Research Article

Generalized Projectively Symmetric Spaces

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We study generalized projectively symmetric spaces. We first study some geometric properties of projectively symmetric spaces and prove that any such space is projectively homogeneous and under certain conditions the projective curvature tensor vanishes. Then we prove that given any regular projective s -space (M, ∇) , there exists a projectively related connection $\bar{\nabla}$, such that $(M, \bar{\nabla})$ is an affine s -manifold.

1. Introduction

Affine and Riemannian s -manifolds were first defined in [1] following the introduction of generalized Riemannian symmetric spaces in [2]. They form a more general class than the symmetric spaces of E. Cartan. More details about generalized symmetric spaces can be found in the monograph [3]. Let M be a connected manifold with an affine connection ∇ , and let $A(M, \nabla)$ be the Lie transformation group of all affine transformation of M . An affine transformation s_x will be called an affine symmetry at a point x if x is an isolated fixed point of s_x . An affine manifold (M, ∇) will be called an affine s -manifold if there is a differentiable mapping $s : M \rightarrow A(M, \nabla)$, such that for each $x \in M$, s_x is an affine symmetry at x .

In [4] Podestà introduced the notion of a projectively symmetric space in the following sense. Let (M, ∇) be a connected C^∞ manifold with an affine torsion free connection ∇ on its tangent bundle; (M, ∇) is said to be projectively symmetric if for every point x of M there is an involutive projective transformation s_x of M fixing x and whose differential at x is $-\text{Id}$. The assignment of a symmetry s_x at each point x of M can be viewed as a map $s : M \rightarrow P(M, \nabla)$, and $P(M, \nabla)$ can be topologised, so that it is a Lie transformation group. In the above definition, however, no further assumption on s is made; even continuity is not assumed.

In this paper we define and state prerequisite results on projective structures and define projective symmetric spaces due to Podestà. Then we generalize them to define projective s -manifolds as manifolds together with more general symmetries and consider the cases where they are essential or inessential. A projective s -manifold is called inessential if it is projectively equivalent to an affine s -manifold and essential otherwise. We prove that these spaces are naturally homogeneous, and moreover under certain conditions the projective curvature tensor vanishes. Later we define regular projective s -manifolds and prove that they are inessential.

2. Preliminaries

Let M be a connected real C^∞ manifold whose tangent bundle TM is endowed with an affine torsion free connection ∇ . We recall that a diffeomorphism s of M is said to be projective transformation if s maps geodesics into geodesics when the parametrization is disregarded [5]; equivalently s is projective if the pull back $s^*\nabla$ of the connection is projectively related to ∇ , that is, if there exists a global 1-form π on M , such that

$$s^*\nabla_X Y = \nabla_X Y + \pi(X)Y + \pi(Y)X \quad \forall X, Y \in \chi(M). \quad (1)$$

If the form π vanishes identically on M , then s is said to be an affine transformation.

Definition 1. (M, ∇) is said to be projectively symmetric if for every point x in M there exists a projective transformation s_x with the following properties:

- (a) $s_x(x) = x$ and x is an isolated fixed point of s_x .
- (b) s_x is involutive.

It is easy to see that conditions (a) and (b) imply $(ds_x)_x = -\text{Id}$. Moreover we recall that a projective transformation is determined if we fix its value at a point, its differential, and its second jet at this point [6], hence a symmetry at x in M is not uniquely determined in general by the condition (a) and (b).

Affine symmetric spaces are affine homogeneous, but in general projectively symmetric spaces are not projective homogeneous; for more detail and examples see [4, 7], but if following Ledger and Obata define the case of a differentiable distribution of projective symmetries in an affine manifold, then this happens.

Let (M, g) be a connected Riemannian manifold. An isometry s_x of (M, g) for which $x \in M$ is an isolated fixed point will be called a Riemannian symmetry of M at x . Clearly, if s_x is a symmetry of (M, g) at x , then the tangent map $S_x = (ds_x)_x$ is an orthogonal transformation of $T_x M$ having no fixed vectors (with the exception of 0). An s -structure on (M, g) is a family $\{s_x \mid x \in M\}$ of symmetries of (M, g) .

A Riemannian s -manifold is a Riemannian manifold (M, g) together with a map $s : M \rightarrow I(M, g)$, such that for each $x \in M$ the image s_x is a Riemannian symmetry at x .

