Research Article

The EA-Dimension of a Commutative Ring

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An elementary annihilator of a ring A is an annihilator that has the form $(0 : a)_A$; $a \in R \setminus (0)$. We define the elementary annihilator dimension of the ring A , denoted by $\text{EAdim}(A)$, to be the upper bound of the set of all integers n such that there is a chain $(0 : a_0) \subset \cdots \subset (0 : a_n)$ of annihilators of A . We use this dimension to characterize some zero-divisors graphs.

1. Introduction

In this paper, all rings are considered to be commutative and unitary.

Let A be a ring and S be a nonempty subset of A . We call the annihilator of S in A denoted by $(0 : S)_A$ or $(0 : S)$ the set $\{a \in A / aS = (0)\}$. If $S = \{a\}$ is a singleton then $(0 : S)$ will be denoted by $(0 : a)$. If $a \neq 0$ then $(0 : a)$ is called an elementary annihilator. An annihilator is said to be maximal if it is maximal in the set of all proper annihilators of A . It is well known that all maximal annihilators are elementary. For $n \in \mathbb{N}$ an elementary annihilator chain $(0 : a_1) \subset (0 : a_2) \subset \cdots \subset (0 : a_{n+1})$ is said to be a chain of elementary annihilators with length n ending in $(0 : a_{n+1})$. The upper bound of the set of all lengths of elementary annihilator chains ending in $(0 : a)$ is called the elementary annihilator height of a_{n+1} (or $(0 : a_{n+1})$). In this paper, we introduce a dimension of a ring R using elementary annihilator chains called *elementary annihilator dimension*, denoted by $\text{EAdim}(R)$. The $\text{EAdim}(R)$ is the upper bound of the set of elementary annihilator heights. We use this dimension to study zero-divisor graphs.

We introduce a class of rings called *isometric maximal elementary annihilator* rings, in short *IMEA-rings*. That is the class of rings with finite EAdim whose all maximal annihilators have the same height.

2. Elementary Annihilator Dimension of a Ring

Definition 1. (1) Let $n \in \mathbb{N}$ and $(0 : a_1) \subset (0 : a_2) \subset \cdots \subset (0 : a_{n+1})$ be chain of elementary annihilators in the ring A .

One says that this chain is an elementary annihilator chain of length n ending in $(0 : a_{n+1})$.

(2) Let a be a nonzero element of A . One defines the elementary annihilator height of a , denoted by $\text{EAht}(a)$, as the upper bound of the set of all lengths of elementary annihilator chains ending in $(0 : a)$.

(3) One calls elementary annihilator dimension of A , denoted by $\text{EAdim}(A)$, the upper bound of the set $\{\text{EAht}(a); a \in A \setminus \{0\}\}$.

Example 2. (1) $\text{EAdim}(\mathbb{Z}/4\mathbb{Z}) = 1$. Indeed, $(0 : 1) \subset (0 : 2)$ is the longest chain of elementary annihilators in $\mathbb{Z}/4\mathbb{Z}$.

(2) $\text{EAht}(1) = 0$.

(3) All nonzero zero-divisors a satisfy $\text{EAht}(a) \geq 1$. Indeed, $(0 : 1) \subset (0 : a)$ is a chain of length one.

It is easy to check the following results.

Remark 3. (1) Let $a \in A \setminus \{0\}$, $\text{EAht}(a) = 0$ if and only if a is regular.

(2) $\text{EAdim}(A) = 0$ if and only if A is a domain.

(3) For an ideal I of A , $\text{EAdim}(A/I) = 0$ if and only if I is prime.

(4) If a is a nonzero noninvertible element $\text{EAdim}(A/(a)) = 0$ if and only if a is prime.

We denote by $\text{nil}(A)$ the set of all nilpotent elements of A . A is said to be reduced if it has no nilpotents other than zero.

Theorem 4. Let $a \in \text{nil}(A) \setminus \{0\}$ and $n(a)$ be its index of nilpotency; one has: $\text{EAht}(a) + n(a) \leq \text{EAdim}(A) + 2$.

Proof. If $\text{EAht}(a) + n(a)$ or $\text{EAdim}(A)$ is infinite the result is obvious. Otherwise, there exists a chain whose length is $\text{EAht}(a)$ and it ends in $(0 : a)$. Let $(0 : 1) \subset (0 : a_1) \subset \cdots \subset (0 : a_{r-1}) \subset (0 : a)$ be this chain. Moreover, we have $(0 : a) \subset \cdots \subset (0 : a^{n(a)-1})$. So we obtain the following chain: $(0 : 1) \subset (0 : a_1) \subset \cdots \subset (0 : a_{r-1}) \subset (0 : a) \subset (0 : a^2) \subset \cdots \subset (0 : a^{n(a)-1})$ whose length is $\text{EAht}(a) + n(a) - 2$. Consequently, $\text{EAht}(a) + n(a) - 2 \leq \text{EAdim}(A)$. $\square$

Corollary 5. *If $a \in \text{nil}(A) \setminus \{0\}$ satisfies $\text{EAht}(a) = \text{EAdim}(A)$ is finite then $n(a) = 2$. In particular, if $\text{EAdim}(A) = 1$ then for all $a \in \text{nil}(A) \setminus \{0\}$, $n(a) = 2$.*

Theorem 6. *Let A_1 and A_2 be two rings; then;*

- (1) $\text{EAdim}(A_1 \times A_2)$ is finite if and only if $\text{EAdim}(A_1)$ and $\text{EAdim}(A_2)$ are finite.
- (2) $\text{EAdim}(A_1 \times A_2) = \text{EAdim}(A_1) + \text{EAdim}(A_2) + 1$.

