

## Conference Paper

# Rigorous Study of the Unbinding Transition of Biomembranes and Strings from Morse Potentials

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The purpose is an exact study of the unbinding transition from two interacting manifolds (strings or bilayer membranes). These systems have similar scaling behavior, and then it is sufficient to consider only the strings' problem. We assume that the manifolds interact via a realistic potential of Morse type. To this end, the use is made of the transfer matrix method, based on the resolution of a Schrödinger equation. We first determine the associated bound states and energy spectrum. Second, the exact ground state energy gives the free energy density, from which we extract the expression of the unbinding temperature. Third, we determine the contact probability between manifolds, from which we compute the (diverging) average separation and roughness of the manifolds. It is found that their critical behavior is close to that obtained using Field-Theoretical Renormalization-Group. The conclusion is that these analytical studies reveal that the Morse potential is a good candidate for the study of the unbinding phenomenon within manifolds. Finally, the discussion is extended to generalized Morse potential.

## 1. Introduction

Unbinding is a phenomenon that occurs in soft elastic manifolds, including strings and bilayer membranes [1–3]. The mechanism that rules adhesion and unbinding of membranes is analog to that of governing interfacial wetting [4] and adsorption-desorption transitions of polymers [5]. The common feature of these interfacial phenomena is that they have a behavior similar to that of 1-dimensional lines or strings of finite tension [6], such as ledges on crystal surfaces, stretched (or directed) polymers, and vortex lines in superconductors.

The major result is that renormalization-group (RG) calculations predicted that fluid membranes possess similar scaling properties as strings in the vicinity of the critical potential depth [7]. Such a property will be the key for the investigation of the unbinding transition in bilayer membranes.

The unbinding transition from strings and bilayer membranes is often driven by steric-shape fluctuations [8] whose

amplitude increases with temperature. These undulation forces not only act between lipid bilayers in the  $L_\alpha$ -phase but also prevent vesicles from coagulation and stabilize emulsions [9, 10]. For a membrane pair, the induced fluctuation interactions are balanced, at short separation, by repulsive hydration [11] and attractive van der Waals forces [12]. The hydration potential (per unit area) acts at small separations of the order of 1 nm. Its amplitude (hydration pressure),  $P_h$ , and potential range,  $\lambda_h$ , are about  $P_h \approx 0.2 \text{ J/m}^2$  and  $\lambda_h \approx 0.3 \text{ nm}$ , respectively. While the attractive van der Waals potential originates from polarizabilities of lipid and water molecules. The associated amplitude (Hamaker constant),  $W$ , is in the range  $10^{-22}$ – $10^{-21} \text{ J}$ , and the bilayer thickness is about  $\delta \approx 4 \text{ nm}$ . The total interaction potential is then the sum of these three interactions. The bilayer membrane remains stable at the minimum of the potential, provided that the potential depth is comparable to the thermal energy. In particular, this

depth depends on the value of amplitude  $W$  of the direct van der Waals energy. In fact, the Hamaker constant  $W$  may be varied changing the polarizability of the aqueous medium.

In a seminal paper, Lipowsky and Leibler [13] demonstrated that there exists a certain threshold beyond which the van der Waals attractive interactions are sufficient to bind the membranes together, while below this characteristic amplitude, the membrane undulations dominate the attractive forces, and then the membranes separate completely. In particular, it was found [2, 3, 13] that when the unbinding temperature,  $T_c$ , is approached from below, the mean-separation between the two membranes,  $\langle l \rangle$ , and their roughness,  $\xi_{\perp}$ , diverge according to  $\langle l \rangle \sim \xi_{\perp} \sim (T_c - T)^{-\psi}$ , where the roughness exponent  $\psi$  is very close to 1. Afterwards, the results were extended to charged membranes [13], where the unbinding is entirely controlled by the ionic concentration of the aqueous medium. Of course, the latter depends on the nature of the lipid system. The theoretical predictions for charged bilayer membranes were tested in some experiment on DPPC in  $\text{CaCl}_2$  solutions [14]. Finally, in all cases, beside  $\psi$ , one defines other exponents [1], such as roughness, roughening, and contact exponents.