For any affine manifold (M, ∇) let $A(M, \nabla)$ denote the Lie group of all affine transformation of (M, ∇) . An affine transformation $s_x \in A(M, \nabla)$ for which $x \in M$ is an isolated fixed point will be called an affine symmetry at x . An affine s -manifold is an affine manifold (M, ∇) together with a differentiable mapping $s : M \rightarrow A(M, \nabla)$, such that for each $x \in M$, the image s_x is an affine symmetry at x .

Let M be an affine s -manifold. Since $s : M \rightarrow A(M, \nabla)$ is assumed to be differentiable, the tensor field S of type (1,1) defined by $S_x = (ds_x)_x$ for each $x \in M$ is differentiable. The tensor field S is defined similarly for a Riemannian s -manifold, although it may not be smooth. For either affine or Riemannian s -manifolds we call S the symmetry tensor field.

Following [3] an s -structure $\{s_x\}$ is called regular if for every pair of points $x, y \in M$ as follows:

$$s_x \circ s_y = s_z \circ s_x, \quad \text{where } z = s_x(y). \quad (2)$$

3. Projective s-Space

Let M be a connected manifold with an affine connection ∇ , and let $P(M, \nabla)$ be the group of all projective transformations of M .

Definition 2. A projective transformation s_x will be called a projective symmetry or simply a symmetry at the point x , if x is an isolated fixed point of s_x and $(ds_x)_x = S$ does not leave any nonzero vector fixed.

Definition 3. A connected affine manifold (M, ∇) will be called a projective s -manifold or simply ps-manifold if for

each $x \in M$ there is a projective symmetry s_x , such that the mapping $s : M \rightarrow P(M, \nabla)$, $x \mapsto s_x$ is smooth.

A symmetry s_x will be called a symmetry of order k at x , if there exist a positive integer k , such that $s_x^k = \text{Id}$, and M will be called ps-manifold of order k , if k is the least positive number such that each symmetry is of order k . Evidently every ps-manifold of order 2 is a projective symmetric space.

Lemma 4. Let G be a topological transformation group acting on a connected topological space M , if for each point x in M , the G -orbit of x contains a neighborhood of x , then G is transitive on M .

Proof. Since G is transitive on each orbit, for each x the G -orbit $G(x)$ of x is open by our assumption. The complement $C(x)$ of $G(x)$ in M is also open, being a union of orbits. Thus $G(x)$ is open and closed. It is nonempty and therefore coincides with the connected space M , thus G is transitive. $\square$

Theorem 5. If M is a ps-manifold, then $P(M, \nabla)$ is transitive on M .

Proof. We fix a point $x_0 \in M$ and consider the C^∞ map $h : M \rightarrow M$ given by $h(x) = s_x(x_0)$; since $s_x(x) = x$ for every x in M , the differential $(dh)_{x_0}$ of h at the point x_0 is given by $(dh)_{x_0} = I - S_{x_0}$, where S_{x_0} is the differential of s_{x_0} at x_0 . $(dh)_{x_0}$ is nonsingular because no eigenvalue of S_{x_0} is equal to 1. Hence h is a diffeomorphism on some neighborhood W of x_0 in M , and $h(W)$ is a neighborhood of x_0 contained in the $P(M, \nabla)$ -orbit $P(M, \nabla)_{x_0}$ of x_0 , therefore from the above lemma $P(M, \nabla)$ is transitive. $\square$

Definition 6. Let M be a ps-manifold; since $s : M \rightarrow P(M, \nabla)$ is assumed to be differentiable, the tensor field S of type (1, 1) defined by $S_x = (ds_x)_x$ is differentiable, we call S the symmetry tensor field.

Lemma 7. If s_x is a projective symmetry of (M, ∇) then there exists a connection $\bar{\nabla}$ projectively equivalent with ∇ which is s_x -invariant.

