Research Article

On Approximation by Algebraic Version of the Trigonometric Jackson Integrals $G_{s,n}$ in Weighted Integral Metric

Teodora Zapryanova

Economic University of Varna, Boulevard. "Knyaz Boris I" 77, 9002 Varna, Bulgaria

Correspondence should be addressed to Teodora Zapryanova; teodorazap@abv.bg

Received 20 June 2013; Accepted 21 November 2013

Academic Editor: Hagen Neidhardt

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We characterize the errors of the algebraic version of trigonometric Jackson integrals $G_{s,n}$ in weighted integral metric. We prove direct and strong converse theorem in terms of the weighted K -functional.

1. Introduction

We study linear approximation process together with the characterization of the rate of convergence of the algebraic version of the trigonometric Jackson integrals $G_{s,n}$ defined by

$$G_{s,n}(f, x) = \int_{-\pi}^{\pi} f(\cos(\arccos x + v)) K_{s,n}(v) dv, \quad (1)$$

where

$$K_{s,n}(v) = \mu_{s,n} \left(\frac{\sin(nv/2)}{\sin(v/2)} \right)^{2s}, \quad \int_{-\pi}^{\pi} K_{s,n}(v) dv = 1. \quad (2)$$

We established the equivalence in uniform norm for the approximation error of the operator $G_{s,n}$, the values of proper K -functional, and modulus of smoothness (see [1, 2]):

$$\|f - G_{s,n}f\| \sim K\left(f, \frac{1}{n^2}; C[-1, 1], C^2, H\right) \sim \Omega_2\left(f, \frac{1}{n}\right). \quad (3)$$

For every $f \in C[-1, 1]$ and $t > 0$ the K -functional is defined by

$$K(f, t; C[-1, 1], C^2, H) := \inf \left\{ \|f - g\| + t \|Hg\| : g \in C^2 \right\}, \quad (4)$$

where the differential operator

$$H := (H_1)^2, \quad (H_1 g)(x) := \sqrt{1-x^2} \frac{d}{dx} g(x). \quad (5)$$

The modulus

$$\Omega_2(f, t) := \sup_{0 < h \leq t} \sup_{-1 \leq x \leq 1} |f(\cos(\arccos x + h)) + f(\cos(\arccos x - h)) - 2f(x)|. \quad (6)$$

The expression $A(n) \sim B(n)$ means that there exists a positive constant c independent of n such that $(1/c)B(n) \leq A(n) \leq cB(n)$. The equivalence $\|f - G_{s,n}f\| \sim K(f, 1/n^2; C[-1, 1], C^2, H)$ consists of a direct inequality and strong converse inequality of type A in the sense of [3]. Ditzian and Ivanov show that a converse inequality follows from several inequalities of Bernstein and Voronovskaya type. We apply their method.

Let $L_p(u)[-1, 1]$ be the weighted L_p space with the norm

$$\begin{aligned} \|f\|_{p,u} &= \|f\|_{L_p(u)[-1,1]} \\ &= \left(\int_{-1}^1 |u(x) f(x)|^p dx \right)^{1/p} \\ &= \left(\int_{-1}^1 \left| (1-x^2)^{-1/2p} f(x) \right|^p dx \right)^{1/p}, \\ u(x) &= (1-x^2)^{-1/2p}. \end{aligned} \quad (7)$$

The approximation error $\|f - G_{s,n}f\|_{p,u}$ of $G_{s,n}$ in $L_p(u)[-1, 1]$ will be compared with the K -functional, which for every $f \in L_p(u)[-1, 1]$ and $t > 0$ is defined by

$$K(f, t; L_p(u)[-1, 1], C^2, H) := \inf \{ \|f - g\|_{p,u} + t \|Hg\|_{p,u} : g \in C^2 \}. \quad (8)$$

Our main result states the following.

Theorem 1. For every $f \in L_p(u)[-1, 1]$, $1 \leq p < \infty$, and $s, n \in \mathbb{N}$, $s \geq 3$, one has

$$\|f - G_{s,n}f\|_{L_p(u)[-1, 1]} \sim K\left(f, \frac{1}{n^2}; L_p(u)[-1, 1], C^2, H\right). \quad (9)$$

In Section 2 we state and prove some auxiliary lemmas. The proof of Theorem 1 is given in Section 3.

