Research Article

Some Common Fixed Point Results in Rectangular Metric Spaces

Muhammad Arshad,¹ Jamshaid Ahmad,² and Erdal Karapınar³

¹ Department of Mathematics, International Islamic University, H-10, Islamabad 44000, Pakistan

² Department of Mathematics, COMSATS Institute of Information Technology, Chak Shahzad, Islamabad 44000, Pakistan

³ Department of Mathematics, Atılım University, İncek, 06836 Ankara, Turkey

Correspondence should be addressed to Jamshaid Ahmad; jamshaid.jasim@yahoo.com

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We obtain sufficient conditions for the existence of unique common fixed point of $(\psi - \phi)$ -weakly contractive mappings on complete rectangular metric spaces. In the process, we generalize several fixed point results from the literature. We also give an example to illustrate our work.

1. Introduction

Fixed point theorems are very important tools in nonlinear functional analysis. Banach Contraction Mapping Principle is the most frequently cited fixed point theorem in the literature. It asserts that if X is a complete metric space and $T : X \rightarrow X$ is a contraction, that is, there exists $\lambda \in [0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \lambda d(x, y), \quad (1)$$

then T has a unique fixed point. The contraction definition (1) implies that T is uniformly continuous, which is a very strong condition. It is quite natural to ask whether the inequality (1) can be replaced with another inequality which does not force T to be continuous. This question was answered affirmatively by Kannan [1]. A self-mapping $T : X \rightarrow X$ has a unique fixed point in a complete metric space (X, d) if there are nonnegative real numbers α, β with $\alpha + \beta < 1$ such that the following inequality is satisfied for all $x, y \in X$:

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty). \quad (2)$$

In 2000, Branciari [2] introduced the notion of a generalized (rectangular) metric space where the triangle inequality of a metric space was replaced by another inequality, the so-called rectangular inequality. In this paper [2], the author

also extended the celebrated Banach Contraction Mapping Principle in the context of generalized metric spaces. Later, Azam and Arshad [3] obtained sufficient conditions for existence of a unique fixed point of Kannan type mappings in the framework of generalized/rectangular metric spaces. Subsequently, Azam et al. [4] proved an analog of Banach Contraction Principle in the setting of rectangular cone metric spaces. Following this trend, a number of authors focused on rectangular metric spaces and proved the existence and uniqueness of a fixed point for certain type of mappings (see e.g., [5–14]).

Recently, Di Bari and Vetro [15] obtained some common fixed point theorems for mappings satisfying a $(\psi - \phi)$ -weakly contractive condition in rectangular metric spaces. In this paper, we prove several fixed point results in rectangular metric spaces that can be considered as a continuation of [15].

We recall some basic definitions and necessary results on the topic in the literature.

Definition 1. Let X be a nonempty set and let $d : X \times X \rightarrow [0, +\infty)$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y , one has

$$d(x, y) = 0 \quad \text{iff} \quad x = y, \quad (\text{RM1})$$

$$d(x, y) = d(y, x), \quad (\text{RM2})$$

$$d(x, y) \leq d(x, u) + d(u, v) + d(v, y) \quad (\text{RM3})$$

(the rectangular inequality).

Then, the map d is called rectangular (generalized) metric. The pair (X, d) is called a rectangular (generalized) metric space.

To avoid confusion, we prefer to use the term “rectangular metric space” for the spaces under consideration in this paper, because there are some other spaces that are also called generalized metric spaces. We abbreviate a rectangular metric spaces with RMS.

Definition 2 (see [2]). Let (X, d) be a RMS.

- (i) A sequence $\{x_n\}$ is called RMS convergent to $x \in X$ if and only if $d(x_n, x) \rightarrow 0$ as $n \rightarrow +\infty$. In this case, we use the notation $x_n \rightarrow x$.
- (ii) A sequence $\{x_n\}$ in X is called a RMS Cauchy if and only if for each $\epsilon > 0$, there exists a natural number $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $n > m > N(\epsilon)$.
- (iii) A RMS (X, d) is called a RMS complete if every RMS Cauchy sequence is RMS convergent in X .

Definition 3. Let F and g be self-mappings of a nonempty set X .