Proof. Let $(a, b) \in A_1 \times A_2$ be a nonzero zero-divisor. If a and b are nonzero then $(0 : (a, b))_{A_1 \times A_2} = (0 : a)_{A_1} \times (0 : b)_{A_2}$.

If one of them is zero, for example, $a = 0$ then $(0 : (a, b))_{A_1 \times A_2} = A_1 \times (0 : b)_{A_2}$.

(1) “ $\Rightarrow$ ” Let $\text{EAdim}(A_1 \times A_2) = n$, suppose that $\text{EAdim}(A_1)$ or $\text{EAdim}(A_2)$ is infinite; for example, $\text{EAdim}(A_1)$ is infinite. Then there exists $r \geq n + 1$ and $(0 : 1) \subset (0 : a_1) \subset \cdots \subset (0 : a_r)$ a chain in A_1 ; then $(0 : (1, 0)) \subset (0 : (a_1, 0)) \subset \cdots \subset (0 : (a_r, 0))$ is a chain of elementary annihilators in $A_1 \times A_2$ whose length is $r > n$, contradiction.

“ $\Leftarrow$ ” If we assume that $n = \text{EAdim}(A_1)$, then there is a chain of length n in A_1 ; let $(0 : 1)_{A_1} \subset (0 : a_1)_{A_1} \subset \cdots \subset (0 : a_n)_{A_1}$ be this chain. In the same way we put $m = \text{EAdim}(A_2)$ and we take $(0 : 1)_{A_2} \subset (0 : b_1)_{A_2} \subset \cdots \subset (0 : b_m)_{A_2}$ as a chain of length m . Then $(0 : (1, 1))_{A_1 \times A_2} \subset (0 : (a_1, 1))_{A_1 \times A_2} \subset \cdots \subset (0 : (a_n, 1))_{A_1 \times A_2} \subset (0 : (a_n, b_1))_{A_1 \times A_2} \subset \cdots \subset (0 : (a_n, b_m))_{A_1 \times A_2} \subset (0 : (0, b_m))_{A_1 \times A_2}$ is an elementary annihilator chain of $A_1 \times A_2$ whose length is $n + m + 1$ that is maximal, because of the inclusion $(0 : (a, b))_{A_1 \times A_2} \subset (0 : (c, d))_{A_1 \times A_2} \Leftrightarrow (0 : a)_{A_1} \subset (0 : c)_{A_1}$ or $(0 : b)_{A_2} \subset (0 : d)_{A_2}$. Then $\text{EAdim}(A_1 \times A_2)$ is finite and $\text{EAdim}(A_1 \times A_2) = n + m + 1$.

(2) If $\text{EAdim}(A_1 \times A_2)$ is infinite (that is $\text{EAdim}(A_1)$ or $\text{EAdim}(A_2)$ is infinite, by (1)) then the result is obvious. The finite case is shown in the proof of (1) “ $\Leftarrow$ ”. $\square$

By induction, we have the following result.

Corollary 7. (1) *Let $A_1, \dots, A_r$ be some rings, one has $\text{EAdim}(A_1 \times \cdots \times A_r) = \sum_{i=1}^r \text{EAdim}(A_i) + r - 1$.*

(2) *If A is a domain and $n \in \mathbb{N}^*$ then $\text{EAdim}(A^n) = n - 1$.*

3. The EAdimension and the Zero-Divisor Graph

Let A be a ring. The zero-divisor graph of A is defined to be the graph whose vertices are the nonzero zero-divisors of A and its edges are the pairs $\{a, b\}$ satisfying $ab = 0$. We denote

this graph by $\Gamma(A)$. For the simplicity of writing we still denote by $\Gamma(A)$ the set of nonzero zero-divisors of A .

$\Gamma(A)$ is said to be connected if for every two different vertices a and b of $\Gamma(A)$ there is a sequence $a_1, \dots, a_n \in \Gamma(A)$ such as $a = a_1$, $b = a_n$ and $\{a_i, a_{i+1}\}$ is an edge, $\forall 1 \leq i \leq n - 1$. This sequence is called a path connecting a and b with length $n - 1$. $\Gamma(A)$ is said to be complete if each two distinct vertices form an edge. We call the distance between a and b the least length of a path connecting them, denoted by $d_A(a, b)$ or $d(a, b)$. We call the diameter of $\Gamma(A)$, denoted $\text{diam}(\Gamma(A))$, the supremum of the set $\{d(a, b); a, b \in \Gamma(A)\}$. In [1], Anderson and Livingston showed that $\Gamma(A)$ is connected and $\text{diam}(\Gamma(A)) \in \{0, 1, 2, 3\}$.