2. Auxiliary Lemmas

Lemma 2. Let f be summable on $\mathbb{R}$ and $g \in L_p[-\pi, \pi]$ 2π -periodic function, $1 \leq p < \infty$. Then the following holds true:

$$\|f * g\|_{L_p[-\pi, \pi]} \leq \|f\|_{L_1(-\infty, \infty)} \|g\|_{L_p[-\pi, \pi]}. \quad (10)$$

Proof. Using the Minkowski inequality we have

$$\begin{aligned} & \|f * g\|_{L_p[-\pi, \pi]} \\ &= \left\| \int_{-\infty}^{\infty} f(x-y) g(y) dy \right\|_{L_p[-\pi, \pi]} \\ &= \left\| \int_{-\infty}^{\infty} g(x-y) f(y) dy \right\|_{L_p[-\pi, \pi]} \\ &\leq \left(\int_{-\pi}^{\pi} \left| \int_{-\infty}^{\infty} |g(x-y) f(y)| dy \right|^p dx \right)^{1/p} \\ &\leq \int_{-\infty}^{\infty} \left(\int_{-\pi}^{\pi} |g(x-y)|^p |f(y)|^p dx \right)^{1/p} dy \\ &= \int_{-\infty}^{\infty} \left(|f(y)|^p \int_{-\pi}^{\pi} |g(x-y)|^p dx \right)^{1/p} dy \\ &= \int_{-\infty}^{\infty} |f(y)| \left(\int_{-\pi}^{\pi} |g(x-y)|^p dx \right)^{1/p} dy \\ &\quad (\text{as } g \text{ is } 2\pi \text{ periodic function}) \\ &= \int_{-\infty}^{\infty} |f(y)| \|g\|_{L_p[-\pi, \pi]} dy = \|g\|_{L_p[-\pi, \pi]} \|f\|_{L_1(-\infty, \infty)}. \end{aligned} \quad (11)$$

Lemma 3. For 2π periodic integrable on $[-\pi, \pi]$ function g and every $v \neq 0$ one has

$$\left\| \frac{1}{v} \int_z^{z+v} |g(\xi)| d\xi \right\|_{L_1[-\pi, \pi]} = \|g\|_{L_1[-\pi, \pi]}. \quad (12)$$

Proof. We consider the function $g(\xi + z)$. By the Fubini theorem we have

$$\int_{-\pi}^{\pi} \int_0^v |g(\xi + z)| d\xi dz = \int_0^v \int_{-\pi}^{\pi} |g(\xi + z)| dz d\xi. \quad (13)$$

As $\int_0^v |g(\xi + z)| d\xi = \int_z^{z+v} |g(x)| dx$ we obtain that the integral on the left-hand side of (13) is equal to

$$v \int_{-\pi}^{\pi} \left(\frac{1}{v} \int_z^{z+v} |g(\xi)| d\xi \right) dz. \quad (14)$$

The integral on the right-hand side of (13) is

$$\begin{aligned} & \int_0^v \int_{-\pi}^{\pi} |g(\xi + z)| dz d\xi \\ &= \int_0^v \left[\int_{-\pi+\xi}^{\pi+\xi} |g(x)| dx \right] d\xi \quad (\text{as } g \text{ is } 2\pi \text{ periodic function}) \\ &= \int_0^v \left[\int_{-\pi}^{\pi} |g(x)| dx \right] d\xi \\ &= v \int_{-\pi}^{\pi} |g(x)| dx. \end{aligned} \quad (15)$$

From (13), (14), and (15) we obtain $\int_{-\pi}^{\pi} \left(\frac{1}{v} \int_z^{z+v} |g(\xi)| d\xi \right) dz = \int_{-\pi}^{\pi} |g(x)| dx = \|g\|_{L_1[-\pi, \pi]}$. If $\int_z^{z+v} |g(\xi)| d\xi \neq 0$ then the sign of $\int_z^{z+v} |g(\xi)| d\xi$ is constant and coincides with the sign of v . Therefore

$$\begin{aligned} \int_{-\pi}^{\pi} \left(\frac{1}{v} \int_z^{z+v} |g(\xi)| d\xi \right) dz &= \int_{-\pi}^{\pi} \left| \frac{1}{v} \int_z^{z+v} |g(\xi)| d\xi \right| dz \\ &= \left\| \frac{1}{v} \int_z^{z+v} |g(\xi)| d\xi \right\|_{L_1[-\pi, \pi]}, \end{aligned} \quad (16)$$

which proves the lemma. $\square$

Lemma 4. For 2π periodic function $g \in L_p[-\pi, \pi]$, $1 \leq p < \infty$ and every $v \neq 0$ one has

$$\left\| \frac{1}{v} \int_z^{z+v} |g(\xi)| d\xi \right\|_{L_p[-\pi, \pi]} \leq \|g\|_{L_p[-\pi, \pi]}. \quad (17)$$

Proof. For $p = 1$ Lemma 4 is fulfilled as equality. Let $1 < p < \infty$. Using the Hölder inequality we get

$$\begin{aligned} & \left| \int_z^{z+v} |g(\xi)| d\xi \right| \\ &\leq \left| \int_z^{z+v} |g(\xi)|^p d\xi \right|^{1/p} \left| \int_z^{z+v} 1^q d\xi \right|^{1/q} \quad \left(\frac{1}{p} + \frac{1}{q} = 1 \right) \quad (18) \\ &= |v|^{1/q} \left| \int_z^{z+v} |g(\xi)|^p d\xi \right|^{1/p}. \end{aligned}$$