An interesting alternative tool is the transfer matrix method (TMM) usually encountered in quantum mechanics [15–17] and critical phenomena [18, 19]. Such an approach was successfully applied to bilayer membranes [20, 21]. Very recently, TMM was applied to the unbinding transition with single and double well potentials [21].

In this paper, the goal is the application of TMM to some popular potential, that is, the Morse potential (MP) [22]. The latter was introduced to describe vibrations of covalent diatomic molecules of very weak van der Waals complexes [23, 24]. The advantage of MP is that it is not singular and simply presents as a difference of two exponential tails (see hereinafter).

We recall that MP was applied recently to unbinding membranes [25] using the traditional variational method. This use can be motivated by the fact that this potential reflects the two essential features described above, namely, a repulsion at short distance and an attraction at intermediate distance. It is characterized by only three parameters, which are its depth (dissociation energy),  $D$ , its range  $\alpha^{-1}$ , and the position of its minimum point,  $l_m$  (see hereinafter). The shape of such a potential models well the real interaction potential discussed above. Another virtue of this potential is that its associated energy spectrum is discrete, and then all its eigenfunctions are bound states. The existence of these states makes it a good candidate for the study of the unbinding transition from strings and membranes. The adjustable parameters  $D, \alpha^{-1}$ , and  $l_m$  can be related to those of the real potential. More precisely, we have functional relations of types  $x = g_x(W, \delta, \kappa, P_h, \lambda_h)$ , with  $x = D, \alpha, l_m$ , where  $\kappa$  is the bending modulus. In addition, MP parameters are temperature dependent.

The variational method is a mean-field theory that underestimates strong fluctuations near criticality. In this paper, however, we present an exact study of the unbinding transition with MP using TMM. In particular, we compute exactly

the unbinding temperature, contact probability, from which we extract all critical properties, namely, the average separation and roughness. Therefore, the unbinding exponents are exactly known. The idea will consist to determine all critical behaviors for strings with a MP; then, the results will be translated to fluid membranes since the two systems are governed by the same scaling.

This paper is organized as follows. In Section 2, we present the general string model. A rigorous study of the unbinding transition is presented in Section 3. Finally, some concluding remarks are drawn in the last section.

## 2. String Model

Strings are commonly defined as one-dimensional lines whose shape fluctuations are governed by finite tension [20]. Consider two interacting strings that move on a two-dimensional space. We assume that, on average, they fluctuate around a line reference, say  $x$ -axis, and in addition, their associated elongations remain perpendicular to this axis. The conformation of strings can be described by the local field separation,  $l(x) > 0$ , which is perpendicular to the line reference. The statistical mechanics of strings is based on the Hamiltonian:

$$H[l] = \int_0^L \left[ \frac{\sigma}{2} \left( \frac{dl}{dx} \right)^2 + V(l) \right] dx, \quad (1)$$

where  $L$  denotes the string length,  $\sigma$  is the effective tension, and  $V(l)$  represents the interaction potential, whose form will be specified hereinafter.

In the thermodynamic limit, that is, for  $L \rightarrow \infty$ , the statistical properties of model (1) can be studied using TMM that is, based on the resolution of a Schrödinger equation type [15–17]:

$$-\frac{(k_B T)^2}{2\sigma} \frac{d^2 \varphi_n}{dl^2} + V(l) \varphi_n = E_n \varphi_n, \quad (2)$$

where  $T$  is the absolute temperature and  $k_B$  is the Boltzmann's constant. In the above differential equation, the parameters  $E_n$  and  $\varphi_n$  denote the set of eigenvalues and wave functions, respectively. The eigenvalues  $E_n$ 's are ordered in such a way that  $E_0 \leq E_1 \leq E_2 \leq \dots$ . The ground state eigenvalue  $E_0$  defines the free energy density,  $f$ , that is,  $f = E_0$ , while the corresponding eigenvector,  $\varphi_0(l)$ , determines the probability distribution,  $P(l)$ . In fact,  $P(l)dl$  represents the probability of finding two strings at a separation between  $l$  and  $l + dl$ . The probability distribution is then given by

$$P(l) = \frac{|\varphi_0(l)|^2}{\int |\varphi_0(l)|^2 dl}. \quad (3)$$

With the help of this distribution, one can calculate, for example, its first and second moments,

$$\langle l \rangle = \int l P(l) dl, \quad \langle l^2 \rangle = \int l^2 P(l) dl. \quad (3a)$$