For an integer $r \geq 2$, Anderson and Livingston defined $\Gamma(A)$ to be r -partite complete if $\Gamma(R) = \Gamma_1 \cup \cdots \cup \Gamma_r$, where the Γ_i s are nonempty disjoint sets and for all $x \neq y$ in $\Gamma(A)$ satisfy $xy \neq 0$ if and only if there exists $1 \leq i \leq r$ such that $x, y \in \Gamma_i$. In this paper we extend the definition of r -partite complete graph to the case when r is infinite.

Lemma 8 (see [2], Theorem 6). (1) *If $(0 : a)$ is an elementary annihilator that is maximal (in the set of proper annihilators of A) then it is prime.*

(2) *Let $a, b \in A \setminus \{0\}$; if $(0 : a)$ is maximal and $(0 : b) \not\subseteq (0 : a)$ then $b \in (0 : a)$.*

Proposition 9. *Let A be a reduced ring that is not a domain and $a \neq b$ be two nonzero zero-divisors such that $(0 : a)$ and $(0 : b)$ are maximal; then $ab \neq 0$ if and only if $(0 : a) = (0 : b)$.*

Proof. “ $\Rightarrow$ ” Immediately, by Lemma 8 “ $\Leftarrow$ ” If $(0 : a) = (0 : b)$, suppose that $ba = 0$. $ba = 0 \Rightarrow b \in (0 : a) = (0 : b) \Rightarrow b^2 = 0$, contradiction. Then $ab \neq 0$. $\square$

Theorem 10. *If A is a nonreduced ring then $\text{EAdim}(A) = 1$ if and only if $\Gamma(A)$ is complete.*

Proof. “ $\Leftarrow$ ” if $\Gamma(A)$ is complete then, according to [1, Theorem 2.8], we have for all $x, y \in Z(A)$, $xy = 0$. Then $Z(A) = (0 : c)$, $\forall c \in Z(A) \setminus \{0\}$. So all nonzero elementary annihilators are equal, and then $\text{EAdim}(A) = 1$.

“ $\Rightarrow$ ” Let $a \in \text{nil}(A) \setminus \{0\}$; then $a^2 = 0$, according to Corollary 5. If $Z(A) = \{0, a\}$ then $\Gamma(A)$ is complete. Otherwise, for all $b \in Z(A) \setminus \{0, a\}$ we have $(0 : a)$ and $(0 : b)$ are maximal. Suppose that $(0 : a) \neq (0 : b)$, according to Lemma 8(2), $ab = 0$ and there, for example, $y \in (0 : a) \setminus (0 : b)$; then $(0 : b) \subset (0 : yb)$, contradiction to the maximality of $(0 : b)$. Then $(0 : a) = (0 : b)$ and $ab = 0$. Consequently, $Z(A) = \text{nil}(A)$ and all nonzero zero-divisors b satisfy $b^2 = 0$. It follows that for all $a, b \in Z(A)$, $ab = 0$. According to [1, Theorem 2.8], $\Gamma(A)$ is complete. $\square$

Theorem 11. *If $\Gamma(A) = \bigcup_{i \in I} \Gamma_i$ is r -partite complete graph with $r = \text{card}(I) \in \mathbb{N} \cup \{\infty\}$, $r \geq 2$ then $r = 2$ under one of the following conditions:*

- (1) *two of the Γ_i 's contain, each one, more than one element;*
- (2) *A is reduced.*

Proof. Suppose that $r \geq 3$.

First case: If two among the Γ_i 's contain, each one, more than one element. Assume that $\Gamma(A) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \dots$, where Γ_1 and Γ_2 have, each one, at least two elements. Let $a_i \in \Gamma_i$, $1 \leq i \leq 3$ and $a_i \neq b_i \in \Gamma_i$, $1 \leq i \leq 2$. We have $(a_1 + a_2)a_3 = 0$; then $a_1 + a_2$ is a divisor of zero in A . Suppose that $a_1 + a_2 = 0$; then $b_1(a_1 + a_2) = 0 \Rightarrow b_1a_1 = 0$, contradiction. Then $a_1 + a_2 \in \Gamma(A)$. Suppose that $a_1 + a_2 \in \Gamma_3 \cup \dots$; then $b_1(a_1 + a_2) = 0 \Rightarrow b_1a_1 = 0$, contradiction; then $a_1 + a_2 \in \Gamma_1 \cup \Gamma_2$. If $a_1 + a_2 \in \Gamma_1$ then $b_2(a_1 + a_2) = 0 \Rightarrow b_2a_2 = 0$, contradiction. Then $a_1 + a_2 \in \Gamma_2$ and $b_1(a_1 + a_2) = 0 \Rightarrow b_1a_1 = 0$, contradiction.