Therefore

$$\begin{aligned}
 \left| \frac{1}{v} \int_z^{z+v} |g(\xi)| d\xi \right|^p &\leq \frac{|v|^{p/q}}{|v|^p} \left| \int_z^{z+v} |g(\xi)|^p d\xi \right| \\
 &= \frac{1}{|v|} \left| \int_z^{z+v} |g(\xi)|^p d\xi \right| \\
 &= \frac{1}{v} \int_z^{z+v} |g(\xi)|^p d\xi \\
 &= \left| \frac{1}{v} \int_z^{z+v} |g(\xi)|^p d\xi \right|.
 \end{aligned} \tag{19}$$

Then

$$\int_{-\pi}^{\pi} \left| \frac{1}{v} \int_z^{z+v} |g(\xi)| d\xi \right|^p dz \leq \int_{-\pi}^{\pi} \left| \frac{1}{v} \int_z^{z+v} |g(\xi)|^p d\xi \right| dz. \tag{20}$$

We apply Lemma 3 to $|g(\xi)|^p$ to obtain

$$\int_{-\pi}^{\pi} \left| \frac{1}{v} \int_z^{z+v} |g(\xi)|^p d\xi \right| dz = \int_{-\pi}^{\pi} |g(\xi)|^p d\xi. \tag{21}$$

Hence

$$\int_{-\pi}^{\pi} \left| \frac{1}{v} \int_z^{z+v} |g(\xi)| d\xi \right|^p dz \leq \int_{-\pi}^{\pi} |g(\xi)|^p d\xi. \tag{22}$$

Thus

$$\left\| \frac{1}{v} \int_z^{z+v} |g(\xi)| d\xi \right\|_{L_p[-\pi, \pi]} \leq \|g\|_{L_p[-\pi, \pi]}. \tag{23}$$

□

Definition 5. Set

$$\begin{aligned}
 Y &= \{g \in C[-1, 1] : H_1 g \in C[-1, 1], \\
 &\quad Hg \in C[-1, 1], H_1 g(\pm 1) = 0\}. \\
 Z &= \{g \in Y : H_1^3 g \in C[-1, 1], \\
 &\quad H^2 g \in C[-1, 1], H_1^3 g(\pm 1) = 0\}.
 \end{aligned} \tag{24}$$

Lemma 6 ([2, page 402]). *Let Y be the space from Definition 5, $g \in Y$, and $\tilde{g}(\sigma) := g(\cos \sigma)$. Then $\tilde{g} \in C^2(\mathbb{R})$ and $\tilde{g}''(\sigma) = Hg(\cos \sigma)$ for $\sigma \in \mathbb{R}$.*

Lemma 7. *Let Y be the space from Definition 5. Then for every $f \in L_p(u)[-1, 1]$ and $t > 0$, one has*

$$\begin{aligned}
 K(f, t; L_p(u)[-1, 1], Y, H) \\
 = K(f, t; L_p(u)[-1, 1], C^2, H).
 \end{aligned} \tag{25}$$

Proof. From $C^2 \subset Y$ we see that $K(f, t; L_p(u)[-1, 1], Y, H) \leq K(f, t; L_p(u)[-1, 1], C^2, H)$. In order to prove $K(f, t; L_p(u)[-1, 1], C^2, H) \leq K(f, t; L_p(u)[-1, 1], Y, H)$ it is sufficient to show (see Lemma 2 in [4, page 116]) that for

every $g \in Y$ and $\varepsilon > 0$ there exists $G \in C^2$ such that $\|G - g\|_{L_p(u)[-1, 1]} \leq \varepsilon$ and $\|HG\|_{L_p(u)[-1, 1]} \leq \|Hg\|_{L_p(u)[-1, 1]} + \varepsilon$. Let $g(x) \in Y$. We put $x = \cos \sigma$ and consider $\tilde{g}(\sigma) := g(\cos \sigma)$. Since $g(x) \in Y$, $\tilde{g}(\sigma) \in C^2$ (see Lemma 6). We use the Jackson integrals of the following type:

$$\begin{aligned}
 J_n(\tilde{g}, \sigma) &:= \int_{-\pi}^{\pi} \tilde{g}(\sigma + v) K_{1s,n}(v) dv \\
 &= \int_{-\pi}^{\pi} \tilde{g}(v) K_{1s,n}(\sigma - v) dv,
 \end{aligned} \tag{26}$$

where $K_{1s,n}(v) := \lambda_{s,n}(\sin(mv/2)/\sin(v/2))^{2s}$, $\int_{-\pi}^{\pi} K_{1s,n}(v) dv = 1$, $n, s > 0$, $m := [n/s] + 1$.