The string roughness is given by

$$\xi_{\perp} = \left[ \langle l^2 \rangle - \langle l \rangle^2 \right]^{1/2}. \quad (3b)$$

Before studying their unbinding transition, it will be convenient to recall some backgrounds concerning the general statistical properties of the string pair.

First, we recall the definition of the roughness exponent. Consider a fluctuating manifold, and notice that the latter makes large transverse excursions from its average position. The manifold is rough if the typical size,  $\xi_{\perp}$ , of its transverse excursions grows with its lateral size,  $\xi_{\parallel}$ . It is admitted that the two lengths obey the following scaling relation [1–3]:

$$\xi_{\perp} \sim \xi_{\parallel}^{\zeta}. \quad (4)$$

This relation then defines the roughness exponent  $\zeta$ . The latter crucially depends on the nature of the considered manifold, for example,  $\zeta = 1/2$ , for string, and  $\zeta = 1$ , for almost flat fluid membranes.

The second exponent of our interest is the unbinding exponent. To definite it, we consider two interacting manifolds of mean separation  $\langle l \rangle$ . When they exhibit an unbinding transition, this average separation diverges according to

$$\langle l \rangle \sim \left( \frac{T_c - T}{T_c} \right)^{-\psi}. \quad (5)$$

This behavior defines the unbinding exponent  $\psi$ , where  $T_c$  is the transition temperature. Also, this exponent generally depends on the nature of the interaction potential experienced by the manifolds. When the unbinding transition is driven by the shape fluctuations, one has  $\langle l \rangle \sim \xi_{\perp}$ . In this case, the unbinding exponent  $\psi$  coincides with the roughening exponent  $\nu_{\perp}$  (we do not recall here), except for special cases as discussed in [1].

The last and interesting exponent to recall is the contact exponent. This characterizes the singular behavior at roughening and unbinding transitions of the contact probability of two adjacent manifolds. The latter scales as [1]

$$P \sim \xi_{\parallel}^{-\zeta_0} \sim \xi_{\perp}^{-\zeta_0/\zeta}. \quad (6)$$

Close to criticality, this scaling relation defines the contact exponent  $\zeta_0$ . Also, the value of this exponent primarily depends on the nature of the interaction potential between manifolds.

On the other hand, we can put the probability distribution on the following scaling form:

$$P(l) = \xi_{\perp}^{-1} \Omega \left( \frac{l}{\xi_{\perp}} \right). \quad (7)$$

Here, the explicit factor  $\xi_{\perp}^{-1}$  arises from normalization. To recover relationship (6), the scaling function  $\Omega(s)$  must behave as

$$\Omega(s) \sim s^{-1+\zeta_0/\zeta}, \quad \text{for small } s \text{ (with } \zeta_0 > 0). \quad (8)$$

This scaling behavior remains valid as long as the mean separation is such that  $\langle l \rangle \sim \xi_{\perp}$ .

In the following section, we shall determine all these critical properties in an exact way for MPs.

### 3. Exact Study of Unbinding Transition

**3.1. Morse Potential Form.** First, we recall the expression of the used one-dimensional MP [21–23]:

$$V(l) = D \left[ e^{-2\alpha(l-l_m)} - 2e^{-\alpha(l-l_m)} \right], \quad (9)$$

where  $D$ ,  $l_m$ , and  $\alpha$  are positive adjustable parameters. The Morse potential has a minimum value of  $-D$  at  $l = l_m$  and vanishes at  $l = \infty$ . The particularity of this potential is that at  $l = 0$ ,  $V(0)$  has finite value of  $D(e^{2\alpha l_m} - 2e^{\alpha l_m})$ , which is positive if only if  $\alpha l_m > \ln 2$ . We recall that the MP energy function is frequently used to describe the vibration motion of diatomic molecules. In this case,  $D$  is the dissociation energy (the potential depth is  $-D$ ), and  $l_m$  accounts for the equilibrium internuclear distance of a diatomic molecules. The former is of the order of  $10^{-5}$  J/m<sup>2</sup> (for fluid membranes), and the second is about ten nanometers. The value  $V(0)$  is always positive for covalent diatomic molecules [22].