Second Case: If A Is Reduced. Assume that $\Gamma(A) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \dots$. Let $a_i \in \Gamma_i$, $1 \leq i \leq 3$, we have $(a_1 + a_2)a_3 = 0$ then $a_1 + a_2$ is a divisor of zero in A . Suppose that $a_1 + a_2 = 0$, then $a_1(a_1 + a_2) = 0 \Rightarrow a_1^2 = 0$, contradiction. Then $a_1 + a_2 \in \Gamma(A)$. Suppose that $a_1 + a_2 \in \Gamma_3 \cup \dots$ then $a_1(a_1 + a_2) = 0 \Rightarrow a_1^2 = 0$, contradiction. Then $a_1 + a_2 \in \Gamma_1 \cup \Gamma_2$. If $a_1 + a_2 \in \Gamma_1$ then $a_2(a_1 + a_2) = 0 \Rightarrow a_2^2 = 0$, contradiction. Then $a_1 + a_2 \in \Gamma_2$ and then $a_1(a_1 + a_2) = 0 \Rightarrow a_1^2 = 0$, contradiction. We conclude that $r = 2$. $\square$

Theorem 12. *If A is reduced then $\text{EAdim}(A) = 1$ if and only if $\Gamma(A)$ is bipartite complete.*

Proof. “ $\Rightarrow$ ” In $\Gamma(A)$, we define the relation “ $\sim$ ” by the following: $x \sim y$ if $(0 : x) = (0 : y)$. $\sim$ is a relation of equivalence. For all $x \in \Gamma(A)$, we denote by Γ_x its equivalence class. The different classes Γ_x form a partition of $\Gamma(A)$ and we write $\Gamma(A) = \bigcup_{i \in I} \Gamma_{x_i}$. Since A is reduced and not a domain, then there exist nonzero elements $b \neq a$ satisfying $ba = 0$. Now, $\text{EAdim}(A) = 1$; then $(0 : a)$ and $(0 : b)$ are maximal. According to Proposition 9, $(0 : a) \neq (0 : b)$; then $\Gamma_a \neq \Gamma_b$; then $r = \text{card}(I) \geq 2$.

If $y \neq z \in \Gamma_x$ then $(0 : z) = (0 : y)$ then $yz \neq 0$, by Proposition 9.

Let $\Gamma_x \neq \Gamma_y$; then $(0 : x) \neq (0 : y)$; then, by Lemma 8, $xy = 0$. And we conclude that $\Gamma(A)$ is r -partite complete. According to Theorem 11, $\Gamma(A)$ is bipartite complete.

“ $\Leftarrow$ ” Assume that $\Gamma(A) = \Gamma_1 \cup \Gamma_2$ is bipartite complete. If $a \in \Gamma_i$ then $(0 : a) = \Gamma_j \cup \{0\}$, for $i \neq j \in \{1, 2\}$. Let $a, b \in \Gamma(A)$, if $a, b \in \Gamma_i$ then $(0 : a) = (0 : b) = \Gamma_j \cup \{0\}$. Otherwise $(0 : a)$ and $(0 : b)$ are incomparable. Then for all $a \in \Gamma(A)$, $\text{EAht}(a) = 1$. It follows that $\text{EAdim}(A) = 1$. $\square$

Theorem 13. *Let R be a ring.*

- (1) *If $\text{EAdim}(R) = 1$ then $\text{diam}(R) \leq 2$.*
- (2) *If $\text{diam}(R) \leq 1$ or $\text{diam}(R) = 2$ and R is reduced then $\text{EAdim}(R) = 1$.*
- (3) *If $\text{diam}(R) = 2$ and R is not reduced or $\text{diam}(R) = 3$ then $\text{EAdim}(R) \geq 2$.*

Proof. (1) If R is reduced then, by Theorem 12, $\Gamma(R)$ is bipartite complete; then $\text{diam}(R) \leq 2$.

If R is not reduced then, by Theorem 10, $\Gamma(R)$ is complete then, by Theorem 2.8 of [1], for all $x, y \in Z(R)$, $xy = 0$. Then, by Theorem 2.6 of [3], $\text{diam}(R) \leq 1$.

(2) If $\text{diam}(R) = 2$ and R is reduced then, by Theorem 2.6 of [3], R is reduced with exactly two minimal primes and at least three nonzero zero-divisors. Then $Z(R) = P_1 \cup P_2$, where P_1, P_2 are the two minimal primes of R ; they satisfy $P_1 \cap P_2 = (0)$. Then for all $p_1 \in P_1$ and $p_2 \in P_2$, $p_1, p_2 = 0$ and for $x \neq y \in P_i \setminus \{0\}$, $xy \neq 0$. Consequently, $\Gamma(R) = [P_1 \setminus \{0\}] \cup [P_2 \setminus \{0\}]$ is bipartite complete graph. According to Theorem 12, $\text{EAdim}(R) = 1$.

If $\text{diam}(R) \leq 1$: if $\text{diam}(R) = 0$, by Theorem 2.6 of [3], R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[y]/(y^2)$ and in both cases R has a unique nonzero elementary annihilator then $\text{EAdim}(R) = 1$.

Now, if $\text{diam}(R) = 1$, using Theorem 2.6 of [3], $xy = 0$ for each distinct pair of zero-divisors and R has at least two nonzero zero-divisors. According to Theorem 2.8 of [1], $\Gamma(R)$ is complete and R is not reduced. By Theorem 10, $\text{EAdim}(R) = 1$.

