Research Article

A Sharp Bound for the Čebyšev Functional of Convex or Concave Functions

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1. Introduction

For two Lebesgue functions $f, g : [a, b] \rightarrow \mathbb{R}$, the Čebyšev functional defined by

$$\mathcal{T}(f, g) = \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt, \quad (1)$$

has interesting applications in various integral approximations as pointed out in the references below.

The problem of bounding the Čebyšev functional has a long history, starting with Grüss [1] inequality in 1935, where he had proved that, for two integrable functions f, g such that $\phi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for any $x \in [a, b]$, the inequality

$$|C(f, g)| \leq \frac{1}{4} (\Phi - \phi) (\Gamma - \gamma) \quad (2)$$

holds, and the constant $1/4$ is the best possible.

After that, many authors have studied the functional (1), and therefore, several bounds under various assumptions for the functions involved have been obtained.

In 1971, Atkinson [2] proved that if f, g are twice differentiable and convex on $[a, b]$ and

$$\int_a^b \left(t - \frac{a+b}{2} \right) g(t) dt = 0, \quad (3)$$

then $\mathcal{T}(f, g) \geq 0$.

This result is implied by that of Lupaş [3] who proved the following: if f, g are convex functions on the interval $[a, b]$, then

$$\mathcal{T}(f, g) \geq \frac{12}{(b-a)^3} \int_a^b \left(t - \frac{a+b}{2} \right) f(t) dt \cdot \int_a^b \left(t - \frac{a+b}{2} \right) g(t) dt, \quad (4)$$

with equality when at least one of the functions f and g is a linear function on $[a, b]$ (see also [4]).

In recent years, several bounds for the Čebyšev functional in various cases including convexity assumptions for the functions involved are proved. For other results for convex integrands, see [5–9] and [10, page 256], [10, page 262] where further references are given.

In this work, a sharp bound for the Čebyšev functional of convex or concave functions is proved.

2. The Result

We may begin with the following lemma.

Lemma 1. *If $F, G : [a, b] \rightarrow \mathbb{R}$ are both nonnegative, decreasing (increasing), and convex on $[a, b]$, then $H(t) := F(t)G(t)$ is nonnegative, decreasing (increasing), and convex function on $[a, b]$.*

Proof. The nonnegativity and monotonicity of H follow directly by assumptions. To prove that H is convex, by the monotonicity of F, G and for $x < y$, we have

$$(F(x) - F(y))(G(y) - G(x)) \leq 0, \quad (5)$$

which implies that

$$\begin{aligned} F(x)G(y) + F(y)G(x) \\ \leq F(x)G(x) + F(y)G(y). \end{aligned} \quad (6)$$

Now, by (6) if $\alpha, \beta > 0$ and $\alpha + \beta = 1$, then

$$\begin{aligned} H(\alpha x + \beta y) &= F(\alpha x + \beta y)G(\alpha x + \beta y) \\ &\leq [\alpha F(x) + \beta F(y)][\alpha G(x) + \beta G(y)] \\ &= \alpha^2 F(x)G(x) + \alpha\beta [F(x)G(y) + F(y)G(x)] \\ &\quad + \beta^2 F(y)G(y) \\ &\leq \alpha^2 F(x)G(x) + \alpha\beta [F(x)G(x) + F(y)G(y)] \\ &\quad + \beta^2 F(y)G(y) \\ &= \alpha F(x)G(x) + \beta F(y)G(y) \\ &= \alpha H(x) + \beta H(y), \end{aligned} \quad (7)$$

which proves that H is convex and the proof is completely finished. $\square$

Throughout this paper, let $f, g : [a, b] \rightarrow \mathbb{R}$ be any functions. Define the functions $F, G : [a, b] \rightarrow \mathbb{R}$, such that

$$\begin{aligned} F(t) &:= f(t) - \frac{f(a) + f(b)}{2}, \\ G(t) &:= g(t) - \frac{1}{b-a} \int_a^b g(s) ds, \\ &\quad \forall t \in [a, b]. \end{aligned} \quad (8)$$

Now, we may state our first result as follows.

Theorem 2. Under the assumptions of Lemma 1, one has

$$\mathcal{T}(f, g) \leq \frac{1}{12} (f(b) - f(a))(g(b) - g(a)). \quad (9)$$

The constant $1/12$ is the best possible in the sense that it cannot be replaced by smaller one.