- (i) A point $x \in X$ is said to be a common fixed point of F and g if $x = Fx = gx$.
- (ii) A point $x \in X$ is called a coincidence point of F and g if $Fx = gx$. And if $w = Fx = gx$, then w is said to be a point of coincidence of F and g .
- (iii) The mappings $F, g : X \rightarrow X$ are said to be weakly compatible if they commute at their coincidence point that is, $Fgx = gFx$ whenever $gx = Fx$.

Lemma 4. Let X be a nonempty set. Suppose that the mappings $F, g : X \rightarrow X$ have a unique coincidence point z in X . If F and g are weakly compatible, then F and g have a unique common fixed point.

Proof. Let $z \in X$ be the coincidence point of $F, g : X \rightarrow X$, that is,

$$Fz = gz = t. \quad (3)$$

Since F and g are weakly compatible, we observe that

$$\begin{aligned} Fz = gz &\implies Fgz = gFz \\ &\iff Ft = gt. \end{aligned} \quad (4)$$

$F, g : X \rightarrow X$ have a unique coincidence point, then $z = t$. Hence, we have $Fz = gz = z$ by (3). $\square$

Let Ψ denote all functions $\phi : [0, \infty) \rightarrow [0, \infty)$ such that

- (a) ϕ is continuous,
- (b) $\phi(t) = 0$ if and only if $t = 0$.

2. Main Result

We start this section with the following theorem.

Theorem 5. Let (X, d) be a Hausdorff RMS and let $F, g : X \rightarrow X$ be self-mappings such that $FX \subset gX$. Assume that (gX, d) is a complete RMS. Suppose that the following condition holds:

$$\psi(d(Fx, Fy)) \leq \psi(M(gx, gy)) - \phi(M(gx, gy)), \quad (5)$$

for all $x, y \in X$ and $\psi, \phi \in \Psi$, where ψ is nondecreasing and

$$M(gx, gy) = \max\{d(gx, gy), d(gx, Fx), d(gy, Fy)\}. \quad (6)$$

Then F and g have a unique coincidence point in X . Moreover, if F and g are weakly compatible, then F and g have a unique common fixed point.

Proof. We first prove that the coincidence point of g and F is unique if it exists. Let z and w be coincidence points of g and F . Thus, there exists some $x, y \in X$ such that $w = Fx = gx$ and $z = Fy = gy$. By (5), we derive that

$$\begin{aligned} \psi(d(w, z)) &= \psi(d(Fx, Fy)) \\ &\leq \psi(M(gx, gy)) - \phi(M(gx, gy)), \end{aligned} \quad (7)$$

where

$$\begin{aligned} M(gx, gy) &= \max\{d(gx, gy), d(gx, Fx), d(gy, Fy)\} \\ &= d(gx, gy) = d(w, z). \end{aligned} \quad (8)$$

Thus, we conclude that $z = w$ by (7).

Remember that g and F are weakly compatible. Since z is the unique coincidence point of g and F , the point z is the unique common fixed point of g and F by Lemma 4.

Now, we will prove the existence of a coincidence point of g and F . Let x_0 be an arbitrary point. Since $FX \subset gX$, we define two iterative sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$y_n = gx_{n+1} = Fx_n \quad (9)$$

for all $n = 0, 1, 2, \dots$. If $y_n = y_{n+1}$ then clearly F and g have a coincidence point in X . Indeed, $y_n = gx_{n+1} = Fx_n = gx_{n+2} = Fx_{n+1} = y_{n+1}$ and x_{n+1} is the desired point. Thus, we assume that $y_n \neq y_{n+1}$, that is, $d(y_n, y_{n+1}) > 0$ for all $n = 0, 1, \dots$. Moreover, if $y_n = Fx_n = Fx_{n+p} = y_{n+p}$, then we choose $x_{n+p+1} = x_{n+1}$, for all $n \geq 0$.

