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*Research Article*

# Remarks on Some Recent Coupled Coincidence Point Results in Symmetric G-Metric Spaces

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We use a method of reducing coupled coincidence point results in (ordered) symmetric G-metric spaces to the respective results for mappings with one variable, even obtaining (in some cases) more general theorems. Our results generalize, extend, unify, and complement recent coupled coincidence point theorems in this frame, established by Cho et al. (2012), Aydi et al. (2011), and Choudhury and Maity (2011). Also, by using our method several recent tripled coincidence point results in ordered symmetric G-metric spaces can be reduced to the coincidence point results with one variable.

## 1. Introduction and Preliminaries

In 2004, Mustafa and Sims introduced a new notion of generalized metric space called G-metric space, where to every triplet of elements a nonnegative real number is assigned [1]. Fixed point theory, as well as coupled and tripled cases, in such spaces were studied in [2–6]. In particular, Banach contraction mapping principle was established in these works.

Fixed point theory has also developed rapidly in metric and cone metric spaces endowed with a partial ordering (see [7, 8] and references therein). Fixed point problems have also been considered in partially ordered G-metric spaces [9–11].

For more details on the following definitions and results concerning G-metric spaces, we refer the reader to [1, 9, 12–20].

*Definition 1.* Let $X$ be a nonempty set, and let $G : X^3 \to \mathbb{R}^+$ be a function satisfying the following properties:

(a) $G(x, y, z) = 0$ if $x = y = z$;

(b) $0 < G(x, y, z)$ for all $x$, $y$, and $z \in X$ with $x \neq y$;

(c) $G(x, x, y) \leq G(x, y, z)$ for all $x$, $y$, and $z \in X$, with $y \neq z$;

(d) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables); and

(e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x$, $y$, $z$, and $a \in X$.

Then the function $G$ is called a G-metric on $X$ and the pair $(X, G)$ is called a G-metric space.

*Definition 2.* Let $(X, G)$ be a G-metric space and let $\{x_n\}$ be a sequence of points in $X$.

(i) A point $x \in X$ is said to be the limit of a sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one says that the sequence $\{x_n\}$ is G-convergent to $x$.

(ii) The sequence $\{x_n\}$ is said to be a G-Cauchy sequence if, for every $\varepsilon > 0$, there is a positive integer $N$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$; that is, $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

(iii) $(X, G)$ is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in $(X, G)$ is G-convergent in $X$.

**Proposition 3** (see [1]). *Let $(X, G)$ be a G-metric space, and let $\{x_n\}$ be a sequence of points in $X$. Then the following are equivalent.*

(1) *The sequence* $\{x_n\}$ *is G-convergent to* $x$.

(2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$.

(3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$.

(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$.

*Definition 4* (see [1, 10]). A G-metric on $X$ is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$. Every G-metric on $X$ defines a metric $d_G$ on $X$ by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \tag{1}$$

For a symmetric G-metric space, one obtains

$$d_G(x, y) = 2G(x, y, y), \quad \forall x, y \in X. \tag{2}$$

However, for an arbitrary G-metric $G$ on $X$, just the following inequality holds: $(3/2)\, G(x, y, y) \le d_G(x, y) \le 3G(x, y, y)$, for all $x, y \in X$.

*Definition 5.* In this work, one will consider the following three classes of mappings [3, 21]:

$\Psi = \{\psi \mid \psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing and $\psi^{-1}(\{0\}) = \{0\}\}$,

$\Phi = \{\phi \mid \phi : [0, \infty) \to [0, \infty)$ is lower semicontinuous and $\phi^{-1}(\{0\}) = \{0\}\}$, and

$\Theta = \{\varphi \mid \varphi : [0, \infty) \to [0, \infty),\ \varphi(t) < t$ for $t > 0$ and $\lim_{r \to t^+} \varphi(r) < t$, for each $t > 0\}$.

For weak $\phi$-contractions in the frame of metric spaces see [19, 21].

At first we need the following well-known definitions and results (see, e.g., [9, 15, 22]).

*Definition 6.* Let $(X, \preceq)$ be a partially ordered nonempty set, and let $F : X^2 \to X$, $g : X \to X$ be two mappings. The mapping $F$ has the mixed $g$-monotone property if for any $x, y \in X$ the following hold:

$$\begin{aligned} gx_1 \preceq gx_2 &\Longrightarrow F(x_1, y) \preceq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \\ gy_1 \preceq gy_2 &\Longrightarrow F(x, y_1) \succeq F(x, y_2), \quad \text{for } y_1, y_2 \in X. \end{aligned} \tag{3}$$

Note that, if $g = i_X$, the identity mapping, then $F$ is said to have the mixed monotone property.