Since $(1/2m)(\sin(mv/2)/\sin(v/2))^2 = 1/2 + \sum_{k=0}^{m-1} (1 - k/m) \cos kv$, it follows that $K_{1s,n}$ is an even nonnegative trigonometric polynomial of degree $\leq n$. $J_n(\tilde{g}, \sigma)$ is a trigonometric polynomial of degree $\leq n$, which is even as $\tilde{g}$ is even. From Jackson theorem (see [5, Chapter 7, Theorem 2.2]) we get

$$\|\tilde{g} - J_n(\tilde{g})\|_{L_p[0, \pi]} \leq c\omega_2\left(\tilde{g}, \frac{1}{n}\right)_{L_p[0, \pi]} = O\left(\frac{1}{n^2}\right). \tag{27}$$

By the substitution $\sigma = \arccos x$ in $J_n(\tilde{g}, \sigma)$ we obtain an algebraic polynomial, which is desired function from C^2 . We set

$$G(x) = J_n(\tilde{g}, \arccos x). \tag{28}$$

From $\|g - G\|_{L_p(u)[-1, 1]} = \|\tilde{g} - J_n(\tilde{g})\|_{L_p[0, \pi]}$ and (27) we get

$$\|g - G\|_{L_p(u)[-1, 1]} \leq c\omega_2\left(\tilde{g}, \frac{1}{n}\right)_{L_p[0, \pi]} = O\left(\frac{1}{n^2}\right). \tag{29}$$

From (26) we get

$$\begin{aligned}
 \frac{d^2}{d\sigma^2} J_n(\tilde{g}, \sigma) &= \int_{-\pi}^{\pi} \tilde{g}''(\sigma + v) K_{1s,n}(v) dv \\
 &= \int_{-\pi}^{\pi} \tilde{g}''(v) K_{1s,n}(\sigma - v) dv \\
 &= J_n(\tilde{g}'', \sigma).
 \end{aligned} \tag{30}$$

Using the Jackson theorem we get

$$\|\tilde{g}'' - J_n(\tilde{g}'')\|_{L_p[0, \pi]} \leq c\omega_2\left(\tilde{g}'', \frac{1}{n}\right)_{L_p[0, \pi]}. \tag{31}$$

Since $(Hg)(x) = (d^2/d\sigma^2)\tilde{g}(\sigma)$ and $(HG)(x) = (d^2/d\sigma^2)J_n(\tilde{g}, \sigma)$ (31) implies

$$\|Hg - HG\|_{L_p(u)[-1, 1]} \leq c\omega_2\left(\tilde{g}'', \frac{1}{n}\right)_{L_p[0, \pi]}. \tag{32}$$

For given $\varepsilon > 0$ we choose n such that $(1 + c)\omega_2(\tilde{g}'', 1/n)_{L_p[0, \pi]} < \varepsilon$ and $(1 + c)\omega_2(\tilde{g}, 1/n)_{L_p[0, \pi]} < \varepsilon$ to obtain

$$\begin{aligned}
 \|HG\|_{L_p(u)[-1, 1]} &< \|Hg\|_{L_p(u)[-1, 1]} + \varepsilon, \\
 \|G - g\|_{L_p(u)[-1, 1]} &< \varepsilon.
 \end{aligned} \tag{33}$$

This completes the proof of the lemma. □

3. Proof of Theorem 1

The theorem will be proved if we show that $\|f - G_{s,n}f\|_{p,u} \sim K(f, 1/n^2; L_p(u)[-1, 1], Y, H)$ (see Lemma 7). First we prove the converse result $K(f, 1/n^2; L_p(u)[-1, 1], Y, H) \leq c\|f - G_{s,n}f\|_{L_p(u)[-1, 1]}$, which is a strong converse inequality of type A in the terminology of [3]. We utilize Theorems 3.1 and 4.1 in [3] with $Q_\alpha = G_{s,n}$, $Df = Hf$, $\Phi(f) = \|D^2 f\|_{L_p(u)[-1, 1]}$, and $\lambda(n) = (1/2) \int_{-\pi}^{\pi} v^2 K_{s,n}(v) dv \sim n^{-2}$ for $s \geq 2$, $\lambda_1(n) = (1/3!) \int_{-\pi}^{\pi} v^4 K_{s,n}(v) dv \sim n^{-4}$ for $s \geq 3$. The result needed for inequality (3.3) from Theorem 3.1 in [3] with $M = 1$ is given by

$$\|G_{s,n}f\|_{L_p(u)[-1, 1]} \leq \|f\|_{L_p(u)[-1, 1]}. \quad (34)$$

We set $f(\cos(\arccos x + v)) := f(\cos(z + v)) := \tilde{f}(z + v)$, $z = \arccos x$. To prove (34) we recall the representation