We assert that

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0, \quad \lim_{n \rightarrow \infty} d(y_n, y_{n+2}) = 0. \quad (10)$$

Now from (5), we have

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &= \psi(d(Fx_n, Fx_{n+1})) \leq \psi(M(gx_n, gx_{n+1})) \\ &\quad - \phi(M(gx_n, gx_{n+1})), \end{aligned} \quad (11)$$

where

$$\begin{aligned}
 & M(gx_n, gx_{n+1}) \\
 &= \max \{d(gx_n, gx_{n+1}), d(gx_n, Fx_n), \\
 &\quad d(gx_{n+1}, Fx_{n+1})\} \\
 &= \max \{d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}), \\
 &\quad d(gx_{n+1}, gx_{n+2})\} \\
 &= \max \{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1})\} \\
 &= \max \{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}.
 \end{aligned} \tag{12}$$

If $M(gx_n, gx_{n+1}) = d(y_n, y_{n+1})$, then we have

$$\psi(d(y_n, y_{n+1})) \leq \psi(d(y_n, y_{n+1})) - \phi(d(y_n, y_{n+1})), \tag{13}$$

which implies that $\phi(d(y_n, y_{n+1})) = 0$, and hence $d(y_n, y_{n+1}) = 0$. Then $y_n = y_{n+1}$, which contradicts with the initial assumption. Thus, we have $M(gx_n, gx_{n+1}) = d(y_{n-1}, y_n)$, and hence

$$\begin{aligned}
 \psi(d(y_n, y_{n+1})) &\leq \psi(d(y_{n-1}, y_n)) - \phi(d(y_{n-1}, y_n)) \\
 &\leq \psi(d(y_{n-1}, y_n)).
 \end{aligned} \tag{14}$$

Since ψ is nondecreasing, then $d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n)$ for all $n \geq 0$, that is, the sequence $\{d(y_n, y_{n+1})\}$ is nonincreasing and bounded below. Hence, it converges to a positive number, say $r > 0$. Taking the limit as $n \rightarrow \infty$ in (14), we get

$$\psi(r) \leq \psi(r) - \phi(r), \tag{15}$$

which leads to $\phi(r) = 0$, and hence $r = 0$. Thus,

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0. \tag{16}$$

In this step, we show that the second limit in (5) is also 0. To prove this claim, we set $x = x_{n-1}$ and $y = x_{n+1}$ in (5) and get that

$$\begin{aligned}
 & \psi(d(y_n, y_{n+2})) \\
 &= \psi(d(Fx_n, Fx_{n+2})) \leq \psi(M(gx_n, gx_{n+2})) \\
 &\quad - \phi(M(gx_n, gx_{n+2})),
 \end{aligned} \tag{17}$$

where

$$\begin{aligned}
 & M(gx_n, gx_{n+2}) \\
 &= \max \{d(gx_n, gx_{n+2}), d(gx_n, Fx_n), d(gx_{n+2}, Fx_{n+2})\} \\
 &= \max \{d(gx_n, gx_{n+2}), d(gx_n, gx_{n+1}), d(gx_{n+2}, gx_{n+3})\} \\
 &= \max \{d(y_{n-1}, y_{n+1}), d(y_{n-1}, y_n), d(y_{n+1}, y_{n+2})\}.
 \end{aligned} \tag{18}$$

We consider all possible cases for $M(gx_n, gx_{n+2})$. If $M(gx_n, gx_{n+2}) = d(y_{n+1}, y_{n+2})$, then we have

$$\psi(d(y_n, y_{n+2})) \leq \psi(d(y_{n+1}, y_{n+2})) - \phi(d(y_{n+1}, y_{n+2})). \tag{19}$$

Letting $n \rightarrow \infty$ in (19), the right hand side of (19) tends to 0. Hence, $\lim_{n \rightarrow \infty} \psi(d(y_n, y_{n+2})) = 0$. Since, ψ is continuous, we find that $\lim_{n \rightarrow \infty} d(y_n, y_{n+2}) = 0$. For the case $M(gx_n, gx_{n+2}) = d(y_{n-1}, y_n)$, we get analogously $\lim_{n \rightarrow \infty} d(y_n, y_{n+2}) = 0$.

Let us consider the last case, that is, $M(gx_n, gx_{n+2}) = d(y_{n-1}, y_{n+1})$. The inequality (17) turns into

$$\begin{aligned}
 0 &\leq \psi(d(y_n, y_{n+2})) \leq \psi(d(y_{n-1}, y_{n+1})) - \phi(d(y_{n-1}, y_{n+1})) \\
 &\leq \psi(d(y_{n-1}, y_{n+1})).
 \end{aligned} \tag{20}$$