Proof. Firstly, we note that, for any convex function h defined on $[a, b]$, we have

$$h(t) \leq \frac{t-a}{b-a} h(b) + \frac{b-t}{b-a} h(a). \quad (10)$$

Using the identity [10, page 246],

$$\begin{aligned} \mathcal{T}(f, g) &= \frac{1}{b-a} \int_a^b F(t)G(t) dt \\ &= \frac{1}{b-a} \int_a^b \left[f(t) - \frac{f(a) + f(b)}{2} \right] \\ &\quad \times \left[g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right] dt, \end{aligned} \quad (11)$$

and since F and G are two nonnegative monotonic convex functions on $[a, b]$, then by Lemma 1, we have

$$\begin{aligned} \mathcal{T}(f, g) &= \frac{1}{b-a} \int_a^b \left[f(t) - \frac{f(a) + f(b)}{2} \right] \\ &\quad \times \left[g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right] dt \\ &\leq \frac{1}{b-a} \int_a^b \left[\frac{t-a}{b-a} f(b) + \frac{b-t}{b-a} f(a) - \frac{f(a) + f(b)}{2} \right] \\ &\quad \times \left[\frac{t-a}{b-a} g(b) + \frac{b-t}{b-a} g(a) \right. \\ &\quad \left. - \frac{1}{b-a} \int_a^b g(s) ds \right] dt \\ &= \frac{1}{b-a} \left\{ \frac{1}{12} a f(a) g(b) - \frac{1}{12} a f(a) g(a) \right. \\ &\quad - \frac{1}{12} a f(b) g(b) + \frac{1}{12} a f(b) g(a) \\ &\quad - \frac{1}{12} b f(a) g(b) + \frac{1}{12} b f(a) g(a) \\ &\quad \left. + \frac{1}{12} b f(b) g(b) - \frac{1}{12} b f(b) g(a) \right\} \\ &= \frac{1}{12} (f(b) - f(a))(g(b) - g(a)), \end{aligned} \quad (12)$$

which gives the desired inequality (6). To prove the sharpness, assume that (6) holds with constant $C > 0$; that is,

$$\mathcal{T}(f, g) \leq C (f(b) - f(a))(g(b) - g(a)). \quad (13)$$

Let $[a, b] = [0, 1]$; consider that $f(x) = g(x) = x$, $x \in [0, 1]$, so that we have $\int_0^1 f(x)g(x)dx = 1/3$ and $\int_0^1 f(x)dx = \int_0^1 g(x)dx = 1/2$. Making use of (9), we get

$$\mathcal{T}(f, g) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \leq C, \quad (14)$$

which shows that the constant $1/12$ is the best possible in both cases, and thus, the proof is completely finished. $\square$

A parallel result for concave functions may be considered as follows.

Lemma 3. If $F : [a, b] \rightarrow \mathbb{R}$ is nonnegative, decreasing (increasing), and concave on $[a, b]$, while $G : [a, b] \rightarrow \mathbb{R}$ is nonnegative, increasing (decreasing), and concave on $[a, b]$, then $H(t) := F(t)G(t)$ is nonnegative, decreasing (increasing), and concave function on $[a, b]$.

Proof. The nonnegativity and monotonicity of H follow directly by assumptions. To prove that H is concave, by the monotonicity of F , G and for $x < y$, we have

$$(F(x) - F(y))(G(y) - G(x)) \geq 0, \quad (15)$$

which implies that

$$F(x)G(y) + F(y)G(x) \geq F(x)G(x) + F(y)G(y). \quad (16)$$

Now, by (16) if $\alpha, \beta > 0$ and $\alpha + \beta = 1$, then

$$\begin{aligned} H(\alpha x + \beta y) &= F(\alpha x + \beta y)G(\alpha x + \beta y) \\ &\geq [\alpha F(x) + \beta F(y)][\alpha G(x) + \beta G(y)] \\ &= \alpha^2 F(x)G(x) + \alpha\beta [F(x)G(y) + F(y)G(x)] \\ &\quad + \beta^2 F(y)G(y) \\ &\geq \alpha^2 F(x)G(x) + \alpha\beta [F(x)G(x) + F(y)G(y)] \\ &\quad + \beta^2 F(y)G(y) \\ &= \alpha F(x)G(x) + \beta F(y)G(y) \\ &= \alpha H(x) + \beta H(y), \end{aligned} \quad (17)$$

which proves that H is concave and the proof is completely finished. $\square$

Now, we may state the reverse of (9) as follows.

Theorem 4. Under the assumptions of Lemma 3, one has

$$\mathcal{T}(f, g) \geq \frac{1}{12} (f(b) - f(a))(g(b) - g(a)). \quad (18)$$

The constant $1/12$ is the best possible in the sense that it cannot be replaced by smaller one.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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