(3) If $\text{diam}(R) = 3$ then, by (1), $\text{EAdim}(R) \neq 1$. Since $\text{diam}(R) = 3$ then R is not a domain then $\text{EAdim}(R) \neq 0$. Consequently, $\text{EAdim}(R) \geq 2$. If $\text{diam}(R) = 2$ and R is not reduced: $\text{diam}(R) = 3$ then R is not a domain then $\text{EAdim}(R) \neq 0$. Suppose that $\text{EAdim}(R) = 1$; then by Theorem 10, $\Gamma(R)$ is complete then $\text{diam}(R) = 1$, contradiction. Then $\text{EAdim}(R) \geq 2$. $\square$

Lemma 14. *Let A be a ring, and $F = \{P_1, \dots, P_r\}$, $r > 1$ a set of distinct prime ideals which are incomparable. For an element of R we denote $E(a) = \{P \in F; a \notin P\}$. Let $R = A/(P_1 \cap \dots \cap P_r)$.*

- (1) *For all subsets E of F there is $a \in A$ such that $E(a) = E$.*
- (2) *Let $a \in A$, $\bar{a} \in Z(R) \Leftrightarrow E(a) \neq F$. In particular, $\bar{a} = \bar{0} \Leftrightarrow E(a) = \emptyset$.*
- (3) *Let E_1 and E_2 be two nonempty subsets of F . If $\bigcap_{P \in E_1} P \subseteq \bigcap_{P \in E_2} P$ then $E_2 \subseteq E_1$.*
- (4) *Let $a, b \in A$. $\bar{b} \in (\bar{0} : \bar{a}) \Leftrightarrow b \in \bigcap_{P \in E(a)} P$.*
- (5) *The map $\varphi : \{(\bar{0} : \bar{a}), \bar{a} \neq \bar{0}\} \rightarrow \mathcal{P}(F) \setminus \{\emptyset\}$, $(\bar{0} : \bar{a}) \mapsto E(a)$ is a decreasing bijection, here $\mathcal{P}(F)$ denotes the set of F subsets.*

Proof. (1) If $E = \emptyset$, take $a = 0$ then $a \in P$, for all $P \in F$ then $E(a) = \emptyset$ and the result is true in this case.

If $E = F$, take $a = 1$ then $a \notin P$, for all $P \in F$ then $E(a) = F$.

Now if $E \neq \{F, \emptyset\}$: $E(a) = E \Leftrightarrow a \in P, \forall P \notin E$ and $a \notin P, \forall P \in E \Leftrightarrow a \in (\bigcap_{P \in F \setminus E} P) \setminus (\bigcup_{P \in E} P)$. Suppose that $(\bigcap_{P \in F \setminus E} P) \setminus (\bigcup_{P \in E} P) = \emptyset \Rightarrow \bigcap_{P \in F \setminus E} P \subseteq \bigcup_{P \in E} P$; then there exists $P_{i_0} \in E$ such that $\bigcap_{P \in F \setminus E} P \subseteq P_{i_0}$; then there exists $P_{i_1} \in F \setminus E$ such that $P_{i_1} \subseteq P_{i_0}$. Since the P 's in F are incomparable under inclusion then $P_{i_1} = P_{i_0}$, then $P_{i_0} \in (F \setminus E) \cap E = \emptyset$, contradiction. Consequently, there exists $a \in A$ such that $E(a) = E$.

(2) Let $a \in A$, $\bar{a} \in Z(R) \Leftrightarrow \exists \bar{b} \neq \bar{0} / \bar{a}\bar{b} = \bar{0} \Leftrightarrow \exists b \in A/b \notin \bigcap_{P \in F} P$ and $ba \in \bigcap_{P \in F} P \Leftrightarrow \exists b \in A/E(b) \neq \emptyset$ and $ba \in P, \forall P \in E(a) \Leftrightarrow \exists b \in A/E(b) \neq \emptyset$ and $b \in P, \forall P \in E(a) \Leftrightarrow \exists b \in A/E(b) \neq \emptyset$ and $E(a) \subseteq F \setminus E(b) \Leftrightarrow E(a) \neq F$. In the last equivalence the indirect sense “ $\Leftarrow$ ” is obtained by (1).

(3) Suppose that $E_1 = \{Q_1, \dots, Q_s\}$ and $E_2 = \{R_1, \dots, R_t\}$, where the Q_i 's (resp., P_i 's) are pairwise different. $Q_1 \cap \dots \cap Q_s \subseteq R_1 \cap \dots \cap R_t \Rightarrow Q_1 \cap \dots \cap Q_s \subseteq R_1$ then one of the Q_i 's is contained in R_1 ; for example, $Q_1 \subseteq R_1$. Since the elements of F are incomparable under inclusion then $R_1 = Q_1$.

$Q_1 \cap \dots \cap Q_s \subseteq R_1 \cap \dots \cap R_t \Rightarrow Q_1 \cap \dots \cap Q_s \subseteq R_2$ then one of the Q_i 's is contained in R_2 . Suppose that $Q_1 \subseteq R_2$; then $Q_1 = R_2$ then $R_1 = R_2$, contradiction. Then there exists $i \geq 2$ such that $Q_i \subseteq R_2$; for example, $Q_2 \subseteq R_2$; then $R_2 = Q_2$.

We repeat this process until reaching the stage number $n = \min(s, t)$.