*Remark 7.* If $(X, \preceq)$ is a partially ordered set, then $(X^2, \sqsubseteq)$ is also partially ordered with

$$(x, y) \sqsubseteq (a, b) \Longleftrightarrow x \preceq a \wedge y \succeq b. \tag{4}$$

*Definition 8.* Let $F : X^2 \to X$ and $g : X \to X$. An element $(x, y) \in X^2$ is called a coupled coincidence point of $F$ and $g$ if

$$F(x, y) = gx, \qquad F(y, x) = gy, \tag{5}$$

while $(gx, gy) \in X^2$ is called a coupled point of coincidence of mappings $F$ and $g$. Moreover, $(x, y)$ is called a coupled common fixed point of $F$ and $g$ if

$$F(x, y) = gx = x, \qquad F(y, x) = gy = y. \tag{6}$$

*Remark 9.* Otherwise, $(x, y)$ is a coupled coincidence point of $F$ and $g$ if and only if $(x, y)$ is a coincidence point of the mappings $T_F : X^2 \to X^2$ and $T_g : X^2 \to X^2$ which are defined by

$$T_F(x, y) = (F(x, y), F(y, x)), \qquad T_g(x, y) = (gx, gy). \tag{7}$$

*Definition 10.* Mappings $f, g : X \to X$ are said to be compatible in a G-metric space $(X, G)$ if

$$G(fgx_n, gfx_n, gfx_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{8}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ in $(X, G)$.

It is easy to prove that $f$ and $g$ are compatible in $(X, G)$ if and only if they are compatible in the associated metric space $(X, d_G)$.

*Definition 11.* Let $(X, G)$ be a G-metric space, and let $F : X^2 \to X$ and $g : X \to X$ be two mappings. One says that $F$ and $g$ are compatible if

$$\begin{aligned} &G(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)) \longrightarrow 0, \\ &G(gF(y_n, x_n), F(gy_n, gx_n), F(gy_n, gx_n)) \longrightarrow 0, \\ &\qquad\qquad \text{as } n \longrightarrow \infty, \end{aligned} \tag{9}$$

whenever $x_n$ and $y_n$ are such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n, y_n) &= \lim_{n\to\infty} gx_n, \\ \lim_{n\to\infty} F(y_n, x_n) &= \lim_{n\to\infty} gy_n. \end{aligned} \tag{10}$$

The proof of the following lemma is immediate (for (2) see [18]).

**Lemma 12.** (1) *Let* $(X, G)$ *be a symmetric G-metric space. Define* $G_1 : X^2 \times X^2 \times X^2 \to \mathbb{R}^+$ *by*

$$G_1((x, y), (u, v), (a, b)) = \max\{G(x, u, a), G(y, v, b)\}. \tag{11}$$

*Then* $(X^2, G_1)$ *is a symmetric G-metric space.*

(2) *Let* $X = \{a, b\}$. *Define*

$$\begin{aligned} G(a, a, a) &= G(b, b, b) = 0, \\ G(a, a, b) &= 1, \qquad G(a, b, b) = 2, \end{aligned} \tag{12}$$

*and extend* $G$ *to* $X^3$ *by using the symmetry in the variables. Then it is clear that* $(X, G)$ *is an asymmetric G-metric space. It is not hard to see that* $(X^3, G_1)$ *is not a G-metric space.*

*Remark 13.* One can prove that mappings $F$ and $g$ are compatible in $(X, G)$ if and only if $T_F$ and $T_g$ are compatible in $(X^2, G_1)$.

*Definition 14.* (1) Let $X$ be a nonempty set. Then $(X, G, \preceq)$ is called an ordered G-metric space if

(i) $(X, G)$ is a G-metric space and

(ii) $(X, \preceq)$ is a partially ordered set.

(2) Let $(X, G, \preceq)$ be a partially ordered G-metric space. One says that $(X, G, \preceq)$ is regular if the following hypotheses hold:

(i) if a nondecreasing sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x \preceq x$ for all $n \in \mathbb{N}$,

(ii) if a nonincreasing sequence $\{y_n\}$ is such that $y_n \to y$ as $n \to \infty$, then $y_n \succeq y \succeq y$ for all $n \in \mathbb{N}$.

## 2. Main Results

Now, we are ready to state and prove our first result.

**Theorem 15.** *Let $(X, G, \preceq)$ be a partially ordered symmetric G-metric space, $F : X^2 \to X$, and $g : X \to X$. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\begin{aligned} &\psi\left(\max\left\{G\left(F\left(x, y\right), F\left(u, v\right), F\left(a, b\right)\right),\right.\right. \\ &\qquad \left.\left. G\left(F\left(y, x\right), F\left(v, u\right), F\left(b, a\right)\right)\right\}\right) \\ &\quad \le \psi\left(\max\left\{G\left(gx, gu, ga\right), G\left(gy, gv, gb\right)\right\}\right) \\ &\qquad - \phi\left(\max\left\{G\left(gx, gu, ga\right), G\left(gy, gv, gb\right)\right\}\right) \end{aligned} \tag{13}$$