In Figure 1, we report MP versus distance for various values of parameters ( $D, l_m, \alpha$ ).

**3.2. Resolution of the Schrödinger Equation.** First, we start by recalling the useful Kato's mathematical theorem [26].

Suppose that the potential  $V(x)$  of the Schrödinger operator  $H = -\Delta + V(x)$  is bounded, and that

$$\lim_{x \rightarrow \infty} V(x) = 0. \quad (10)$$

Then  $H$  has no positive eigenvalues; that is,  $H\varphi = E\varphi$ , where  $E > 0$ ,  $\varphi \in L^2(\mathbf{R}^n)$ ; then,  $\varphi = 0$ .

Therefore, all eigenfunctions of the Schrödinger equation are bound states, of negative eigenvalues. The eigenvalues spectrum is then discrete.

As remark [26], the point  $E = 0$  can be an eigenvalue of the operator  $H = -\Delta + V(x)$  even when  $V(x) \in C_0^{\infty}(\mathbf{R}^n)$ .

The considered MP effectively satisfies the above theorem conditions. First, it is locally bounded, since  $V(l) < V(0)$ , for  $l < l_0$ , and  $|V(l)| < D$ , for  $l > l_0$ , where  $l_0 = l_m - \alpha^{-1} \ln 2$  is the unique positive zero of this potential. Second, the limit (10) is also fulfilled due to the presence of exponential tails being the potential expression (9). In addition, according to the above remark, this potential may have a ground state since it is infinitely differentiable. Thus, the system can undergo an unbinding transition that takes place when a bound state appears. The conclusion is that the Schrödinger equation with a Morse potential has only bound states as solutions and a discrete spectrum.

Now to solve the differential equation (2), we first note that this is very similar to the traditional Schrödinger equation:

$$-\frac{\hbar^2}{2\mu} \frac{d^2 \varphi_n}{dl^2} + V(l) \varphi_n = E_n \varphi_n, \quad (11)$$

with  $\mu$  is the reduced mass and  $\hbar$  is the renormalized Planck's constant, making the substitution

$$\frac{(k_B T)^2}{\sigma} \longrightarrow \frac{\hbar^2}{\mu}. \quad (11a)$$

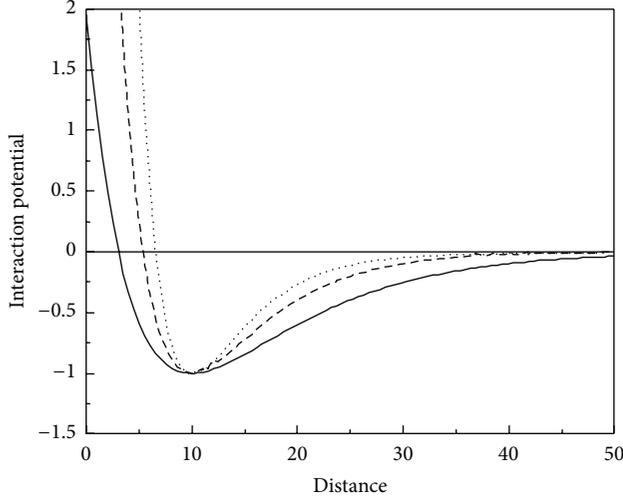


FIGURE 1: Reduced Morse potential,  $V(l)/D$ , versus distance (in nanometer unit), at fixed minimum point ( $l_m = 10$  nm), for three values of parameter  $\alpha$ : 0.1 (solid line), 0.15 (dashed line), and 0.2 (dotted line).

The Schrödinger equation has been exactly solved [27], and it will be convenient to recall briefly the essential steps of its resolution.