Suppose that $s < t$; then $Q_1 \cap \dots \cap Q_s = R_1 \cap \dots \cap R_s \subseteq R_{s+1}$; then there exists $k \leq s$ such that $R_k \subseteq R_{s+1}$ that is, $R_k = R_{s+1}$, contradiction. Consequently, $s \geq t$ and we get $E_1 = \{R_1, \dots, R_t, Q_{t+1}, \dots, Q_s\}$; then $E_2 \subseteq E_1$.

(4) Let $a, b \in A$, $\bar{b} \in (\bar{0} : \bar{a}) \Leftrightarrow ba \in \bigcap_{P \in E} P \Leftrightarrow b \in \bigcap_{P \in E(a)} P$.

(5) We check that φ is well defined: put $(\bar{0} : \bar{a}) = (\bar{0} : \bar{b})$. According to (4), $\bar{x} \in (\bar{0} : \bar{a}) \Leftrightarrow x \in \bigcap_{P \in E(a)} P$. Then $x \in \bigcap_{P \in E(a)} P \Leftrightarrow x \in \bigcap_{P \in E(b)} P$; then $\bigcap_{P \in E(a)} P = \bigcap_{P \in E(b)} P$. According to (3), $E(b) \subseteq E(a)$.

For $\bar{a} \neq \bar{0}$, $E(a) \neq \emptyset$; then $E(a) \in \mathcal{P}(F) \setminus \{\emptyset\}$. Then φ is well defined.

φ is injective, by (4).

We show that φ is surjective: let $E \in \mathcal{P}(F) \setminus \{\emptyset\}$. According to (1), there exists $a \in A$ such that $E(a) = E$. Since $E \neq \emptyset$ then $\bar{a} \neq \bar{0}$ and $\varphi(\bar{0} : \bar{a}) = E$.

$(\bar{0} : \bar{a}) \subseteq (\bar{0} : \bar{b}) \Rightarrow \bigcap_{P \in E(a)} P \subseteq \bigcap_{P \in E(b)} P$. Then, according to (3), $E(b) \subseteq E(a)$. Thus φ is a decreasing bijection. $\square$

Theorem 15. Let A be a ring and $P_1, \dots, P_r, r \geq 1$ be different incomparable prime ideals of A . Then $\text{EAdim}(A/(P_1 \cap \dots \cap P_r)) = r - 1$.

Proof. If $r = 1$ then A/P_1 is a domain and the result is checked.

Now let $r \geq 2$. $\{P_1\} \subset \{P_1, P_2\} \subset \dots \subset \{P_1, \dots, P_{r-1}\} \subset \{P_1, \dots, P_r\}$ is decreasing sequence in $\mathcal{P}(F) \setminus \{\emptyset\}$ with maximal length. Using the bijection φ defined in Lemma 14, $\varphi^{-1}(\{P_1, \dots, P_r\}) \subset \varphi^{-1}(\{P_1, \dots, P_{r-1}\}) \subset \dots \subset \varphi^{-1}(\{P_1, P_2\}) \subset \varphi^{-1}(\{P_1\})$ is a chain of elementary annihilators in $A/(P_1 \cap \dots \cap P_r)$ that has a maximal length. Then $\text{EAdim}(A/(P_1 \cap \dots \cap P_r)) = r - 1$. $\square$

Example 16. A ring is said to be semilocal ring if it has a finite number of maximal ideals. Let A be a semilocal ring with n maximal ideal; then $\text{EAdim}(A/J) = n - 1$, where J is the Jacobson radical of A . We obtain this result by using the previous theorem.

A ring is called a noetherian spectrum ring if it satisfies the ascending chain condition (acc) on radical ideals; equivalently each radical ideal is a radical of finitely generated ideal. The set of prime ideals of a ring A which are minimal over an ideal I , denoted by $\min_I(A)$, is finite in the case when A is a noetherian spectrum ring. If $I = (0)$, we denote by $\min(A)$

instead of $\min_I(A)$. For more information about noetherian spectrum rings see [4, Chapter 2].

Proposition 17. Let A be a noetherian spectrum ring, for all ideals I , $\text{EAdim}(A/\sqrt{I}) = |\min_I(A)| - 1$.

Proof. Since A is a noetherian spectrum ring then $\min_I(A)$ is finite ([4], Chapter 2, Corollary 2.1.10). Assume that $\min_I(A) = \{P_1, \dots, P_r\}$, $r = |\min_I(A)| \in \mathbb{N}^*$. The P_i 's are incomparable, then we get the result by using Theorem 15. $\square$

Definition 18 (according to [5]). Let R be a ring.

- (1) One calls the chromatic number of R the minimal number of colors used to color the elements of R such that each two adjacent elements (with zero product) have different colors, denoted by $\chi(R)$.
- (2) One says that the ring R is a coloring if its chromatic number is finite.

Theorem 19. If R is a reduced coloring then $\text{EAdim}(R) = \chi(R) - 2$.

Proof. If R is a reduced coloring, according to [5, Theorem 3.8], $\min(R)$ is finite. And if $|\min(R)| = n$ then $\chi(R) = n + 1$. Let $\min(R) = \{P_1, \dots, P_n\}$; then $R = R/(P_1 \cap \dots \cap P_n)$. According to Theorem 15, $\text{EAdim}(R) = n - 1 = \chi(R) - 2$.