*for all $x$, $y$, $u$, $v$, $a$, and $b \in X$ for which $gx \preceq gu \preceq ga \wedge gy \succeq gv \succeq gb$ or $gx \succeq gu \succeq ga \wedge gy \preceq gv \preceq gb$. Assume that $F$ and $g$ satisfy the following conditions:*

(1) *$F(X^2) \subset g(X)$;*

(2) *$F$ has the mixed $g$-monotone property;*

(3) *$F$ and $g$ are continuous and compatible and $(X, G)$ is G-complete, or*

(3′) *$(X, G, \preceq)$ is regular and one of $F(X^2)$ or $g(X)$ is G-complete;*

(4) *there exist $x_0, y_0 \in X$ such that*

$$\begin{aligned} &gx_0 \preceq F\left(x_0, y_0\right) \wedge gy_0 \succeq F\left(y_0, x_0\right) \\ &\text{or } gx_0 \succeq F\left(x_0, y_0\right) \wedge gy_0 \preceq F\left(y_0, x_0\right). \end{aligned} \tag{14}$$

*Then $F$ and $g$ have a coupled coincidence point.*

*Remark 16.* (a) Obviously, the condition (1) from [23] is equivalent to the condition (13). Hence, by using a new symmetric G-metric space $(X^2, G_1)$ we have obtained a method of reducing coupled coincidence and coupled fixed point results in (ordered) symmetric G-metric spaces to the respective results for mappings with one variable, even obtaining (in some cases) more general theorems. We note that this method cannot be applied in the case of asymmetric G-metric spaces (see (2) of Lemma 12). For other details of coupled case in ordered metric spaces see also [22].

(b) Also, we note that Theorem 3.1. from [23] holds if $F$ and $g$ are compatible instead of commuting (see Step 3 in [23]). Indeed, since

$$\begin{aligned} \lim_{n\to\infty} gx_n &= \lim_{n\to\infty} F\left(x_n, y_n\right) = x, \\ \lim_{n\to\infty} gy_n &= \lim_{n\to\infty} F\left(y_n, x_n\right) = y, \end{aligned} \tag{15}$$

then

$$\begin{aligned} &\lim_{n\to\infty} G\left(gF\left(x_n, y_n\right), F\left(gx_n, gy_n\right), F\left(gx_n, gy_n\right)\right) = 0, \\ &\lim_{n\to\infty} G\left(gF\left(y_n, x_n\right), F\left(gy_n, gx_n\right), F\left(gy_n, gx_n\right)\right) = 0, \end{aligned} \tag{16}$$

because $F$ and $g$ are compatible.

Therefore, now we have

$$\begin{aligned} &G\left(gx, F\left(x, y\right)\right) \\ &\quad \le G\left(gx, ggx_{n+1}, ggx_{n+1}\right) \\ &\qquad + G\left(ggx_{n+1}, F\left(x, y\right), F\left(x, y\right)\right) \\ &\quad \le G\left(gx, ggx_{n+1}, ggx_{n+1}\right) \\ &\qquad + G\left(ggx_{n+1}, F\left(gx_n, gy_n\right), F\left(gx_n, gy_n\right)\right) \\ &\qquad + G\left(F\left(gx_n, gy_n\right), F\left(x, y\right), F\left(x, y\right)\right) \\ &\quad \longrightarrow G\left(gx, gx, gx\right) + 0 \\ &\qquad + G\left(F\left(x, y\right), F\left(x, y\right), F\left(x, y\right)\right) = 0, \\ &\qquad\qquad \text{as } n \longrightarrow \infty, \end{aligned} \tag{17}$$

that is, $F(x, y) = gx$. Similarly, we obtain that $F(y, x) = gy$.

Assertions similar to the following lemma were used in the frame of metric spaces in the course of proofs of several fixed point results in various papers (see, e.g., [9, 21]). This lemma holds in every G-metric space.

**Lemma 17.** *Let $(X, G)$ be a G-metric space, and let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$. If $\{x_n\}$ is not a G-Cauchy sequence in $(X, G)$, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to $\varepsilon^+$ when $k \to \infty$:*

$$\begin{aligned} &G\left(x_{m_k}, x_{n_k}, x_{n_k}\right), \qquad G\left(x_{m_k}, x_{n_k-1}, x_{n_k-1}\right), \\ &G\left(x_{m_k+1}, x_{n_k}, x_{n_k}\right), \qquad G\left(x_{n_k-1}, x_{m_k+1}, x_{m_k+1}\right). \end{aligned} \tag{18}$$

The following lemma is crucial for the proof of Theorem 15, and it holds in every G-metric space.