The first step consists of rewriting it in terms of a new variable  $r$ ,

$$-\frac{d^2\varphi_n}{dr^2} + V(r)\varphi_n = \varepsilon_n\varphi_n, \quad (12)$$

with the dimensionless quantities

$$r = \alpha l, \quad r_e = \alpha l_m, \quad (12a)$$

$$\lambda = \frac{\sqrt{2\mu D}}{\alpha\hbar}, \quad \varepsilon_n = \frac{2\mu}{\alpha^2\hbar^2}E_n, \quad (12b)$$

and the rescaled Morse potential

$$V(r) = \lambda^2 [e^{-2(r-r_e)} - 2e^{-(r-r_e)}]. \quad (12c)$$

The minimal value of  $V(r)$  is  $-\lambda^2$ , reached for  $r = r_e$ . The reduced energies  $\varepsilon_n$  (or  $E_n$ ) are negative for bound states.

In the second step, the variable

$$z = 2\lambda e^{-(r-r_e)} \quad (13)$$

is introduced. It was found in [27] that the orthonormalized bound state wave functions  $\varphi_n(r)$  are as follows:

$$\varphi_n(r) = A_n z^{\lambda-n-1/2} e^{-z/2} L_n^{2\lambda-2n-1}(z), \quad (14)$$

with the normalization constant

$$A_n = [\Gamma(n+1)\Gamma(2\lambda-n)]^{-1/2} \quad (14a)$$

and the orthogonal Laguerre polynomial

$$L_n^\gamma(z) = z^{-\gamma} e^z \frac{d^n}{dz^n} (z^{n+\gamma} e^{-z}). \quad (14b)$$

The associated eigenenergies are

$$\varepsilon_n = -\left(\lambda - n - \frac{1}{2}\right)^2. \quad (15)$$

These were obtained using TMM [27]. The existence of solutions to the Schrödinger equation requires that the quantum number  $n$  must be smaller than  $\lambda$ . Therefore, the vibrational quantum number  $n$  is

$$n = 0, 1, 2, \dots, n_{\max}, \quad (15a)$$

where  $n_{\max}$  is the largest integer less than  $\lambda$ .

In terms of the original variables and using substitution (11a), we find that the energies  $E_n$  (per unit length) write

$$E_n = -\frac{D}{\lambda^2} \left(\lambda - n - \frac{1}{2}\right)^2, \quad (16)$$

with the notation

$$\lambda = \frac{\sqrt{2D\sigma}}{\alpha k_B T}. \quad (16a)$$

**3.3. Probability Density.** Applying the general formula (14) to the particular value  $n = 0$  and using the fact that  $L_0^\gamma(z) = 1$ , whatever are the values of indice  $\gamma$  and argument  $z$ , we find that the ground state is exactly given by

$$\varphi_0(l) = A_0 e^{-\alpha(\lambda-1/2)(l-l_m) - \lambda e^{-\alpha(l-l_m)}}. \quad (17)$$

This gives the probability density

$$P(l) = K_0 e^{-\alpha(2\lambda-1)(l-l_m) - 2\lambda e^{-\alpha(l-l_m)}}, \quad (18)$$

with the normalization constant

$$K_0 = \frac{\alpha}{(2\lambda)^{1-2\lambda} \gamma(2\lambda-1, 2\lambda e^{\alpha l_m})}, \quad (18a)$$

with the condition  $\lambda > 1/2$ . We have used the established formula

$$\int_0^c y^{\nu-1} e^{-\mu y} (\ln y)^m dy = \frac{d^m}{d\nu^m} [\mu^{-\nu} \gamma(\nu, c\mu)], \quad (18b)$$

$$m = 0, 1, \dots, \operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0,$$

where  $\gamma(\nu, x)$  is the incomplete gamma function [28].

With the help of the determined probability, we compute quantities of our interest, such as  $\langle l \rangle$  and  $\xi_\perp = [\langle l^2 \rangle - \langle l \rangle^2]^{1/2}$ .