Proof. Since s_x is a projective symmetry of (M, ∇) then there is a 1-form α on M , such that

$$(s_x^* \nabla)_X Y = \nabla_X Y + \alpha(X) Y + \alpha(Y) X. \quad (3)$$

We are looking for a connection $\bar{\nabla}$ with the following properties:

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(X) Y + \pi(Y) X. \quad (4)$$

As $\bar{\nabla}$ should be s_x -invariant we need

$$(s_x^* \bar{\nabla})_X Y = \bar{\nabla}_X Y \quad (5)$$

that is, s_x is an affine transformation of $(M, \bar{\nabla})$. We have

$$\begin{aligned} (s_x^* \bar{\nabla})_X Y &= s_{x_*}^{-1} \bar{\nabla}_{s_{x_*} X} s_{x_*} Y \\ &= s_{x_*}^{-1} (\nabla_{s_{x_*} X} s_{x_*} Y + \pi(s_{x_*} X) s_{x_*} Y \\ &\quad + \pi(s_{x_*} Y) s_{x_*} X) \\ &= (s_x^* \nabla)_X Y + \pi(s_{x_*} X) Y + \pi(s_{x_*} Y) X. \end{aligned} \quad (6)$$

It follows from (3)

$$\begin{aligned} (s_x^* \bar{\nabla})_X Y &= \nabla_X Y + \alpha(X) Y + \alpha(Y) X \\ &\quad + \pi(s_{x_*} X) Y + \pi(s_{x_*} Y) X. \end{aligned} \quad (7)$$

From (5) we have

$$\begin{aligned} \nabla_X Y + \alpha(X) Y + \alpha(Y) X + \pi(s_{x_*} X) Y + \pi(s_{x_*} Y) X \\ = \nabla_X Y + \pi(X) Y + \pi(Y) X, \end{aligned} \quad (8)$$

thus it is enough to have for every vector field Z as follows:

$$\alpha(Z) + \pi(s_{x_*} Z) = \pi(Z) \quad (9)$$

which is equivalent to

$$\pi(Z - s_{x_*} Z) = \alpha(Z) \quad (10)$$

or simply

$$\pi \circ (I - s_{x_*}) Z = \alpha(Z) \quad (11)$$

since s_x is symmetry, then $I - s_x$ is invertible; hence we obtain

$$\pi = \alpha \circ (I - s_{x_*})^{-1} \quad (12)$$

thus if we choose π as (12), then (4) and (5) are true, and $\bar{\nabla}$ is the required connection. $\square$

So it would be convenient to introduce the following definition for connection $\bar{\nabla}$ and 1-form $\pi = \alpha \circ (I - s_{x_*})^{-1}$.

Definition 8. Let (M, ∇) be a ps-manifold, and let s_x be the projective symmetry at the point x . Then we call the associate connection $\bar{\nabla}$ the fundamental connection of s_x . Also the 1-form

$$\pi = \alpha \circ (I - s_{x_*})^{-1} \quad (13)$$

will be called the fundamental 1-form of s_x , where α is the 1-form on M , such that

$$s^* \nabla_X Y = \nabla_X Y + \pi(X) Y + \pi(Y) X \quad \forall X, Y \in \chi(M). \quad (14)$$

Definition 9. The projective curvature tensor of (M, ∇) is defined as follows [5, 8]:

$$W_{jkl}^i = \Pi_{jkl}^i - \frac{1}{n-1} (\delta_k^i \Pi_{jl} - \delta_l^i \Pi_{jk}), \quad (15)$$

where

$$\begin{aligned} \Pi_{jk}^i &= \Gamma_{jk}^i - \frac{2}{n+1} \delta_{(j}^i \Gamma_{k)l}^l, \\ \Pi_{jkl}^i &= \partial_k \Pi_{jl}^i - \partial_l \Pi_{jk}^i + \Pi_{jl}^h \Pi_{hk}^i - \Pi_{jk}^h \Pi_{hl}^i, \quad \Pi_{jk} = \Pi_{jkh}^h. \end{aligned} \quad (16)$$

The projective curvature tensor W is invariant with respect to projective transformations [5, 8].

Theorem 10. In a ps-manifold (M, ∇) , let s_x be a symmetry, and let $\bar{\nabla}$ be the fundamental connection of s_x , if $\bar{\nabla} S = 0$; that is, $((\nabla_X S)(Y) = \pi(Y)S(X) - \pi(S(Y))X)$, then $(\bar{\nabla} W)_x = 0$.