$$\begin{aligned} (G_{s,n}f)(x) &= \int_{-\pi}^{\pi} f(\cos(\arccos x + v)) K_{s,n}(v) dv \\ &= \int_{-\pi}^{\pi} \tilde{f}(z + v) K_{s,n}(v) dv \\ &= \int_{-\pi}^{\pi} \tilde{f}(z - v) K_{s,n}(v) dv \\ &= \tilde{f} * K(z) \\ &= K * \tilde{f}(z) \\ &= (\tilde{G}_{s,n}\tilde{f})(z), \end{aligned} \quad (35)$$

where

$$K(v) = K_1(v) = \begin{cases} \mu_{s,n} \left(\frac{\sin(nv/2)}{\sin(v/2)} \right)^{2s}, & \text{if } |v| \leq \pi, \\ 0, & \text{otherwise.} \end{cases} \quad (36)$$

And the convolution is defined by

$$F * G(x) := \int_{-\infty}^{\infty} F(x - v) G(v) dv. \quad (37)$$

We have

$$\begin{aligned} \|G_{s,n}f\|_{L_p(u)[-1, 1]} &= \left\{ \int_{-1}^1 |(G_{s,n}f)(x)|^p \frac{1}{\sqrt{1-x^2}} dx \right\}^{1/p} (x = \cos z) \\ &= \left\{ \int_{-\pi}^{\pi} |(\tilde{G}_{s,n}\tilde{f})(z)|^p \frac{1}{\sqrt{1-\cos^2 z}} (-\sin z) dz \right\}^{1/p} \\ &= \left\{ \int_0^{\pi} |(\tilde{G}_{s,n}\tilde{f})(z)|^p dz \right\}^{1/p} = \|\tilde{G}_{s,n}\tilde{f}\|_{L_p[0, \pi]}. \end{aligned} \quad (38)$$

Using Lemma 2 as $\tilde{f}$ and $\tilde{G}_{s,n}\tilde{f}$ are even, 2π periodic functions, we estimate

$$\begin{aligned} \|\tilde{G}_{s,n}\tilde{f}\|_{L_p[0, \pi]} &= \|K * \tilde{f}\|_{L_p[0, \pi]} \\ &\leq \|K\|_{L_1(-\infty, \infty)} \|\tilde{f}\|_{L_p[0, \pi]} \\ &= \|\tilde{f}\|_{L_p[0, \pi]} \\ &= \|f\|_{L_p(u)[-1, 1]} \end{aligned} \quad (39)$$

to complete the proof of (34).

We will show now that for $f \in Z$ the following Voronovskaya-type estimate holds true:

$$\|f - G_{s,n}f + \lambda(n) Df\|_{L_p(u)[-1, 1]} \leq \lambda_1(n) \Phi(f), \quad (40)$$

with $\Phi(f) = \|\tilde{f}^{(4)}\|_{L_p[0, \pi]} = \|H^2 f\|_{L_p(u)[-1, 1]}$, $Df = Hf$, and $\lambda(n) = (1/2) \int_{-\pi}^{\pi} v^2 K_{s,n}(v) dv$, $\lambda_1(n) = (1/3!) \int_{-\pi}^{\pi} v^4 K_{s,n}(v) dv$. Inequality (40) will serve for (3.4) from Theorem 3.1 in [3]. We note that $G_{s,n}^m f \in Z$ as $G_{s,n}f$ is an algebraic polynomial. Let $f \in Z$. From $Z \subset Y$ and Lemma 6 $\tilde{f} \in C^2(\mathbb{R})$. We apply Lemma 6 for Hf to obtain that $\tilde{f} \in C^4(\mathbb{R})$. With $\lambda(n) = (1/2) \int_{-\pi}^{\pi} v^2 K_{s,n}(v) dv$ we have

$$\begin{aligned} f(x) - (G_{s,n}f)(x) + \lambda(n) Df(x) &= \tilde{f}(z) - (\tilde{G}_{s,n}\tilde{f})(z) + \lambda(n) \tilde{f}''(z) \\ &= \int_{-\pi}^{\pi} \left[\tilde{f}(z) - \tilde{f}(z + v) + \frac{1}{2} v^2 \tilde{f}''(z) \right] K_{s,n}(v) dv. \end{aligned} \quad (41)$$

Expanding $\tilde{f}(z + v)$ by Taylor's formula and using that $\int_{-\pi}^{\pi} v K_{s,n}(v) dv = 0$, $\int_{-\pi}^{\pi} v^3 K_{s,n}(v) dv = 0$, we now have

$$\begin{aligned} f(x) - (G_{s,n}f)(x) + \lambda(n) Df(x) &= - \int_{-\pi}^{\pi} \int_z^{z+v} \frac{1}{3!} \tilde{f}^{(4)}(\xi) (z + v - \xi)^3 d\xi K_{s,n}(v) dv. \end{aligned} \quad (42)$$