**3.4. Average Separation.** We start from its definition

$$\langle l \rangle = \int_0^\infty l P(l) dl = l_m + \langle l - l_m \rangle, \quad (19)$$

with

$$\langle l - l_m \rangle = K_0 \int_{-l_m}^\infty x e^{-\alpha(2\lambda-1)x - 2\lambda e^{-\alpha x}} dx. \quad (19a)$$

To calculate this integral, we introduce the new variable  $y = e^{-\alpha x}$ . Then, we have

$$\langle l - l_m \rangle = -\frac{K_0}{\alpha^2} \int_0^{e^{\alpha l_m}} y^{2\lambda-2} \ln ye^{-2\lambda y} dy. \quad (19b)$$

Using the general formula (18b), we find

$$\begin{aligned} \langle l \rangle &= l_m - \frac{1}{\alpha(2\lambda)^{1-2\lambda} \gamma(2\lambda-1, 2\lambda e^{\alpha l_m})} \\ &\times \frac{d}{d\nu} [\mu^{-\nu} \gamma(\nu, \mu e^{\alpha l_m})] \Big|_{\mu=2\lambda, \nu=2\lambda-1}. \end{aligned} \quad (20)$$

The above exact expression has a mathematical sense only if  $\lambda > 1/2$ .

**3.5. String Roughness.** The starting point is its definition:

$$\xi_{\perp}^2 = \int_0^{\infty} l^2 P(l) dl - \left( \int_0^{\infty} l P(l) dl \right)^2. \quad (21)$$

Using the same techniques yields

$$\begin{aligned} \xi_{\perp}^2 &= \frac{1}{\alpha^2(2\lambda)^{1-2\lambda} \gamma(2\lambda-1, 2\lambda e^{\alpha l_m})} \\ &\times \frac{d^2}{d\nu^2} [\mu^{-\nu} \gamma(\nu, \mu e^{\alpha l_m})] \Big|_{\mu=2\lambda, \nu=2\lambda-1} \\ &- \left( \frac{1}{\alpha(2\lambda)^{1-2\lambda} \gamma(2\lambda-1, 2\lambda e^{\alpha l_m})} \right. \\ &\times \left. \frac{d}{d\nu} [\mu^{-\nu} \gamma(\nu, \mu e^{\alpha l_m})] \Big|_{\mu=2\lambda, \nu=2\lambda-1} \right)^2. \end{aligned} \quad (22)$$

This exact expression makes sense if and only if  $\lambda > 1/2$ .

**3.6. Discussion and Scaling.** Let us first note that the system undergoes an unbinding transition if the ground state energy,  $E_0 = -(D/\lambda^2)(\lambda - 1/2)^2$ , vanishes. This produces for the typical value  $\lambda_c = 1/2$ . This value then defines the unbinding temperature

$$T_c = \frac{2\sqrt{2D\sigma}}{k_B\alpha} \quad (23)$$

that naturally depends on the parameters of the problem, which are  $\sigma$ ,  $D$ , and  $\alpha$ . At this temperature, the system exhibits a transition from the bound state ( $T < T_c$  or  $D > D_c$ ) to the unbound one ( $T > T_c$  or  $D < D_c$ ). The above definition then defines a critical line in the plane of variables  $T$  and  $\sigma$  (Figure 2), which separates unbound and bound states domains.

The above expression also defines a critical potential amplitude,  $D_c$ , at fixed temperature,

$$D_c = \frac{\alpha^2(k_B T)^2}{8\sigma}. \quad (24)$$

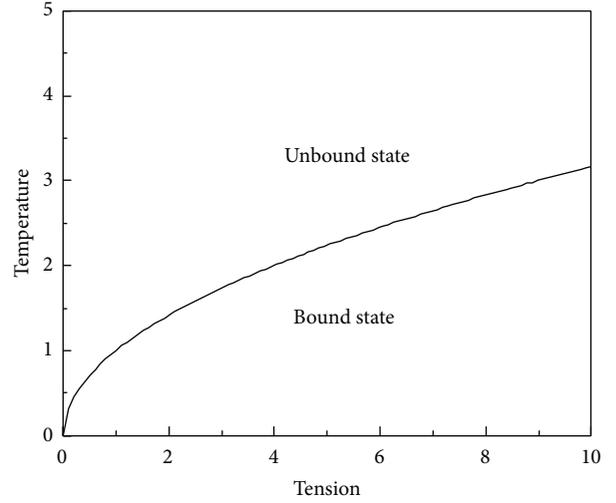


FIGURE 2: Critical unbinding line in the  $(\sigma, T)$ -plane, where the parameters  $D$  and  $\sigma$  are fixed to some values.