Proof. Let $s : M \rightarrow P(M, \nabla)$ be the ps-structure and $\bar{\nabla} S = 0$. Let $X, Y, Z \in T_x M$ be tangent vectors, and let $\omega \in T_x^* M$ be a covector at $x \in M$. By parallel translation along each geodesics through x , X, Y, Z , and ω can be extended to local vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$, and $\tilde{\omega}$ with vanishing covariant derivative with respect to $\bar{\nabla}$ at x . Because S is parallel, the local vector fields $S\tilde{X}, S\tilde{Y}, S\tilde{Z}$, and $S^* \tilde{\omega}$ have also vanishing covariant derivative at x . (Here S^* denotes the transpose map to S .) As W is invariant with respect to the projective transformation s_p , $p \in M$, we have

$$W(S^* \tilde{\omega}, \tilde{X}, \tilde{Y}, \tilde{Z}) = W(\tilde{\omega}, S\tilde{X}, S\tilde{Y}, S\tilde{Z}). \quad (17)$$

Now we show that $\bar{\nabla} W(S^* \tilde{\omega}, \tilde{X}, \tilde{Y}, \tilde{Z}, U)$ and $\bar{\nabla} W(\tilde{\omega}, S\tilde{X}, S\tilde{Y}, S\tilde{Z}, SU)$ are equal at x . These are equal if and only if $(S^* \omega)(\bar{\nabla}_U W(\tilde{X}, \tilde{Y}, \tilde{Z}))$ and $\omega(\bar{\nabla}_{SU} W(S\tilde{X}, S\tilde{Y}, S\tilde{Z}))$ are equal, which follows from the assumption on $\bar{\nabla}$. That is

$$\bar{\nabla} W(S_x^* \omega, X, Y, Z, U) = \bar{\nabla} W(\omega, S_x X, S_x Y, S_x Z, S_x U) \quad (18)$$

or

$$\bar{\nabla} W(\omega, X, Y, Z, U) = \bar{\nabla} W(S^{*-1} \omega, SX, SY, SZ, SU). \quad (19)$$

Differentiating covariantly (17) with respect to $\bar{\nabla}$ in the direction of SU at x and using (19) we get

$$\bar{\nabla} W(\omega, X, Y, Z, SU) = \bar{\nabla} W(\omega, X, Y, Z, U), \quad (20)$$

thus

$$(\bar{\nabla} W)_x(\omega, X, Y, Z, (I - S)U) = 0, \quad (21)$$

for all $X, Y, Z, U \in T_x M$ and $\omega \in T_x^* M$, and because $(I - S)_x$ is a nonsingular transformation we obtain

$$(\bar{\nabla} W)_x = 0. \quad (22)$$

$\square$

Theorem 11. Let (M, ∇) be a ps-manifold of dimension $n > 2$; if there exist two different projective symmetries σ_1, σ_2 at a point q of M , such that $\sigma_{1*q} = \sigma_{2*q}$ and $\bar{\nabla} S = 0$, where $\bar{\nabla}$ is the fundamental connection corresponding to σ_1 , then the projective curvature tensor W vanishes that is, M is projectively flat.

Proof. By a similar method used in Proposition 1.1 of [7] the proof follows from Lemma 7 and Theorem 10. $\square$

Corollary 12. *If (M, ∇) is a ps-manifold of order 2, and two different projective symmetry σ_1, σ_2 can be defined at a point q , then M is projectively flat.*

Proof. It is evident from the fact that $\sigma_{1*q} = \sigma_{2*q}$. $\square$

Proposition 13. *Let (M, ∇) be ps-manifold, such that at every point x of M the projective symmetry is uniquely determined. Then the linear isotropy representation $\rho : P(M, \nabla)_x \rightarrow GL(n, R)$ is faithful for every $x \in M$.*

Proof. Since s_x and s_x^{-1} both are projective symmetry at x , then we have $s_x = s_x^{-1}$; that is, $s_x^2 = \text{Id}$, thus (M, ∇) is a ps-manifold of order 2. Now, our assertion follows from Theorem 1.1 of [7]. $\square$

4. Regular Projective s-Space

Definition 14. A ps-manifold (M, ∇) is called regular ps-manifold or simply rps-manifold if for all $p, q \in M$, $s_p \circ s_q = s_z \circ s_p$, where $z = s_p(q)$.

Lemma 15. *Let (M, ∇) be a regular ps-manifold, then the $(1, 1)$ tensor field S is invariant under all symmetries s_x ; that is*

$$ds_x(SX) = S(ds_x X), \quad (23)$$

for all $X \in \chi(M)$.

