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Research Article

Hankel Determinant for p-Valent Alpha-Convex Functions

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The objective of the present paper is to obtain the sharp upper bound of $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for p-valent α-convex functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ in the unit disc $E = \{z : |z| < 1\}$.

1. Introduction

Let A_p be the class of analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \tag{1}$$

in the unit disc $E = \{z : |z| < 1\}$ with $p \in N = \{1, 2, 3, \ldots\}$. Let S be the subclass of $A_1 = A$, consisting of univalent functions.

S_p^* is the class consisting of functions of the form (1) and satisfying the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{pf(z)}\right\} > 0, \quad z \in E. \tag{2}$$

The functions of the class S_p^* are called p-valent starlike functions. In particular, $S_1^* \equiv S^*$, the class of starlike functions.

K_p is the class of functions of the form (1), satisfying the condition

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{pf'(z)}\right\} > 0, \quad z \in E. \tag{3}$$

The functions of the class K_p are known as p-valent convex functions. Particularly, $K_1 \equiv K$, the class of convex functions.

Obviously $f(z) \in K_p$ if and only if $zf'(z)/p \in S_p^*$.

Let $M_p(\alpha)$ $(\alpha \geq 0)$ be the class of functions of the form (1), satisfying the condition

$$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{pf(z)} + \alpha\frac{(zf'(z))'}{pf'(z)}\right\} > 0, \quad z \in E. \tag{4}$$

Functions in the class $M_p(\alpha)$ are known as p-valent alpha-convex functions. For $p = 1$, the class $M_p(\alpha)$ reduces to the class $M(\alpha)$ of alpha-convex functions introduced by Mocanu [1]. Also $M_p(0) \equiv S_p^*$ and $M_p(1) \equiv K_p$.

In 1976, Noonan and Thomas [2] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{5}$$

This determinant has also been considered by several authors. For example, Noor [3] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions given by (1) with bounded boundary. Ehrenborg [4] studied the Hankel determinant of exponential polynomials. Also Hankel determinant was studied by various authors including Hayman [5] and Pommerenke [6]. In [7], Janteng et al. studied the Hankel determinant for the classes of starlike and convex functions. Again Janteng et al. discussed the Hankel determinant problem for the classes of starlike functions with respect to symmetric points and convex functions with respect to symmetric

points in [8] and for the functions whose derivative has a positive real part in [9]. Also Hankel determinant for various subclasses of p-valent functions was investigated by various authors including Krishna and Ramreddy [10] and Hayami and Owa [11].

Easily, one can observe that the Fekete and Szegö functional is $H_2(1)$. Fekete and Szegö [12] then further generalised the estimate $|a_3 - \mu a_2^2|$, where μ is real and $f \in S$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$:

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}. \tag{6}$$

In this paper, we seek sharp upper bound of the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for functions belonging to the class $M_p(\alpha)$. The results due to Janteng et al. [7] follow as special cases.

2. Preliminary Results

Let Q be the family of all functions q analytic in E for which $\mathrm{Re}(q(z)) > 0$ and

$$q(z) = 1 + q_1 z + q_2 z^2 + \cdots \tag{7}$$

for $z \in E$.

Lemma 1 (see [6]). *If $q \in Q$, then $|q_k| \le 2$ $(k = 1, 2, 3, \ldots)$.*

Lemma 2 (see [13, 14]). *If $q \in Q$, then*

$$\begin{aligned} 2q_2 &= q_1^2 + \left(4 - q_1^2\right) x, \\ 4q_3 &= q_1^3 + 2q_1\left(4 - q_1^2\right) x - q_1\left(4 - q_1^2\right) x^2 \\ &\quad + 2\left(4 - q_1^2\right)\left(1 - |x|^2\right) z, \end{aligned} \tag{8}$$

for some x and z satisfying $|x| \le 1$, $|z| \le 1$ and $q_1 \in [0, 2]$.

3. Main Result

Theorem 3. *If $f \in M_p(\alpha)$, then*

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \le \frac{p^4}{(p+2\alpha)^2}\left[1 - \frac{12\alpha^2(p+\alpha)^4}{(p+3\alpha)A(\alpha)}\right], \tag{9}$$

where

$$\begin{aligned} A(\alpha) = &-24p(p+2\alpha)\left[\alpha(p+1)(p+2) - (\alpha-1)p^2\right] \\ &\times\left[(p+\alpha)^2 + p\left(p^2+\alpha+2\alpha p\right)\right] \\ &+ 12p(p+3\alpha)\left(p^2+\alpha+2\alpha p\right) \\ &\times\left[p\left(p^2+\alpha+2\alpha p\right) + 2(p+\alpha)^2\right] \\ &+ 4(p+\alpha)^3\left[3(p+\alpha)(p+3\alpha) - 4(p+2\alpha)^2\right] \\ &- 16p^2(p+2\alpha)^2\left[(\alpha-1)p^3 - \alpha(p+1)^3\right]. \end{aligned} \tag{10}$$