In particular, this expression indicates that the unbinding transition is more accentuated for those high tension strings (and high-bending modulus fluid membranes).

Now, come back to the probability distribution expression (18), and notice that its discussion depends on the values of the parameter  $\lambda$  (at fixed  $\alpha$ ). First, the condition that the probability must fail at infinity ( $l \rightarrow \infty$ ) requires the inequality  $\lambda > 1/2$ . Second, this probability reaches its maximal value for the particular distance

$$l_{\max} = l_m + \frac{1}{\alpha} \ln \left( \frac{2\lambda}{2\lambda-1} \right). \quad (25)$$

The fact that  $l_{\max} > l_m$  expresses the tendency of the system to unbind. But as the temperature transition is approached from above ( $T \rightarrow T_c^-$  or  $\lambda \rightarrow \lambda_c^+$ ), the maximum point is shifted towards its higher values, and the probability distribution then becomes more and more boarder; hence the average or mean string position as well as the string roughness increase (see discussion hereinafter). At the transition temperature, the probability vanishes, in agreement with behavior (6) in the  $\xi_{\perp} \rightarrow \infty$  limit.

In Figure 3, we report the probability distribution upon distance for some values of  $\lambda$  close to the critical value  $\lambda_c = 1/2$ .

Consider now the mean separation defined in (20). Using the limit  $\gamma(\nu, x) \sim \nu^{-1} x^{\nu}$  ( $\nu = 2\lambda - 1 \rightarrow 0$ ) [28], we find that near the transition temperature, this average position scales exactly as

$$\langle l \rangle = \frac{1}{\alpha} \frac{1}{2\lambda-1} = \frac{1}{\alpha} \left( \frac{T_c - T}{T_c} \right)^{-1}, \quad T \rightarrow T_c^-. \quad (26)$$

Therefore the unbinding exponent  $\psi$  has the exact value

$$\psi = 1 \quad (26a)$$

that is very close to the computed one by RG [13].

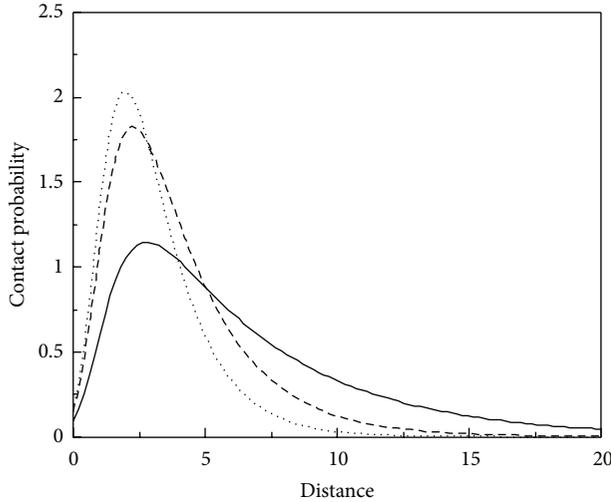


FIGURE 3: Contact probability versus renormalized distance,  $\alpha l$ , for some values of  $\lambda$  : 0.6 (solid line), 0.7 (dashed line), and 0.8 (dotted line). These curves are drawn choosing  $\alpha = 0.1 \text{ nm}^{-1}$  and  $l_m = 10 \text{ nm}$ . Remark that the probability is more and more boarder as  $\lambda$  approaches its critical value  $\lambda_c = 1/2$ .

On the other hand, from relation (22), we find that the string roughness is given by

$$\xi_{\perp} = \frac{1}{\alpha} \frac{1}{2\lambda - 1} = \frac{1}{\alpha} \left( \frac{T_c - T}{T_c} \right)^{-1}, \quad T \rightarrow T_c^- \quad (27)$$

Therefore, asymptotically, we have the exact equality  $\xi_{\perp} = \langle l \rangle$ .