Let A be a ring such that $\text{EAdim}(A) = n \geq 1$. For all vertices a in $\Gamma(A)$, we denote $i(a) = \text{EAht}(a)$ and $j(a) = \max\{\text{EAht}(b); (0 : a) \subseteq (0 : b)\}$. For all $i \leq j \leq n$, we denote by $\Gamma_{i,j}(A)$ the subgraph of $\Gamma(A)$ whose vertices form the following set $\{a; i(a) = i \text{ et } j(a) = j\}$.

Theoretically we can write $\Gamma(A) = \bigcup_{1 \leq i \leq j \leq n} \Gamma_{i,j}(A)$. $\square$

Remark 20. Let A be a ring such that $\text{EAdim}(A) = n \geq 2$. If $\Gamma_{1,1}(A) \neq \emptyset$ then $\text{diam}(\Gamma(A)) = 2$. Indeed, consider $a \in \Gamma_{1,1}(A)$ and $b \in \Gamma(A) \setminus \Gamma_{1,1}(A)$. There exists $i \geq 1$ and $j \geq 2$ such that $b \in \Gamma_{i,j}(A)$. Suppose that $ab \neq 0$ then $b \notin (0 : a)$. According to Lemma 8(2) $(0 : b) \subseteq (0 : a)$, this contradicts the fact that $a \in \Gamma_{1,1}(A)$. Then $ab = 0$ and $\{a, b\}$ is an edge. Now take $x \neq y \in \Gamma(A)$, three cases are possible. If $x, y \in \Gamma_{1,1}(A)$ then take $b \in \Gamma(A) \setminus \Gamma_{1,1}(A)$ and the chain $x - b - y$ is of length 2 then $d(x, y) \leq 2$. If only x is in $\Gamma_{1,1}(A)$ then $x - y$ is an edge. If $x, y \in \Gamma(A) \setminus \Gamma_{1,1}(A)$ then take $a \in \Gamma_{1,1}(A)$ and $x - a - y$ is a chain of length 2. Then, in all cases, $d(x, y) \leq 2$ that is $\text{diam}(\Gamma(A)) \leq 2$. Now $\text{EAdim}(A) = n \geq 2$, then there exists a, b such that $(0 : 1) \subset (0 : a) \subset (0 : b)$. Let $x \in (0 : b) \setminus (0 : a)$ then $xa \neq 0$ then $d(x, a) \geq 2$. Consequently, $\text{diam}(\Gamma(A)) = 2$.

4. Isometric Maximal Elementary Annihilator Rings

Definition 21. Let R be a ring with finite EA dimension; one says that R is an isometric maximal elementary annihilator ring, in short an *IMEA-ring* if its all maximal elementary annihilators have the same height.

Example 22. The ring $\mathbb{Z}/8\mathbb{Z}$ is an IMEA-ring. Indeed, the elementary annihilators of $\mathbb{Z}/8\mathbb{Z}$ are $(\bar{0} : \bar{1})$, $(\bar{0} : \bar{2})$, $(\bar{0} : \bar{4})$

and $(\bar{0} : \bar{6})$. They satisfy $(\bar{0} : \bar{1}) \subset (\bar{0} : \bar{2}) \subset (\bar{0} : \bar{4})$ and $(\bar{0} : \bar{1}) \subset (\bar{0} : \bar{2}) \subset (\bar{0} : \bar{6})$.

Theorem 23. Let R_1 and R_2 be two rings; then $R_1 \times R_2$ is an IMEA-ring if and only if R_1 and R_2 are two IMEA-rings.

Proof. By Theorem 6(1), $\text{EAdim}(R_1 \times R_2)$ is finite if and only if $\text{EAdim}(R_1)$ and $\text{EAdim}(R_2)$ are finite.

$(0 : (a, b))$ is a maximal elementary annihilator in $R_1 \times R_2$ if and only if $(0 : (a, b)) = R_1 \times (0 : b)$, and $(0 : b)$ is a maximal elementary annihilator in R_2 or $(0 : (a, b)) = (0 : a) \times R_2$, and $(0 : a)$ is a maximal elementary annihilator in R_1 , then all maximal elementary annihilators of $R_1 \times R_2$ have the same height if and only if all maximal elementary annihilators of R_1 have the same height and also for the maximal elementary annihilators of R_2 . We get the following result, inductively. $\square$

Corollary 24. Let $R_1, \dots, R_n$ be some rings. We have $R_1 \times R_2 \times \dots \times R_n$ is an IMEA-ring if and only if each R_i is an IMEA-ring.

Let R be a domain, we say that R is atomic if each nonzero nonunit element of R decomposes into a finite product of irreducibles, according to [6]. An atomic domain is called a *half factorial domain*, in short a *HFD* if $x_1 \cdots x_n = y_1 \cdots y_m$ are two decompositions into irreducibles then $n = m$. This concept was introduced by Zaks in [7]. A *HFD* is called a *unique factorization domain*, in short a *UFD* if $x_1 \cdots x_n = y_1 \cdots y_m$ are two decompositions into irreducibles then the x_i 's and the y_i 's are associates after reordering them. It is well known that a *UFD* is an atomic domain in which each irreducible is primed by [8, Theorem 1].