**Lemma 18.** *Let $(X, G, \preceq)$ be a partially ordered G-metric space, and let $f$ and $g$ be two self-mappings on $X$. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\begin{aligned}\psi(G(fx, fy, fz)) &\le \psi(\max\{G(gx, gy, gz), G(gu, gv, gw), G(ga, gb, gc)\}) \\ &\quad - \phi(\max\{G(gx, gy, gz), G(gu, gv, gw), G(ga, gb, gc)\})\end{aligned} \tag{19}$$

*for all $x$, $y$, $z$, $u$, $v$, $w$, $a$, $b$, and $c \in X$ for which $gx \preceq gu \preceq ga \wedge gy \succeq gv \succeq gb \wedge gz \preceq gw \preceq gc$ or $gx \succeq gu \succeq ga \wedge gy \preceq gv \preceq gb \wedge gz \succeq gw \succeq gc$. If the following conditions hold:*

(i) *$f$ is a $g$-nondecreasing with respect to $\preceq$ and $fX \subset gX$;*

(ii) *there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$;*

(iii) *$f$ and $g$ are continuous and compatible, and $(X, G)$ is G-complete or*

(iii′) *$(X, G, \preceq)$ is regular, and one of $fX$ or $gX$ is G-complete.*

*Then $f$ and $g$ have a coincidence point in $X$.*

*Proof.* If $gx_0 = fx_0$, then $x_0$ is a coincidence point of $f$ and $g$. Therefore, let $gx_0 \prec fx_0$. Since $fX \subset gX$, we obtain a Jungck sequence $y_n = fx_n = gx_{n+1}$ for all $n = 0, 1, 2, \ldots$, where $x_n \in X$, and by induction we get that $y_n \preceq y_{n+1}$. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then $x_{n+1}$ is a coincidence point of $f$ and $g$. Therefore, suppose that $y_n \neq y_{n+1}$ for each $n$. Now, we will prove the following:

(1) $G(y_n, y_{n+1}, y_{n+1}) \to 0$ as $n \to \infty$;

(2) $\{y_n\}$ is a G-Cauchy sequence.

Indeed, by putting $x = u = a = x_n$, $y = v = b = x_{n+1}$, and $z = u = c = x_{n+1}$ in (19) we get

$$\begin{aligned}\psi(G(y_n, y_{n+1}, y_{n+1})) &= \psi(G(fx_n, fx_{n+1}, fx_{n+1})) \\ &\le \psi(G(gx_n, gx_{n+1}, gx_{n+1})) \\ &\quad - \phi(G(gx_n, gx_{n+1}, gx_{n+1})) \\ &= \psi(G(y_{n-1}, y_n, y_n)) \\ &\quad - \phi(G(y_{n-1}, y_n, y_n)) \\ &< \psi(G(y_{n-1}, y_n, y_n)),\end{aligned} \tag{20}$$

and since the function $\psi$ is nondecreasing, it follows that $G(y_n, y_{n+1}, y_{n+1}) \le G(y_{n-1}, y_n, y_n)$; that is, there exists $\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = G^* \ge 0$. If $G^* > 0$, we get from the previous relation $\psi(G^*) \le \psi(G^*) - \phi(G^*)$; that is, $G^* = 0$ which is a contradiction. Hence, we obtain that $\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = 0$.

Further, using Lemma 17 we shall prove that $\{y_n\}$ is a G-Cauchy sequence. Suppose this is not the case. Then, by Lemma 17 there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences tend to $\varepsilon^+$ when $k \to \infty$:

$$\begin{aligned}&G(x_{m_k}, x_{n_k}, x_{n_k}), \qquad G(x_{m_k}, x_{n_k-1}, x_{n_k-1}), \\ &G(x_{m_k+1}, x_{n_k}, x_{n_k}), \qquad G(x_{n_k-1}, x_{m_k+1}, x_{m_k+1}).\end{aligned} \tag{21}$$

Putting $x = x_{m_k+1}$, $y = x_{n_k}$, and $z = x_{n_k}$ in (19) we have

$$\begin{aligned}\psi(G(fx_{m_k+1}, fx_{n_k}, fx_{n_k})) &\le \psi(G(gx_{m_k+1}, gx_{n_k}, gx_{n_k})) \\ &\quad - \phi(G(gx_{m_k+1}, gx_{n_k}, gx_{n_k})),\end{aligned} \tag{22}$$

that is,

$$\begin{aligned}\psi(G(y_{m_k+1}, y_{n_k}, y_{n_k})) &\le \psi(G(y_{m_k}, y_{n_k-1}, y_{n_k-1})) \\ &\quad - \phi(G(y_{m_k}, y_{n_k-1}, y_{n_k-1})).\end{aligned} \tag{23}$$

Letting $k \to \infty$, we get $\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon)$; that is, $\phi(\varepsilon) = 0$. Since $\phi \in \Phi$, we get $\varepsilon = 0$, which is a contradiction. We have proved that $\{y_n\}$ is a G-Cauchy sequence in $(X, G)$.