We recall that $\tilde{f}^{(4)}(\xi) = (H^2 f)(\cos \xi)$. As for $\xi \in [z, z + v]$, $z + v - \xi \in [0, v]$, and the sign of $\int_z^{z+v} |\tilde{f}^{(4)}(\xi)| d\xi$ is constant and coincides with the sign of v (if the integral is not zero); via Minkowski's inequality, we get

$$\begin{aligned} \|f - G_{s,n}f + \lambda(n) Df\|_{L_p(u)[-1, 1]} &= \|\tilde{f} - \tilde{G}_{s,n}\tilde{f} + \lambda(n) \tilde{f}''\|_{L_p[0, \pi]} \end{aligned}$$

$$\begin{aligned}
&= \left\{ \int_0^\pi \left| \int_{-\pi}^\pi \int_z^{z+\nu} \frac{1}{3!} \tilde{f}^{(4)}(\xi) \right. \right. \\
&\quad \left. \left. \times (z + \nu - \xi)^3 d\xi K_{s,n}(\nu) d\nu \right|^p dz \right\}^{1/p} \\
&\leq \left\{ \int_0^\pi \left(\int_{-\pi}^\pi K_{s,n}(\nu) \frac{1}{3!} \right. \right. \\
&\quad \left. \left. \times \left| \int_z^{z+\nu} \tilde{f}^{(4)}(\xi) (z + \nu - \xi)^3 d\xi \right| d\nu \right)^p dz \right\}^{1/p} \\
&\leq \left\{ \int_0^\pi \left(\int_{-\pi}^\pi K_{s,n}(\nu) \frac{1}{3!} \right. \right. \\
&\quad \left. \left. \times \left| \int_z^{z+\nu} \tilde{f}^{(4)}(\xi) |\nu^3| d\xi \right| d\nu \right)^p dz \right\}^{1/p} \\
&= \left\{ \int_0^\pi \left(\int_{-\pi}^\pi K_{s,n}(\nu) \frac{|\nu^3|}{3!} \left| \int_z^{z+\nu} \tilde{f}^{(4)}(\xi) d\xi \right| d\nu \right)^p dz \right\}^{1/p} \\
&= \frac{1}{3!} \left\{ \int_0^\pi \left(\int_{-\pi}^\pi K_{s,n}(\nu) \nu^4 \frac{1}{|\nu|} \right. \right. \\
&\quad \left. \left. \times \left| \int_z^{z+\nu} \tilde{f}^{(4)}(\xi) d\xi \right| d\nu \right)^p dz \right\}^{1/p} \\
&= \frac{1}{3!} \left\{ \int_0^\pi \left(\int_{-\pi}^\pi \nu^4 K_{s,n}(\nu) \right. \right. \\
&\quad \left. \left. \times \underbrace{\frac{1}{\nu} \int_z^{z+\nu} \tilde{f}^{(4)}(\xi) d\xi d\nu}_{\varphi(z,\nu)} \right)^p dz \right\}^{1/p} \\
&= \frac{1}{3!} \left\{ \int_0^\pi \left(\int_{-\pi}^\pi \nu^4 K_{s,n}(\nu) \varphi(z,\nu) d\nu \right)^p dz \right\}^{1/p} \\
&\leq \frac{1}{3!} \int_{-\pi}^\pi \left\{ \int_0^\pi \left[\nu^4 K_{s,n}(\nu) \varphi(z,\nu) \right]^p dz \right\}^{1/p} d\nu \\
&= \frac{1}{3!} \int_{-\pi}^\pi \nu^4 K_{s,n}(\nu) \left\{ \int_0^\pi [\varphi(z,\nu)]^p dz \right\}^{1/p} d\nu \\
&= \frac{1}{3!} \int_{-\pi}^\pi \nu^4 K_{s,n}(\nu) \left\{ \int_0^\pi |\varphi(z,\nu)|^p dz \right\}^{1/p} d\nu \\
&= \frac{1}{3!} \int_{-\pi}^\pi \nu^4 K_{s,n}(\nu) \left\| \frac{1}{\nu} \int_z^{z+\nu} \tilde{f}^{(4)}(\xi) d\xi \right\|_{L_p[0,\pi]} d\nu \\
&\leq \frac{1}{3!} \int_{-\pi}^\pi \nu^4 K_{s,n}(\nu) \|\tilde{f}^{(4)}\|_{L_p[0,\pi]} d\nu \\
&= \lambda_1(n) \Phi(f).
\end{aligned} \tag{43}$$

We apply Lemma 4 (as $\tilde{f}^{(4)}$ is even 2π periodic function) to obtain $\|(1/\nu) \int_z^{z+\nu} \tilde{f}^{(4)}(\xi) d\xi\|_{L_p[0,\pi]} \leq \|\tilde{f}^{(4)}\|_{L_p[0,\pi]}$. This establishes (40).

