar, “On the fine spectrum of the difference operator Δ on c_0 and c ,” *Information Sciences*, vol. 168, no. 1–4, pp. 217–224, 2004.
- [13] A. M. Akhmedov and S. R. El-Shabrawy, “On the fine spectrum of the operator $\Delta_{a,b}$ over the sequence space c ,” *Computers and Mathematics with Applications*, vol. 61, no. 10, pp. 2994–3002, 2011.
- [14] B. L. Panigrahi and P. D. Srivastava, “Spectrum and fine spectrum of generalized second order difference operator Δ_{uv}^2 on sequence space c_0 ,” *Thai Journal of Mathematics*, vol. 9, no. 1, pp. 57–74, 2011.
- [15] S. Dutta and P. Baliarsingh, “On the spectrum of 2-nd order generalized difference operator Δ^2 over the sequence space c_0 ,” *Boletim da Sociedade Paranaense de Matemática*, vol. 31, no. 2, pp. 235–244, 2013.

- [16] H. Furkan, H. Bilgiç, and F. Başar, “On the fine spectrum of the operator $B(r, s, t)$ over the sequence spaces ℓ_p and bv_p , ($1 \leq p < \infty$),” *Computers and Mathematics with Applications*, vol. 60, no. 7, pp. 2141–2152, 2010.
- [17] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New York, NY, USA, 1978.
- [18] S. Goldberg, *Unbounded Linear Operators*, Dover Publications, New York, NY, USA, 1985.
- [19] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, London, UK, 1970.
- [20] B. Choudhary and S. Nanda, *Functional Analysis with Applications*, John Wiley & Sons, New York, NY, USA, 1989.