In case (iii), since $(X, G)$ is G-complete, there exists $z \in X$ such that $y_n \to z$. Then we have

$$G(fx_n, z, z) \longrightarrow 0, \quad G(gx_n, z, z) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{24}$$

Further, according to Definition 1 (e) and since $f$ and $g$ are continuous and compatible, we get

$$\begin{aligned}G(fz, gz, gz) &\le G(fz, fgx_n, fgx_n) + G(fgx_n, gz, gz) \\ &\le G(fz, fgx_n, fgx_n) + G(fgx_n, gfx_n, gfx_n) \\ &\quad + G(gfx_n, gz, gz) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{25}$$

It follows that $z$ is a coincidence point for $f$ and $g$.

In case (iii′), it follows that $y_n = fx_n = gx_{n+1} \to gz$, $z \in X$ (in both cases when $fX$ or $gX$ is G-complete), and then $gx_n \preceq gz \preceq gz$, and by the contractive condition (19) we have

$$\begin{aligned}\psi(G(fx_n, fz, fz)) &\le \psi(G(gx_n, gz, gz)) \\ &\quad - \phi(G(gx_n, gz, gz)).\end{aligned} \tag{26}$$

By taking limit as $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned}\psi(G(gz, fz, fz)) &\le \psi(G(gz, gz, gz)) \\ &\quad - \phi(G(gz, gz, gz)) \\ &= 0 - 0 = 0,\end{aligned} \tag{27}$$

and hence $fz = gz$. □

*Proof of Theorem 15.* Firstly, (13) implies

$$\begin{aligned}\psi&\left(G_1\left(T_F(Y),T_F(V),T_F(A)\right)\right)\\&\le\psi\left(G_1\left(T_g(Y),T_g(V),T_g(A)\right)\right)\\&\quad-\phi\left(G_1\left(T_g(Y),T_g(V),T_g(A)\right)\right),\end{aligned}\tag{28}$$

for all $Y=(x,y)$, $V=(u,v)$, and $A=(a,b)$ from $X^2$ for which $T_g(Y)\sqsubseteq T_g(V)\sqsubseteq T_g(A)$ or $T_g(A)\sqsubseteq T_g(V)\sqsubseteq T_g(Y)$.

Further,

(1) implies that $T_F(X^2)\subset T_g(X^2)$;

(2) implies that $T_F$ is $T_g$-nondecreasing with respect to $\sqsubseteq$ and $T_F(X^2)\subset T_g(X^2)$;

(3) implies that $T_F$ and $T_g$ are continuous and compatible and $(X^2,G_1)$ is G-complete,

or

(3′) implies that $(X^2,G_1,\sqsubseteq)$ is regular and one of $T_F(X^2)$ or $T_g(X^2)$ is G-complete;

(4) implies that there exists $Y_0=(x_0,y_0)\in X^2$ such that $T_g(Y_0)\sqsubseteq T_F(Y_0)$ or $T_F(Y_0)\sqsubseteq T_g(Y_0)$.

All conditions of Lemma 18 for the ordered G-metric space $(X^2,G_1)$ are satisfied. Therefore, the mappings $T_F$ and $T_g$ have a coincidence point in $X^2$. According to Remark 9 the mappings $F$ and $g$ have a coupled coincidence point. The proof of Theorem 15 is complete. □

Our second main result is the following.

**Theorem 19.** *Let $(X,G,\preceq)$ be a partially ordered symmetric G-metric space, $F:X^2\to X$, and $g:X\to X$. Assume that there exists $\varphi\in\Theta$ such that*

$$\begin{aligned}\max&\{G(F(x,y),F(u,v),F(a,b)),\\&\quad G(F(y,x),F(v,u),F(b,a))\}\\&\le\varphi(\max\{G(gx,gu,ga),G(gy,gv,gb)\}),\end{aligned}\tag{29}$$

*for all $x$, $y$, $u$, $v$, $a$, and $b\in X$ for which $gx\preceq gu\preceq ga\wedge gy\succeq gv\succeq gb$ or $gx\succeq gu\succeq ga\wedge gy\preceq gv\preceq gb$. Assume that $F$ and $g$ satisfy the following conditions:*

(1) *$F(X^2)\subset g(X)$;*

(2) *$F$ has the mixed $g$-monotone property;*

(3) *$F$ and $g$ are continuous and compatible and $(X,G)$ is G-complete, or*

(3′) *$(X,G,\preceq)$ is regular and one of $F(X^2)$ or $g(X)$ is G-complete;*

(4) *there exist $x_0,y_0\in X$ such that*

$$\begin{aligned}&gx_0\preceq F(x_0,y_0)\wedge gy_0\succeq F(y_0,x_0)\\&\text{or } gx_0\succeq F(x_0,y_0)\wedge gy_0\preceq F(y_0,x_0).\end{aligned}\tag{30}$$

*Then $F$ and $g$ have a coupled coincidence point.*

For the proof of Theorem 19 we use the following.