To obtain results corresponding to (3.5) and (3.6) from Theorem 3.1 in [3] we need a weighted Bernstein-type inequality for the power of the operator $G_{s,n}$ like (6.10) in [3].

We use representations

$$\begin{aligned}
(G_{s,n}f)(x) &= \int_{-\pi}^\pi \tilde{f}(z - \nu) K_{s,n}(\nu) d\nu \\
&= \tilde{f} * K(z) = K * \tilde{f}(z) = (\tilde{G}_{s,n}\tilde{f})(z), \\
G_{s,n}(G_{s,n}f)(x) &= K * K * \tilde{f}(z) = K_2 * \tilde{f}(z), \dots
\end{aligned} \tag{44}$$

$$(G_{s,n}^m f)(x) = K * K_{m-1} * \tilde{f}(z) = K_m * \tilde{f}(z)$$

($K_m = K * K_{m-1}$ for $m = 2, 3, 4, \dots$) to obtain

$$H_1 G_{s,n}^m f = -\{K'\} * K_{m-1} * \tilde{f},$$

where

$$\{K'\}(\nu) = \begin{cases} \mu_{s,n} \frac{d}{d\nu} \left[\left(\frac{\sin(n\nu/2)}{\sin(\nu/2)} \right)^{2s} \right], & \text{if } |\nu| \leq \pi, \\ 0, & \text{otherwise.} \end{cases} \tag{45}$$

We estimate the action of H_1 on the m th degree $G_{s,n}^m$ of the operator. Using Lemma 2 we have

$$\begin{aligned}
\|H_1 G_{s,n}^m f\|_{L_p(u)[-1,1]} &= \|\{K'\} * K_{m-1} * \tilde{f}\|_{L_p[0,\pi]} \\
&\leq \|\{K'\} * K_{m-1}\|_{L_1(-\infty,\infty)} \|\tilde{f}\|_{L_p[0,\pi]} \\
&= \|\{K'\} * K_{m-1}\|_{L_1(-\infty,\infty)} \|f\|_{L_p(u)[-1,1]}.
\end{aligned} \tag{46}$$

We have already proved earlier (see Assertion 1.2 in [1]) that

$$\|\{K'\} * K_{m-1}\|_{L_1(-\infty,\infty)} \leq c \frac{n}{\sqrt{m}} \tag{47}$$

and therefore

$$\|H_1 G_{s,n}^m f\|_{L_p(u)[-1,1]} \leq c \frac{n}{\sqrt{m}} \|f\|_{L_p(u)[-1,1]}. \tag{48}$$

As $G_{s,n}$ commutes with the operator H_1 using estimation (48) we now observe that

$$\begin{aligned} \|D^2 G_{s,n}^{4m} f\|_{L_p(u)[-1,1]} &= \|H^2 G_{s,n}^{4m} f\|_{L_p(u)[-1,1]} \\ &\leq cn^2 m^{-1} \|HG_{s,n}^{2m} f\|_{L_p(u)[-1,1]} \\ &= A \frac{\lambda(n)}{\lambda_1(n)} \|DG_{s,n}^{2m} f\|_{L_p(u)[-1,1]}, \end{aligned} \quad (49)$$

$$\begin{aligned} \|DG_{s,n}^{2m} f\|_{L_p(u)[-1,1]} &= \|HG_{s,n}^{2m} f\|_{L_p(u)[-1,1]} \\ &= \|H_1 G_{s,n}^m H_1 G_{s,n}^m f\|_{L_p(u)[-1,1]} \\ &\leq cnm^{-1/2} \|H_1 G_{s,n}^m f\|_{L_p(u)[-1,1]} \\ &\leq cn^2 m^{-1} \|f\|_{L_p(u)[-1,1]} \\ &= B \frac{1}{\lambda(n)} \|f\|_{L_p(u)[-1,1]}. \end{aligned} \quad (50)$$

Estimations (49) and (50) correspond to (3.5) and (3.6) from Theorem 3.1 in [3]. To match the conditions of Theorem 4.1 in [3] we need

$$A \frac{\lambda(n)}{\lambda_1(n)} = cn^2 m^{-1} \quad (51)$$

to be satisfied with $A < 1$ or

$$A = cm^{-1} < 1 \quad (52)$$

which is true for large m . From (34), (40), (49), and (50) we obtain

$$K\left(f, \frac{1}{n^2}; L_p(u)[-1, 1], Y, H\right) \leq c \|f - G_{s,n} f\|_{L_p(u)[-1,1]}. \quad (53)$$

To prove the direct result we need inequality (34) and

$$\|g - G_{s,n} g\|_{L_p(u)[-1,1]} \leq c \frac{1}{n^2} \|Hg\|_{L_p(u)[-1,1]} \quad \text{for } g \in Y \quad (54)$$

to be satisfied (see Theorem 3.4 in [3]).