Therefore, the sequence $\{d(y_n, y_{n+2})\}$ is non-increasing and bounded below. Hence, the sequence $\{d(y_n, y_{n+2})\}$ converges to a number, $s \geq 0$. Taking limit as $n \rightarrow \infty$ in (20), we get

$$0 \leq \psi(d(s)) \leq \psi(s) - \phi(s), \tag{21}$$

which implies that $\phi(s) = 0$, and hence $s = 0$. In other words,

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+2}) = 0. \tag{22}$$

Suppose that $y_n \neq y_m$ for all $m \neq n$ and prove that $\{y_n\}$ is a RMS Cauchy sequence. If possible, let $\{y_n\}$ be not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ with $n_k > m_k \geq k$ such that

$$d(y_{m_k}, y_{n_k}) \geq \epsilon. \tag{23}$$

Furthermore, corresponding to m_k , we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (23). Then,

$$d(y_{m_k}, y_{n_k-1}) < \epsilon. \tag{24}$$

Using (23), (24), and the rectangular inequality (RM3), we have

$$\begin{aligned}
 \epsilon &\leq d(y_{m_k}, y_{n_k}) \\
 &\leq d(y_{n_k}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k-1}) + d(y_{n_k-1}, y_{m_k}) \\
 &\leq d(y_{n_k}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k-1}) + \epsilon.
 \end{aligned} \tag{25}$$

Taking limit as $k \rightarrow \infty$ in (23) and using (16), (22) we get

$$\lim_{k \rightarrow \infty} d(y_{m_k}, y_{n_k}) = \epsilon. \tag{26}$$

Again, using the rectangular inequality (RM3), we obtain

$$\begin{aligned}
 & d(y_{n_k}, y_{m_k}) - d(y_{m_k}, y_{m_k-1}) - d(y_{n_k-1}, y_{n_k}) \\
 &\leq d(y_{n_k-1}, y_{m_k-1}) \\
 &\leq d(y_{n_k-1}, y_{n_k}) + d(y_{n_k}, y_{m_k}) + d(y_{m_k}, y_{m_k-1}).
 \end{aligned} \tag{27}$$

Letting $k \rightarrow \infty$ in (27), and by using (16) and (22) we get

$$\lim_{k \rightarrow \infty} d(y_{n_k-1}, y_{m_k-1}) = \epsilon. \tag{28}$$

Now, we substitute $x = x_{n_k}$ and $y = x_{m_k}$ in (5). Consider

$$\begin{aligned} \psi(d(y_{n_k}, y_{m_k})) &= \psi(d(Fx_{n_k}, Fx_{m_k})) \\ &\leq \psi(M(gx_{n_k}, gx_{m_k})) \\ &\quad - \phi(M(gx_{n_k}, gx_{m_k})) \end{aligned} \quad (29)$$

where

$$\begin{aligned} M(gx_{n_k}, gx_{m_k}) &= \max \{d(gx_{n_k}, gx_{m_k}), d(gx_{n_k}, Fx_{n_k}), \\ &\quad d(gx_{m_k}, Fx_{m_k})\} \\ &= \max \{d(y_{n_k-1}, y_{m_k-1}), d(y_{n_k-1}, y_{n_k}), \\ &\quad d(y_{m_k-1}, y_{m_k})\}. \end{aligned} \quad (30)$$

Clearly, as $k \rightarrow \infty$, we have

$$M(gx_{n_k}, gx_{m_k}) \rightarrow \max \{\epsilon, 0, 0\} = \epsilon. \quad (31)$$

Then letting $k \rightarrow \infty$ in (29), we have

$$0 \leq \psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon). \quad (32)$$

This implies that

$$\phi(\epsilon) = 0, \quad \text{hence } \epsilon = 0, \quad (33)$$

which contradicts the fact that $\epsilon > 0$. Thus, $\{y_n\}$ is a RMS Cauchy sequence. Since (gX, d) is RMS complete, there exists $z \in gX$ such that $y_n \rightarrow z$ as $n \rightarrow \infty$. Let $y \in X$ such that $gy = z$. Applying the inequality (5), with $x = x_n$, we obtain

$$\psi(d(Fx_n, Fy)) \leq \psi(M(gx_n, gy)) - \phi(M(gx_n, gy)), \quad (34)$$

where

$$\begin{aligned} M(gx_n, gy) &= \max \{d(gx_n, gy), d(gx_n, Fx_n), d(gy, Fy)\} \\ &= \max \{d(gx_n, gy), d(gx_n, gx_{n+1}), \\ &\quad d(gy, Fy)\}. \end{aligned} \quad (35)$$