**Lemma 20.** *Let $(X,G,\preceq)$ be a partially ordered G-metric space, and let $f$ and $g$ be two self-mappings on $X$. Assume that there exists $\varphi\in\Theta$ such that*

$$\begin{aligned}G&(fx,fy,fz)\\&\le\varphi(\max\{G(gx,gy,gz),\\&\qquad G(gu,gv,gw),\ G(ga,gb,gc)\})\end{aligned}\tag{31}$$

*for all $x$, $y$, $z$, $u$, $v$, $w$, $a$, $b$, and $c\in X$ for which $gx\preceq gu\preceq ga\wedge gy\succeq gv\succeq gb\wedge gz\preceq gw\preceq gc$ or $gx\succeq gu\succeq ga\wedge gy\preceq gv\preceq gb\wedge gz\succeq gw\succeq gc$. If the following conditions hold:*

(i) *$f$ is $g$-nondecreasing with respect to $\preceq$ and $fX\subset gX$;*

(ii) *there exists $x_0\in X$ such that $gx_0\preceq fx_0$;*

(iii) *$f$ and $g$ are continuous and compatible, and $(X,G)$ is G-complete or*

(iii′) *$(X,G,\preceq)$ is regular and one of $fX$ or $gX$ is G-complete,*

*Then $f$ and $g$ have a coincidence point in $X$.*

*Proof.* If $gx_0=fx_0$, then $x_0$ is a coincidence point of $f$ and $g$. Therefore, let $gx_0\prec fx_0$. Since $fX\subset gX$, we obtain a Jungck sequence $y_n=fx_n=gx_{n+1}$ for all $n=0,1,2,\ldots$, where $x_n\in X$, and by induction we get that $y_n\preceq y_{n+1}$. If $y_n=y_{n+1}$ for some $n\in\mathbb{N}$, then $x_{n+1}$ is a coincidence point of $f$ and $g$. Therefore, suppose that $y_n\neq y_{n+1}$ for each $n$. Now, we will prove that $G(y_n,y_{n+1},y_{n+1})\to 0$ as $n\to\infty$.

Indeed, by putting $x=u=a=x_n$, $y=v=b=x_{n+1}$, and $z=u=c=x_{n+1}$ in (31) we get

$$\begin{aligned}G(y_n,y_{n+1},y_{n+1})&=G(fx_n,fx_{n+1},fx_{n+1})\\&\le\varphi(G(gx_n,gx_{n+1},gx_{n+1}))\\&=\varphi(G(y_{n-1},y_n,y_n))\\&<G(y_{n-1},y_n,y_n).\end{aligned}\tag{32}$$

That is, there exists $\lim_{n\to\infty}G(y_n,y_{n+1},y_{n+1})=G^*\ge 0$. If $G^*>0$, we get

$$\begin{aligned}G^*&=\lim_{n\to\infty}G(y_n,y_{n+1},y_{n+1})\\&\le\lim_{n\to\infty}\varphi(G(y_{n-1},y_n,y_n))\\&=\lim_{G(y_{n-1},y_n,y_n)\to(G^*)^+}\varphi(G(y_{n-1},y_n,y_n))<G^*,\end{aligned}\tag{33}$$

which is a contradiction. Hence, we obtain that $\lim_{n\to\infty}G(y_n,y_{n+1},y_{n+1})=0$.

Further, by using Lemma 17, we shall prove that $\{y_n\}$ is a G-Cauchy sequence. Suppose this is not the case. Then, by Lemma 17 there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences tend to $\varepsilon^+$ when $k \to \infty$:

$$\begin{aligned} &G\left(y_{m_k}, y_{n_k}, y_{n_k}\right), \qquad G\left(y_{m_k}, y_{n_k-1}, y_{n_k-1}\right), \\ &G\left(y_{m_k+1}, y_{n_k}, y_{n_k}\right), \qquad G\left(y_{n_k-1}, y_{m_k+1}, y_{m_k+1}\right). \end{aligned} \tag{34}$$

Putting $x = x_{m_k+1}$, $y = x_{n_k}$, and $z = x_{n_k}$ in (31) we have

$$G\left(fx_{m_k+1}, fx_{n_k}, fx_{n_k}\right) \le \varphi\left(G\left(gx_{m_k+1}, gx_{n_k}, gx_{n_k}\right)\right), \tag{35}$$

that is,

$$G\left(y_{m_k+1}, y_{n_k}, y_{n_k}\right) \le \varphi\left(G\left(y_{m_k}, y_{n_k-1}, y_{n_k-1}\right)\right). \tag{36}$$

Letting $k \to \infty$, we obtain

$$\begin{aligned} \varepsilon &\le \lim_{k\to\infty} \varphi\left(G\left(y_{m_k}, y_{n_k-1}, y_{n_k-1}\right)\right) \\ &= \lim_{G(y_{m_k}, y_{n_k-1}, y_{n_k-1}) \to \varepsilon^+} \varphi\left(G\left(y_{m_k}, y_{n_k-1}, y_{n_k-1}\right)\right) < \varepsilon. \end{aligned} \tag{37}$$

Hence, we get $\varepsilon < \varepsilon$, which is a contradiction. We have proved that $\{y_n\}$ is a G-Cauchy sequence in $(X, G)$.