Proof. Since $f \in M_p(\alpha)$, so from (4)

$$(1-\alpha)\frac{zf'(z)}{pf(z)} + \alpha\frac{\left(zf'(z)\right)'}{pf'(z)} = q(z). \tag{11}$$

On expanding and equating the coefficients of z, z^2, and z^3 in (11), we get

$$\begin{aligned} a_{p+1} &= \frac{p^2 q_1}{p+\alpha}, \\ a_{p+2} &= \frac{p^2 q_2}{2(p+2\alpha)} + \frac{p^3\left(p^2+\alpha+2\alpha p\right)q_1^2}{2(p+2\alpha)(p+\alpha)^2}, \\ a_{p+3} &= \frac{p^2 q_3}{3(p+3\alpha)} \\ &\quad + \frac{\left[3\alpha(p+1)(p+2) - 3(\alpha-1)p^2\right]p^3 q_1 q_2}{6(p+\alpha)(p+2\alpha)(p+3\alpha)} \\ &\quad + p^4\left[\left(p^2+\alpha+2\alpha p\right)\right. \\ &\qquad \times\left(3\alpha(p+1)(p+2) - 3(\alpha-1)p^2\right) \\ &\qquad \left. + 2(p+2\alpha)\left((\alpha-1)p^3 - \alpha(p+1)^3\right)\right] q_1^3 \\ &\quad \times\left(6(p+2\alpha)(p+3\alpha)(p+\alpha)^3\right)^{-1}. \end{aligned} \tag{12}$$

Equation (12) yields:

$$\begin{aligned} &a_{p+1}a_{p+3} - a_{p+2}^2 \\ &= \frac{p^4}{C(\alpha)} \\ &\quad \times\left\{4(p+2\alpha)^2(p+\alpha)^3 q_1 q_3\right. \\ &\qquad + \left[2p(p+2\alpha)(p+\alpha)^2\right. \\ &\qquad\quad \times\left(3\alpha(p+1)(p+2) - 3(\alpha-1)p^2\right) \\ &\qquad\quad - 6p(p+3\alpha)(p+\alpha)^2 \\ &\qquad\quad \left.\times\left(p^2+\alpha+2\alpha p\right)\right] q_1^2 q_2 \\ &\qquad + \left[2p^2(p+2\alpha)\right. \\ &\qquad\quad \times\left(\left(p^2+\alpha+2\alpha p\right)\right. \\ &\qquad\qquad \times\left(3\alpha(p+1)(p+2) - 3(\alpha-1)p^2\right) \\ &\qquad\qquad \left.+2(p+2\alpha)\left((\alpha-1)p^3 - \alpha(p+1)^3\right)\right) \end{aligned}$$

$$-3p^2(p+3\alpha)\left(p^2+\alpha+2\alpha p\right)^2\Big]q_1^4$$
$$-3(p+3\alpha)(p+\alpha)^4q_2^2\Big\}, \tag{13}$$

where $C(\alpha)=12(p+3\alpha)(p+2\alpha)^2(p+\alpha)^4$.

Using Lemmas 1 and 2 in (13), we obtain

$$\begin{aligned}
&\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| \\
&=\frac{p^4}{4C(\alpha)} \\
&\times\Big|-\Big[-4(p+2\alpha)^2(p+\alpha)^3+3(p+\alpha)^4 \\
&\quad\times(p+3\alpha)-4p(p+2\alpha)(p+\alpha)^2 \\
&\quad\times\left(3\alpha(p+1)(p+2)-3(\alpha-1)p^2\right) \\
&\quad+12p(p+3\alpha)(p+\alpha)^2\left(p^2+\alpha+2\alpha p\right) \\
&\quad+12p^2(p+3\alpha)\left(p^2+\alpha+2\alpha p\right)^2 \\
&\quad-8p^2(p+2\alpha)\left(p^2+\alpha+2\alpha p\right) \\
&\quad\times\left[3\alpha(p+1)(p+2)-3(\alpha-1)p^2\right] \\
&\quad-16p^2(p+2\alpha)^2\left[(\alpha-1)p^3-\alpha(p+1)^3\right]\Big]q_1^4 \\
&+(p+\alpha)^2\Big[8(p+2\alpha)^2(p+\alpha)+4p(p+2\alpha) \\
&\quad\times\left(3\alpha(p+1)(p+2)-3(\alpha-1)p^2\right) \\
&\quad-6(p+\alpha)^2(p+3\alpha)-12p(p+3\alpha) \\
&\quad\times\left(p^2+\alpha+2\alpha p\right)\Big]q_1^2\left(4-q_1^2\right)x \\
&-(p+\alpha)^3\Big[4(p+2\alpha)^2q_1^2+3(p+\alpha)(p+3\alpha) \\
&\quad\times\left(4-q_1^2\right)\Big]\left(4-q_1^2\right)x^2 \\
&+8(p+\alpha)^3(p+2\alpha)^2q_1\left(4-q_1^2\right)\left(1-|x|^2\right)z\Big|.
\end{aligned} \tag{14}$$