Now, if $M(gx_n, gy) = d(gx_n, gy)$ or $M(gx_n, gy) = d(gx_n, gx_{n+1})$, we have

$$\begin{aligned} d(Fx_n, Fy) &\leq d(gx_n, gy) \quad \text{or} \\ d(Fx_n, Fy) &\leq d(gx_n, gx_{n+1}), \end{aligned} \quad (36)$$

since ψ is nondecreasing. In either case, letting $n \rightarrow \infty$, we get $gx_{n+1} = Fx_n \rightarrow Fy$. Since X is Hausdorff, we deduce that $gy = Fy$. If, on the other hand, $M(gx_n, gy) = d(gy, Fy)$, then taking limit as $n \rightarrow \infty$ in

$$\psi(d(Fx_n, Fy)) \leq \psi(d(gy, Fy)) - \phi(d(gy, Fy)), \quad (37)$$

we get $\phi(d(gy, Fy)) = 0$, hence $d(gy, Fy) = 0$, that is, $gy = Fy$. Let $z = gy = Fy$. Then z is a point of coincidence of F and g . Suppose that there exists $n, p \in \mathbb{N}$ such that $y_n = y_{n+p}$, we can choose in such a way that it is the smallest positive integer satisfying $y_n = y_{n+p}$. We aim to prove that $p = 1$, then

$$gx_{n+1} = Fx_n = Fx_{n+1} = y_{n+1}, \quad (38)$$

and so y_{n+1} is a point of coincidence of g and F . Assume that $p > 1$. This implies that $d(y_{n+p-1}, y_{n+p}) > 0$. Now we have

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &= \psi(d(y_{n+p}, y_{n+p+1})) = \psi(d(Fx_{n+p}, Fx_{n+p+1})) \\ &\leq \psi(M(gx_{n+p}, gx_{n+p+1})) - \phi(M(gx_{n+p}, gx_{n+p+1})), \end{aligned} \quad (39)$$

where

$$\begin{aligned} M(gx_{n+p}, gx_{n+p+1}) &= \max \{d(gx_{n+p}, gx_{n+p+1}), d(gx_{n+p}, Fx_{n+p}), \\ &\quad d(gx_{n+p+1}, Fx_{n+p+1})\} \\ &= \max \{d(gx_{n+p}, gx_{n+p+1}), d(gx_{n+p}, gx_{n+p+1}), \\ &\quad d(gx_{n+p+1}, gx_{n+p+2})\} \\ &= \max \{d(gx_{n+p}, gx_{n+p+1}), d(gx_{n+p+1}, gx_{n+p+2})\} \\ &= \max \{d(y_{n+p-1}, y_{n+p}), d(y_{n+p}, y_{n+p+1})\} \\ &= \max \{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \end{aligned} \quad (40)$$

If $M(gx_{n+p}, gx_{n+p+1}) = d(y_n, y_{n+1})$, then we have

$$\psi(d(y_n, y_{n+1})) = \psi(d(y_{n+p}, y_{n+p+1})) < \psi(d(y_n, y_{n+1})), \quad (41)$$

which is a contradiction. Thus $p = 1$. $\square$

Corollary 6. Let (X, d) be a Hausdorff and complete RMS and let $F, g : X \rightarrow X$ be self-mappings such that $FX \subseteq gX$ satisfying

$$d(Fx, Fy) \leq k \max \{d(gx, gy), d(gx, Fx), d(gy, Fy)\} \quad (42)$$

for all $x, y \in X$ and $0 \leq k < 1$. Then F and g have a unique common fixed point in X .

Proof. Let $\psi(t) = t$ and $\phi(t) = (1-k)t$. Then by Theorem 5, F and g have a unique common fixed point. $\square$

Corollary 7. Let (X, d) be a Hausdorff and complete RMS and let $F, g : X \rightarrow X$ be self-mappings such that $FX \subseteq gX$ satisfying

$$d(Fx, Fy) \leq k(d(gx, gy) + d(gx, Fx) + d(gy, Fy)), \quad (43)$$

for all $x, y \in X$ and $0 \leq k < 1$. Then F and g have a unique common fixed point in X .