Now, in case (iii′), since $(X, G)$ is G-complete, there exists $z \in X$ such that $y_n \to z$. Then we have

$$G\left(fx_n, z, z\right) \longrightarrow 0, \quad G\left(gx_n, z, z\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{38}$$

and because

$$\begin{aligned} &G\left(fz, gz, gz\right) \\ &\quad \le G\left(fz, fgx_n, fgx_n\right) + G\left(fgx_n, gz, gz\right) \\ &\quad \le G\left(fz, fgx_n, fgx_n\right) + G\left(fgx_n, gfx_n, gfx_n\right) \\ &\qquad + G\left(gfx_n, gz, gz\right) \\ &\quad \longrightarrow G\left(fz, fz, fz\right) + 0 + G\left(gz, gz, gz\right) = 0, \\ &\qquad\qquad \text{as } n \longrightarrow \infty, \end{aligned} \tag{39}$$

it follows that $z$ is a coincidence point for $f$ and $g$.

In case (iii′), it follows that $y_n = fx_n = gx_{n+1} \to gz$, $z \in X$ (in both cases when $fX$ or $gX$ is G-complete), and then $gx_n \preceq gz \preceq gz$, and by the contractive condition (31) we have

$$G\left(fx_n, fz, fz\right) \le \varphi\left(G\left(gx_n, gz, gz\right)\right). \tag{40}$$

By taking the limit as $n \to \infty$ in the above inequality we obtain

$$\begin{aligned} G\left(gz, fz, fz\right) &\le \lim_{n\to\infty} \varphi\left(G\left(gx_n, gz, gz\right)\right) \\ &\le \lim_{n\to\infty} G\left(gx_n, gz, gz\right) \\ &= 0, \end{aligned} \tag{41}$$

and hence $fz = gz$. □

*Proof of Theorem 19.* The proof is very similar to the proof of Theorem 15. Namely, the contractive condition (29) for the mappings $F$ and $g$ is equivalent to the following condition:

$$\begin{aligned} &G_1\left(T_F(Y), T_F(V), T_F(A)\right) \\ &\quad \le \varphi\left(G_1\left(T_g(Y), T_g(V), T_g(A)\right)\right), \end{aligned} \tag{42}$$

for the mappings $T_F$ and $T_g$. The proof is further an immediate consequence of Lemma 20. □

*Remark 21.* We have obtained that, in case of symmetric G-metric spaces, Theorem 19 generalizes both results (Theorems 3.1 and 3.2) from [3]. Also, Example 22 shows that this generalization is proper. It is clear that our new method with ordered symmetric G-metric spaces $(X^2, G_1)$ and with mappings $T_F$ and $T_g$ implies that all results from [3] can be reduced to known results with one variable.

The following example supports both of our theorems, the first with $\psi(t) = t$ and $\phi(t) = (1/2)t$ and the second with $\varphi(t) = (1/2)t$.

*Example 22.* Let $X = \mathbb{R}$ be endowed with the complete G-metric

$$G(x, y, z) = |x - y| + |y - z| + |z - x|, \tag{43}$$

for all $x$, $y$, and $z \in X$ and with the usual order. Consider the mappings $F(x, y) = (x^3 - 2y^3)/8$ and $g(x) = x^3$. All the conditions of Theorems 15 and 19 are satisfied. In particular, the mapping $F$ has the mixed $g$-monotone property, and we will check that $F$ and $g$ are compatible.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $X$ such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n, y_n) &= \lim gx_n = A, \\ \lim_{n\to\infty} F(y_n, x_n) &= \lim gy_n = B. \end{aligned} \tag{44}$$

Then $(A - 2B)/8 = A$ and $(B - 2A)/8 = B$, wherefrom it follows that $A = B = 0$. Then

$$\begin{aligned} &G\left(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)\right) \\ &\quad = \left|\left(\frac{x_n^3 - 2y_n^3}{8}\right)^3 - \frac{x_n^9 - 2y_n^9}{8}\right| \\ &\qquad + 0 + \left|\frac{x_n^9 - 2y_n^9}{8} - \left(\frac{x_n^3 - 2y_n^3}{8}\right)^3\right| \\ &\quad = 2\left|\left(\frac{x_n^3 - 2y_n^3}{8}\right)^3 - \frac{x_n^9 - 2y_n^9}{8}\right| \longrightarrow 2 \\ &\qquad \cdot |0 - 0| = 0, \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{45}$$

and similarly

$$\begin{aligned} &G\left(gF(y_n, x_n), F(gy_n, gx_n), F(gy_n, gx_n)\right) \longrightarrow 0, \\ &\qquad\qquad \text{as } n \longrightarrow \infty. \end{aligned} \tag{46}$$

Also, $F$ and $g$ do not commute, and therefore a coupled coincidence point of $F$ and $g$ cannot be obtained by Theorem 3.1 from [3].