Let $g \in Y$. We set $g(\cos(\arccos x + v)) := g(\cos(z + v)) := \tilde{g}(z + v)$, $z = \arccos x$. We have

$$(G_{s,n} g - g)(x) = \int_{-\pi}^{\pi} [\tilde{g}(z + v) - \tilde{g}(z)] K_{s,n}(v) dv. \quad (55)$$

Expanding $\tilde{g}(z + v)$ by Taylor's formula (by Lemma 6 $\tilde{g} \in C^2(\mathbb{R})$)

$$\tilde{g}(z + v) - \tilde{g}(z) = v \tilde{g}'(z) + \int_z^{z+v} \tilde{g}''(\xi) (z + v - \xi) d\xi, \quad (56)$$

and using that $\int_{-\pi}^{\pi} v K_{s,n}(v) dv = 0$ we have

$$(G_{s,n} g - g)(x) = \int_{-\pi}^{\pi} \int_z^{z+v} \tilde{g}''(\xi) (z + v - \xi) d\xi K_{s,n}(v) dv. \quad (57)$$

We recall that $\tilde{g}''(\xi) = (Hg)(\cos \xi)$. As for $\xi \in [z, z + v]$, $z + v - \xi \in [0, v]$, and the sign of $\int_z^{z+v} |\tilde{g}''(\xi)| d\xi$ is constant and coincides with the sign of v (if the integral is not zero); we get

$$\begin{aligned} \|G_{s,n} g - g\|_{L_p(u)[-1,1]} &= \|\tilde{G}_{s,n} \tilde{g} - \tilde{g}\|_{L_p[0,\pi]} \\ &= \left\{ \int_0^{\pi} \left| \int_{-\pi}^{\pi} \int_z^{z+v} \tilde{g}''(\xi) (z + v - \xi) d\xi K_{s,n}(v) dv \right|^p dz \right\}^{1/p} \\ &\leq \left\{ \int_0^{\pi} \left(\int_{-\pi}^{\pi} |v| \left| \int_z^{z+v} |\tilde{g}''(\xi)| d\xi \right| K_{s,n}(v) dv \right)^p dz \right\}^{1/p} \\ &= \left\{ \int_0^{\pi} \left(\int_{-\pi}^{\pi} v^2 K_{s,n}(v) \underbrace{\frac{1}{v} \int_z^{z+v} |\tilde{g}''(\xi)| d\xi}_{\psi(z,v)} dv \right)^p dz \right\}^{1/p} \\ &= \left\{ \int_0^{\pi} \left(\int_{-\pi}^{\pi} v^2 K_{s,n}(v) \psi(z, v) dv \right)^p dz \right\}^{1/p}. \end{aligned} \quad (58)$$

Using the Minkowski inequality we have

$$\begin{aligned} &\left\{ \int_0^{\pi} \left(\int_{-\pi}^{\pi} v^2 K_{s,n}(v) \psi(z, v) dv \right)^p dz \right\}^{1/p} \\ &\leq \int_{-\pi}^{\pi} \left\{ \int_0^{\pi} [v^2 K_{s,n}(v) \psi(z, v)]^p dz \right\}^{1/p} dv \\ &= \int_{-\pi}^{\pi} v^2 K_{s,n}(v) \left\{ \int_0^{\pi} [\psi(z, v)]^p dz \right\}^{1/p} dv \\ &= \int_{-\pi}^{\pi} v^2 K_{s,n}(v) \left\| \frac{1}{v} \int_z^{z+v} |\tilde{g}''(\xi)| d\xi \right\|_{L_p[0,\pi]} dv \\ &\leq \int_{-\pi}^{\pi} v^2 K_{s,n}(v) \|\tilde{g}''\|_{L_p[0,\pi]} dv \\ &\leq c \frac{1}{n^2} \|\tilde{g}''\|_{L_p[0,\pi]} \\ &= c \frac{1}{n^2} \|Hg\|_{L_p(u)[-1,1]}. \end{aligned} \quad (59)$$

We applied Lemma 4 (as $\tilde{g}''$ is even 2π periodic function) to obtain $\|(1/v) \int_z^{z+v} |\tilde{g}''(\xi)| d\xi\|_{L_p[0,\pi]} \leq \|\tilde{g}''\|_{L_p[0,\pi]}$. The proof of the direct result

$$\|f - G_{s,n} f\|_{L_p(u)[-1,1]} \leq cK\left(f, 1/n^2; L_p(u)[-1, 1], C^2, H\right) \quad (60)$$

is completed. This concludes the proof of Theorem 1.

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