Proof. It is obvious that

$$\begin{aligned} d(gx, gy) + d(gx, Fx) + d(gy, Fy) \\ \leq 3 \max \{d(gx, gy), d(gx, Fx), d(gy, Fy)\}. \end{aligned} \quad (44)$$

Let $\psi(t) = t$ and $\phi(t) = (1 - 3k)t$. Then by Theorem 5, F and g have a unique common fixed point. $\square$

Corollary 8. Let (X, d) be a Hausdorff and complete RMS and let $F, g : X \rightarrow X$ be self-mappings such that $FX \subseteq gX$ satisfying

$$d(Fx, Fy) \leq M(gx, gy) - \phi(M(gx, gy)), \quad (45)$$

for all $x, y \in X$ where

$$\begin{aligned} M(gx, gy) \\ = \max \{d(gx, gy), d(gx, Fx), d(gy, Fy)\}. \end{aligned} \quad (46)$$

Then F and g have a unique common fixed point.

Proof. Let $\psi(t) = t$. Then by Theorem 5, F and g have a unique common fixed point. $\square$

Theorem 9. Let (X, d) be a Hausdorff and complete RMS and let $F, g : X \rightarrow X$ be self-mappings such that $FX \subseteq gX$ satisfying

$$\psi(d(Fx, Fy)) \leq \psi(M(gx, gy)) - \phi(M(gx, gy)), \quad (47)$$

for all $x, y \in X$ and $\psi, \phi \in \Psi$, where ψ is nondecreasing and

$$\begin{aligned} M(gx, gy) \\ = \max \left\{ d(gx, gy), d(gy, Fy) \frac{1 + d(gx, Fx)}{1 + d(gy, gy)} \right\}. \end{aligned} \quad (48)$$

Then F and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $FX \subseteq gX$, we define the sequence $\{x_n\} \subset X$ as follows $gx_n = Fx_{n-1}$ for all $n \geq 1$. Assume that $gx_n \neq gx_{n+1} = Fx_n$ for all $n \geq 0$. Now from (5), we get

$$\begin{aligned} \psi(d(gx_n, gx_{n+1})) &= \psi(d(Fx_{n-1}, Fx_n)) \\ &\leq \psi(M(gx_{n-1}, gx_n)) \\ &\quad - \phi(M(gx_{n-1}, gx_n)), \end{aligned} \quad (49)$$

where

$$\begin{aligned} M(gx_{n-1}, gx_n) \\ = \max \left\{ d(gx_{n-1}, gx_n), \right. \\ \left. d(gx_n, Fx_n) \frac{1 + d(gx_{n-1}, Fx_{n-1})}{1 + d(gx_{n-1}, gx_n)} \right\} \end{aligned} \quad (50)$$

$$\begin{aligned} M(gx_{n-1}, gx_n) \\ = \max \{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\}. \end{aligned}$$

The rest of the proof is the same as the proof of Theorem 5. $\square$

By Λ , we denote the class of functions $f : [0, \infty) \rightarrow [0, \infty)$ satisfying the following.

- (a) f is Lebesgue integrable function on each compact subset of $[0, \infty)$,
- (b) $\int_0^\varepsilon f(s)ds > 0$ for any $\varepsilon > 0$.

Theorem 10. Let (X, d) be a Hausdorff RMS and let $F, g : X \rightarrow X$ be self-mappings such that $FX \subset gX$. Assume that (gX, d) is a complete RMS and that the following condition holds:

$$\begin{aligned} \int_0^{d(Fx, Fy)} f(r) dr \\ \leq \int_0^{M(gx, gy)} f(r) dr - \int_0^{M(gx, gy)} h(r) dr \end{aligned} \quad (51)$$

for all $x, y \in X$ and $f, h \in \Lambda$ where

$$M(gx, gy) = \max \{d(gx, gy), d(gx, Fx), d(gy, Fy)\}. \quad (52)$$

Then F and g have a unique common fixed point.