Assume that $q_1=q$ and $q\in[0,2]$; using triangular inequality and $|z|\le 1$, we have

$$\begin{aligned}
&\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| \\
&\le\frac{p^4}{4C(\alpha)} \\
&\times\Big\{\Big[-4(p+2\alpha)^2(p+\alpha)^3+3(p+\alpha)^4 \\
&\quad\times(p+3\alpha)-4p(p+2\alpha)(p+\alpha)^2 \\
&\quad\times\left(3\alpha(p+1)(p+2)-3(\alpha-1)p^2\right) \\
&\quad+12p(p+3\alpha)(p+\alpha)^2\left(p^2+\alpha+2\alpha p\right) \\
&\quad+12p^2(p+3\alpha)\left(p^2+\alpha+2\alpha p\right)^2 \\
&\quad-8p^2(p+2\alpha)\left(p^2+\alpha+2\alpha p\right) \\
&\quad\times\left[3\alpha(p+1)(p+2)-3(\alpha-1)p^2\right] \\
&\quad-16p^2(p+2\alpha)^2\left[(\alpha-1)p^3-\alpha(p+1)^3\right]\Big]q^4 \\
&+(p+\alpha)^2\Big[8(p+2\alpha)^2(p+\alpha)+4p(p+2\alpha) \\
&\quad\times\left(3\alpha(p+1)(p+2)-3(\alpha-1)p^2\right) \\
&\quad-6(p+\alpha)^2(p+3\alpha)-12p(p+3\alpha) \\
&\quad\times\left(p^2+\alpha+2\alpha p\right)\Big]q^2\left(4-q^2\right)\delta \\
&+8(p+\alpha)^3(p+2\alpha)^2q\left(4-q^2\right) \\
&+(p+\alpha)^3\Big[\left(4(p+2\alpha)^2-3(p+\alpha)(p+3\alpha)\right)q^2 \\
&\quad-8(p+2\alpha)^2q+12(p+\alpha) \\
&\quad\times(p+3\alpha)\Big]\left(4-q^2\right)\delta^2\Big\} \\
&=\frac{p^4}{4C(\alpha)}F(\delta),\quad\text{where } \delta=|x|\le 1.
\end{aligned} \tag{15}$$

It is easy to verify that $F(\delta)$ is an increasing function and so Max. $F(\delta)=F(1)$.

Consequently

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right|\le\frac{p^4}{4C(\alpha)}G(q), \tag{16}$$

where

$$G(q)=F(1). \tag{17}$$

So

$$G(q)=A(\alpha)q^4+48\alpha(p+\alpha)^4q^2+48(p+3\alpha)(p+\alpha)^4, \tag{18}$$

where $A(\alpha)$ is defined in (10).

Now

$$\begin{aligned}
G'(q)&=4A(\alpha)q^3+96\alpha(p+\alpha)^4q, \\
G''(q)&=12A(\alpha)q^2+96\alpha(p+\alpha)^4.
\end{aligned} \tag{19}$$

$G'(q)=0$ gives

$$4q\left[A(\alpha)q^2+24\alpha(p+\alpha)^4\right]=0. \tag{20}$$

$G''(q)$ is negative at $q=\sqrt{-24\alpha(p+\alpha)^4/A(\alpha)}=q'$.

So

$$\text{Max.}\, G(q) = G(q'). \tag{21}$$

Hence from (15), we obtain (9). □

The result is sharp for $q_1 = q'$, $q_2 = q_1^2 - 2$, and $q_3 = q_1(q_1^2 - 3)$.

For $\alpha = 0$, Theorem 3 gives the following result.

Corollary 4. *If $f(z) \in S_p^*$, then*

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \le p^2. \tag{22}$$

For $\alpha = 1$, Theorem 3 yields.

Corollary 5. *If $f(z) \in K_p$, then*

$$\begin{aligned}&\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \\ &\quad \le \frac{p^4}{(p+2)^2}\left[1 + \frac{3}{(p+3)(-p^3 - 3p^2 + 3p + 7)}\right].\end{aligned} \tag{23}$$

Putting $\alpha = 0$ and $p = 1$ in Theorem 3, we obtain the following result due to Janteng et al. [7].

Corollary 6. *If $f(z) \in S^*$, then*

$$\left|a_2a_4 - a_3^2\right| \le 1. \tag{24}$$

Putting $\alpha = 1$ and $p = 1$ in Theorem 3, we obtain the following result due to Janteng et al. [7].

Corollary 7. *If $f(z) \in K$, then*

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{8}. \tag{25}$$

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