The contractive condition (13) is satisfied with $\psi(t) = t$ and $\phi(t) = (1/2)t$ which follows from

$$\begin{aligned}
&G(F(x,y),F(u,v),F(a,b)) \\
&\quad = \left|\frac{x^3-2y^3}{8} - \frac{u^3-2v^3}{8}\right| + \left|\frac{u^3-2v^3}{8} - \frac{a^3-2b^3}{8}\right| \\
&\qquad + \left|\frac{a^3-2b^3}{8} - \frac{x^3-2y^3}{8}\right| \\
&\quad = \frac{1}{8}\left(\left|\left(x^3-u^3\right) - 2\left(y^3-v^3\right)\right|\right. \\
&\qquad\qquad + \left|\left(u^3-a^3\right) - 2\left(v^3-b^3\right)\right| \\
&\qquad\qquad \left. + \left|\left(a^3-x^3\right) - 2\left(b^3-y^3\right)\right|\right) \\
&\quad \le \frac{1}{8}\left(G(gx,gu,ga) + 2G(gy,gv,gb)\right) \\
&\quad \le \frac{4}{8}\cdot\frac{G(gx,gu,ga)+G(gy,gv,gb)}{2} \\
&\quad \le \frac{1}{2}\max\{G(gx,gu,ga),G(gy,gv,gb)\} \\
&\quad = \max G(gx,gu,ga),G(gy,gv,gb) \\
&\qquad - \frac{1}{2}\max\{G(gx,gu,ga),G(gy,gv,gb)\}, \\
&G(F(y,x),F(v,u),F(b,a)) \\
&\quad = \left|\frac{y^3-2x^3}{8} - \frac{v^3-2u^2}{8}\right| + \left|\frac{v^3-2u^3}{8} - \frac{b^3-2a^3}{8}\right| \\
&\qquad + \left|\frac{b^3-2a^3}{8} - \frac{y^3-2x^3}{8}\right| \\
&\quad = \frac{1}{8}\left(\left|\left(y3-v^3\right) - 2\left(x^3-u^3\right)\right|\right. \\
&\qquad\qquad + \left|\left(v^3-b^3\right) - 2\left(u^3-a^3\right)\right| \\
&\qquad\qquad \left. + \left|\left(b^3-y^3\right) - 2\left(a^3-x^3\right)\right|\right) \\
&\quad \le \frac{1}{8}\left(2G(gx,gu,ga) + G(gy,gv,gb)\right) \\
&\quad \le \frac{4}{8}\cdot\frac{G(gx,gu,ga)+G(gy,gv,gb)}{2} \\
&\quad \le \frac{1}{2}\max\{G(gx,gu,ga),G(gy,gv,gb)\} \\
&\quad = \max\{G(gx,gu,ga),G(gy,gv,gb)\} \\
&\qquad - \frac{1}{2}\max\{G(gx,gu,ga),G(gy,gv,gb)\},
\end{aligned} \tag{47}$$

for all $x$, $y$, $u$, $v$, $a$, and $b \in X$ for which $gx \le gu \le ga \wedge gy \ge gv \ge gb$ or $gx \ge gu \ge ga \wedge gy \le gv \le gb$.

Hence,

$$\begin{aligned}
&\psi\left(\max\{G(F(x,y),F(u,v),F(a,b)),\right. \\
&\qquad \left. G(F(y,x),F(v,u),F(b,a))\}\right) \\
&\quad \le \psi\left(\max\{G(gx,gu,ga),G(gy,gv,gb)\}\right) \\
&\qquad - \phi\left(\max\{G(gx,gu,ga),G(gy,gv,gb)\}\right).
\end{aligned} \tag{48}$$

Similarly, the condition (29) is satisfied with $\varphi(t) = (1/2)t$; that is,

$$\begin{aligned}
&\max\{G(F(x,y),F(u,v),F(a,b)), \\
&\qquad G(F(y,x),F(v,u),F(b,a))\} \\
&\quad \le \varphi\left(\max\{G(gx,gu,ga),G(gy,gv,gb)\}\right),
\end{aligned} \tag{49}$$

for all $x$, $y$, $u$, $v$, $a$, and $b \in X$ for which $gx \le gu \le ga \wedge gy \ge gv \ge gb$ *or* $gx \ge gu \ge ga \wedge gy \le gv \le gb$. There exists a coupled coincidence point $(0,0)$ of the mappings $F$ and $g$.

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