Proof. Let $\psi(r) = \int_0^r f(v)dv$ and $\phi(r) = \int_0^r h(v)dv$. Then ψ and ϕ are function in Ψ . By Theorem 5, F and g have a unique common fixed point. $\square$

Theorem 11. Let (X, d) be a Hausdorff RMS and let $F, g : X \rightarrow X$ be self-mappings such that $FX \subset gX$. Assume that (gX, d) is a complete RMS and that the following condition holds:

$$\int_0^{d(Fx, Fy)} f(s) ds \leq \lambda \int_0^{M(gx, gy)} f(s) ds \quad (53)$$

for all $x, y \in X$ and $f \in \Lambda$ and some $0 \leq \lambda < 1$, where

$$M(gx, gy) = \max \{d(gx, gy), d(gx, Fx), d(gy, Fy)\}. \quad (54)$$

Then F and g have a unique common fixed point.

Proof. Let $h(r) = (1 - \lambda)f(r)$. Then by Theorem 10, F and g have a unique common fixed point. $\square$

Theorem 12. Let (X, d) be a Hausdorff RMS and let $F, g : X \rightarrow X$ be a self-mappings such that $FX \subset gX$. Assume that (gX, d) is a complete RMS and that the following condition holds:

$$\int_0^{d(Fx, Fy)} f(r) dr \leq \int_0^{M(gx, gy)} f(r) dr - \int_0^{M(gx, gy)} h(r) dr \quad (55)$$

for all $x, y \in X$ and $f, h \in \Lambda$ where

$$M(gx, gy) = \max \left\{ d(gx, gy), d(gy, Fy) \frac{1 + d(gx, Fx)}{1 + d(gy, gy)} \right\}. \quad (56)$$

Then F and g have a unique common fixed point.

Proof. Let $\psi(r) = \int_0^r f(v)dv$ and $\phi(r) = \int_0^r h(v)dv$. Then ψ and ϕ are function in Ψ . By Theorem 9, F and g have a unique common fixed point. $\square$

Example 13. Let $X = A \cup B$, where $A = \{1/2, 2/3, 3/4, 4/5\}$ and $B = [1, 3]$. Define the generalized metric d on X as follows:

$$d\left(\frac{1}{2}, \frac{2}{3}\right) = d\left(\frac{3}{4}, \frac{4}{5}\right) = 0.2, \quad (57)$$

$$d\left(\frac{1}{2}, \frac{4}{5}\right) = d\left(\frac{2}{3}, \frac{3}{4}\right) = 0.3, \quad (58)$$

$$d\left(\frac{1}{2}, \frac{3}{4}\right) = d\left(\frac{2}{3}, \frac{4}{5}\right) = 0.6, \quad (59)$$

$$\left(\frac{1}{2}, \frac{1}{2}\right) = d\left(\frac{2}{3}, \frac{2}{3}\right) = d\left(\frac{3}{4}, \frac{3}{4}\right) = d\left(\frac{4}{5}, \frac{4}{5}\right) = 0, \quad (60)$$

$$d(x, y) = |x - y| \quad \text{if } x, y \in B \text{ or } x \in A, y \in B \text{ or } x \in B, y \in A. \quad (61)$$

It is easy to show that d does not satisfy the triangle inequality on A . Indeed,

$$0.6 = d\left(\frac{1}{2}, \frac{3}{4}\right) \geq d\left(\frac{1}{2}, \frac{2}{3}\right) + d\left(\frac{2}{3}, \frac{3}{4}\right) = 0.2 + 0.3 = 0.5. \quad (62)$$

Thus (RM3) holds, so d is a rectangular metric. Notice that $(X|_B, d)$ is usual metric space, and hence it is Hausdorff. On the other hand, each singleton is closed and open in $(X|_A, d)$, and hence (X, d) is Hausdorff rectangular metric space.

Let $F, g : X \rightarrow X$ be defined as follows:

$$Fx = \begin{cases} \frac{4}{5} & \text{if } x \in [1, 3], \\ \frac{3}{4} & \text{if } x \in \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\}, \\ \frac{2}{3} & \text{if } x = \frac{4}{5}, \end{cases} \quad (63)$$

$$gx = \begin{cases} \frac{2}{3} & \text{if } x \in [1, 3], \\ \frac{3}{4} & \text{if } x \in \left\{\frac{1}{2}, \frac{3}{4}\right\}, \\ \frac{4}{5} & \text{if } x = \frac{2}{3}, \\ \frac{1}{2} & \text{if } x = \frac{4}{5}. \end{cases}$$

Define $\psi(t) = t$ and $\phi(t) = t/3$. Then F and g satisfy the condition of Theorem 5 and have a unique common fixed point of X , that is, $x = 3/4$.

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