We show that the probability distribution obeys the exact scaling law

$$P(l) = \frac{1}{\xi_{\perp}} \Omega \left( \frac{l - l_m}{\xi_{\perp}} \right), \quad (28)$$

with the scaling function

$$\Omega(s) = e^{-s - e^{-\alpha \xi_{\perp} s}}. \quad (28a)$$

Its behavior as  $s \rightarrow 0$  is  $\Omega(s) = e^{-1} s^0 = e^{-1}$ . Thus, the associated contact exponent exactly reads

$$\zeta_0 = \zeta. \quad (28b)$$

Then,  $\zeta_0 = 1/2$ , for strings, and  $\zeta_0 = 1$ , for fluid membranes. Incidentally, the above equality is that found for those interfaces characterized by a Gaussian probability distribution  $\xi_{\perp}^{-1} \exp(-l^2/2\xi_{\perp}^2)$  for the fluctuating field  $l$ .

Near criticality, the maximum point of the probability distribution is

$$l_{\max} = l_m - \frac{1}{\alpha} \ln \left( \frac{T_c - T}{T_c} \right), \quad T \rightarrow T_c^- \quad (28c)$$

Thus, the shift  $l_{\max} - l_m$  diverges logarithmically as the unbinding temperature is approached. Incidentally, this shift plays the role of the wetting-layer size [4].

Finally, the free energy density scales as

$$f = -\frac{D}{\alpha^2} \frac{1}{\xi_{\perp}^2} = -D \left( \frac{T_c - T}{T_c} \right)^2, \quad T \rightarrow T_c^- \quad (29)$$

The corresponding disjoining pressure,  $\Pi$ , can be obtained from the standard definition  $\Pi = \partial f / \partial \xi_{\perp}$ . Then, we have  $\Pi = 2(D/\alpha^2)\xi_{\perp}^{-3}$ . The latter can be interpreted as the pressure required to maintain the two strings at the average distance  $\langle l \rangle = \xi_{\perp}$ . Naturally, this pressure is directly proportional to the potential amplitude  $D$ . This result reveals that the unbinding phenomenon is entirely governed by the shape fluctuations of strings and bilayer membranes.

## 4. Conclusions

The goal of this paper is an analytical study of the unbinding transition undergone by strings or bilayer membranes, with a potential of Morse type. Using the transfer matrix method, based on the resolution of a Schrödinger equation, their solutions were found to be bound states. From the exact ground state expression, we first derived the expression of the unbinding temperature in terms of the parameters of the potential. Second, we computed the contact probability that is defined as the probability to find the two interacting manifolds at a (finite) distance apart each other. This probability allowed us to extract all length scales, which are the average separation and roughness, and the associated exponents were exactly deduced. These were found to be very close to those derived using RG machineries.

The main conclusion is that our analytical studies reveal that the Morse potential is a good candidate for the description of the unbinding phenomenon within manifolds (strings and membranes). This potential choice overcomes the difficulty related to the singularity of the real potential at the origin. As a matter of fact, this singularity cannot be seen since one is usually interested in the high scales critical behavior.

We emphasize that once our work is achieved, we learned that the Morse potential was already used for the study of DNA denaturation problem [29, 30].

Such a study can be extended to a generalized Morse potential (GMP). The latter was introduced by Deng and Fan [31] and related to the Manning-Rosen potential [32, 33] (also called Eckart potential by some authors [34, 35]). It is defined by  $V(l) = D[(1 - b/(e^{al} - 1))^2 - 1]$ , with the notation  $b \equiv e^{al_m} - 1$ , where  $l_m$  is the maximum of the potential. The specificity of GMP is that it behaves as a hard-sphere potential at the origin  $l = 0$ . It has been demonstrated [36] that the Schrödinger equation with this potential is exactly solvable. In our context, in particular, we find that the unbinding temperature reads  $T_c = k_B^{-1} \sqrt{b/(b+2)} \sqrt{2D\sigma}$ . Other developments will be presented elsewhere [37].

Finally, the present analysis may be extended to more than two manifolds.

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