Proposition 25. If R is a *UFD*, then for all nonzero nonunit a of R we have $R/(a)$ is an IMEA-ring. Moreover if $a = p_1^{m_1} \cdots p_r^{m_r}$ is the decomposition of a into prime elements then $\text{EAdim}(R/(a)) = m_1 + \dots + m_r - 1$.

Proof. Let $a = p_1^{m_1} \cdots p_r^{m_r} \in R$. Suppose that $x \in R$; then $\bar{x} \in Z(R/(a)) \setminus \{\bar{0}\}$

$$\begin{aligned} \iff \begin{cases} \bar{x} \neq \bar{0} \\ \exists \bar{y} \neq \bar{0} \\ \bar{x}\bar{y} = \bar{0} \end{cases} \\ \iff \begin{cases} x = p_1^{\alpha_1} \cdots p_r^{\alpha_r} x_1; \alpha_i \in \mathbb{N}, x_1 \notin (p_i), \\ \forall i, \frac{\exists i}{\alpha_i} < m_i, \\ \exists y = p_1^{\beta_1} \cdots p_r^{\beta_r} y_1; \beta_i \in \mathbb{N}, y_1 \notin (p_i), \\ \forall i, \frac{\exists i}{\beta_i} < m_i, \\ \alpha_i + \beta_i \geq m_i, \forall i. \end{cases} \end{aligned} \quad (1)$$

Then

$$\begin{aligned} Z\left(\frac{R}{(a)}\right) = \left\{ \overline{p_1^{\alpha_1} \cdots p_r^{\alpha_r} x_1}; (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r \setminus \{(0, \dots, 0)\}, \right. \\ \left. x_1 \in R \setminus \bigcup_i (p_i) \right\}. \end{aligned} \quad (2)$$

And for $\bar{x} = \overline{p_1^{\alpha_1} \cdots p_r^{\alpha_r} x_1} \neq \bar{0}$ (one among the α_i 's is $< m_i$ and $x_1 \in R \setminus \bigcup_i (p_i)$), we have the following:

$$\begin{aligned} (\bar{0} : \bar{x}) = \left\{ \overline{p_1^{\beta_1} \cdots p_r^{\beta_r} y_1}; (\beta_1, \dots, \beta_r) \in \mathbb{N}^r \setminus \{(0, \dots, 0)\} / \beta_i \right. \\ \left. \geq m_i - \alpha_i, \forall i, y_1 \in R \setminus \bigcup_i (p_i) \right\}. \end{aligned} \quad (3)$$

It is easy to check that the set of all elementary annihilators of $R/(a)$ is

$$\begin{aligned} \left\{ (\bar{0} : \overline{p_1^{\alpha_1} \cdots p_r^{\alpha_r}}); \right. \\ \left. (\alpha_1, \dots, \alpha_r) \in [[0, m_1]] \times \dots \times [[0, m_r]] \setminus (m_1, \dots, m_r) \right\}. \end{aligned} \quad (4)$$

The maximal ones among them are

$$\begin{aligned} (\bar{0} : \overline{p_1^{m_1-1} p_2^{m_2} \cdots p_r^{m_r}}), (\bar{0} : \overline{p_1^{m_1} p_2^{m_2-1} p_3^{m_3} \cdots p_r^{m_r}}), \\ \dots, (\bar{0} : \overline{p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r-1}}). \end{aligned} \quad (5)$$

A longest chain ending in one of them, for example, $(\bar{0} : \bar{1}) \subset (\bar{0} : \overline{p_1}) \subset \dots \subset (\bar{0} : \overline{p_1^{m_1-1}}) \subset (\bar{0} : \overline{p_1^{m_1-1} p_2}) \subset \dots \subset (\bar{0} : \overline{p_1^{m_1-1} p_2^{m_2}}) \subset \dots \subset (\bar{0} : \overline{p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r-1} p_r}) \subset \dots \subset (\bar{0} : \overline{p_1^{m_1-1} p_2^{m_2} \cdots p_r^{m_r}})$, has the length $m_1 + \dots + m_r - 1$. Thus all maximal elementary annihilators have the same height $m_1 + \dots + m_r - 1$ then $R/(a)$ is an IMEA-ring and $\text{EAdim}(R/(a)) = m_1 + \dots + m_r - 1$. $\square$

Proposition 26. Let R be a *HFD*. R is a *UFD* if and only if $\text{EAdim}(R/(a)) = \lambda(a) - 1$, for all nonzero nonunit a . Where $\lambda(a)$ is the number of factors in a decomposition of a into irreducibles (counted with multiplicities).

Proof. " $\Rightarrow$ " is due to the previous proposition.

" $\Leftarrow$ " Let a be an irreducible of R then $\lambda(a) = 1$; then $\text{EAdim}(R/(a)) = \lambda(a) - 1 = 0$. According to Remark 3, $R/(a)$ is a domain; then (a) is prime. According to [8, Theorem 1], R is a *UFD*. $\square$

Question 1. Are all finite *EAdimensional* rings IMEA-rings?

Question 2. Are all finite rings IMEA-rings?

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