Proof. Since M is regular ps-manifold, then for all $X \in T_y M$ we have $d(s_x \circ s_y)X = d(s_z \circ s_x)X$ and so $ds_x(SX) = S_z(ds_x X)$. Thus S is s_x invariant for all s_x . $\square$

Lemma 16. *Let (M, ∇) be a connected ps-manifold, such that at every point x of M the projective symmetry is uniquely determined, then (M, ∇) is rps-manifold.*

Proof. Suppose $p, q \in M$ and $z = s_p(q)$; then from the uniqueness of the projective symmetry, we have $s_p \circ s_q = s_z \circ s_p$, so (M, ∇) is regular ps-manifold. $\square$

Remark 17. A general question is to find condition under which, given a ps-manifold (M, ∇) , there exists a projectively related connection $\bar{\nabla}$, such that $(M, \bar{\nabla})$ is an affine s-manifold; we shall call such spaces inessential ps-manifold and essential otherwise.

Definition 18. A ps-manifold (M, ∇) is called inessential ps-manifold if there exists a projectively related connection $\bar{\nabla}$ such that $(M, \bar{\nabla})$ is an affine s-manifold.

Let us denote by Φ_h the 1-form corresponding to an element h of $P(M, \nabla)$. We want to see when (M, ∇) is inessential, in order to show that (M, ∇) is inessential, we must find a connection $\bar{\nabla}$ which is projectively related to ∇

and is invariant under all symmetries. Let s_q be a symmetry at q , we must find a one-form π , such that

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(X)Y + \pi(Y)X. \quad (24)$$

As s_q is a projective transformation for (M, ∇) and leaves the connection $\bar{\nabla}$ invariant, we find that

$$\Phi_{s_q}(X) + \pi(s_q^* X) = \pi(X) \quad \forall X \in \chi(M) \quad (25)$$

and hence at q we have

$$\pi|_q \left(I - (s_q)_{*q} \right) X = \Phi_{s_q}(X). \quad (26)$$

So we define a 1-form π through the following formula:

$$\pi|_x(X) = \Phi_{s_x|x} \circ (I - (s_x)_{*x})^{-1} X \quad \forall X \in T_x M. \quad (27)$$

Theorem 19. *Let (M, ∇) be an rps-manifold, then (M, ∇) is inessential.*

Proof. We define a torsion free affine connection $\bar{\nabla}$ projectively related to ∇ through the fundamental 1-form of s_x , π as follows:

$$\pi|_x(X) = \Phi_{s_x|x} \circ (I - (s_x)_{*x})^{-1} X \quad \forall X \in T_x M \quad (28)$$

and prove that the connection $\bar{\nabla}$ is invariant under all the symmetries of M .

Let s_q be a symmetry at q of M , the condition that $\bar{\nabla}$ is invariant under s_q is equivalent to

$$\pi(X) - \pi(s_q^* X) = \Phi_{s_q}(X). \quad (29)$$

We verify (29) at a point p of M , we have to prove that by (28)

$$\Phi_{s_p|p} \circ (I - (s_p)_{*p})^{-1} X - \pi_{s_q(p)}(s_q^*|_p X) = \Phi_{s_q|p}(X) \quad (30)$$

so if we put $z = s_q(p)$, (30) reduces to

$$\begin{aligned} & \Phi_{s_p|p} \circ (I - (s_p)_{*p})^{-1} X - \Phi_{s_z|z} \circ (I - (s_z)_{*z})^{-1} (s_q^*|_p X) \\ &= \Phi_{s_q|p}(X). \end{aligned} \quad (31)$$

But since $s_q \circ s_p = s_z \circ s_q$, we have

$$\Phi_{s_p}(Y) + \Phi_{s_q}(s_p^* Y) = \Phi_{s_q}(Y) + \Phi_{s_z}(s_q^* Y). \quad (32)$$

Now evaluate (32) at p , and let $Y = (I - (s_p)_{*p})^{-1} X$; then as $s_q \circ s_p = s_z \circ s_q$, we have (31), and we are done. $\square$

Remark 20. The authors have studied Finsler homogeneous and symmetric spaces [9]; recently Habibi and the second author generalized them to Finsler s-manifolds and weakly Finsler symmetric spaces [10, 11]. Therefore these concepts can be mixed and find more generalizations which will be the content of other papers.

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