Research Article

On Certain Classes of Harmonic p -Valent Functions Defined by an Integral Operator

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We obtain coefficient characterization, extreme points, and distortion bounds of certain classes of harmonic p -valent functions defined by an integral operator.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D . In any simply connected domain, we can write

$$f = h + \bar{g}, \quad (1)$$

where h and g are analytic in D . We call h the analytic part and g the coanalytic part of f . A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [1]).

Denote by S_H the class of functions f of the form (1) that are harmonic univalent and sense preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$.

Recently, Jahangiri and Ahuja [2] defined the class $\mathcal{H}_p(p \in \mathbb{N} = \{1, 2, 3, \dots\})$, consisting of all p -valent harmonic functions $f = h + \bar{g}$ that are sense preserving in U and h , and g are of the form

$$h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad g(z) = \sum_{k=p}^{\infty} b_k z^k, \quad |b_p| < 1. \quad (2)$$

For $f = h + \bar{g}$ given by (2), we define the modified p -valent Salagean integral operator $I_{p,\lambda}^n$ of f (see [3] and also [4] when $p = 1$) as follows:

$$I_{p,\lambda}^n f(z) = I_{p,\lambda}^n h(z) + (-1)^n \overline{I_{p,\lambda}^n g(z)}, \quad (3)$$

where

$$\begin{aligned} I_{p,\lambda}^n h(z) &= z^p + \sum_{k=p+1}^{\infty} \left(\frac{p}{p + \lambda(k-p)} \right)^n a_k z^k \\ &\quad (p \in \mathbb{N}; \lambda > 0; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \\ I_{p,\lambda}^n g(z) &= \sum_{k=p}^{\infty} \left(\frac{p}{p + \lambda(k-p)} \right)^n b_k z^k \\ &\quad (p \in \mathbb{N}; \lambda > 0; n \in \mathbb{N}_0). \end{aligned} \quad (4)$$

For $p \in \mathbb{N}$, $\lambda > 0$, $n \in \mathbb{N}_0$, $0 \leq \alpha < 1$, and $z \in U$, we let $\mathcal{H}_{p,\lambda}(n; \alpha)$ denote the family of harmonic functions f of the form (2) such that

$$\operatorname{Re} \left\{ \frac{I_{p,\lambda}^n f(z)}{I_{p,\lambda}^{n+1} f(z)} \right\} > \alpha, \quad (5)$$

where $I_{p,\lambda}^n f$ is defined by (3).

We let the subclass $\mathcal{H}_{p,\lambda}^-(n; \alpha)$ consists of harmonic functions $f_n = h + \bar{g}_n$ in $\mathcal{H}_{p,\lambda}(n; \alpha)$ so that h and g_n are of the form

$$\begin{aligned} h(z) &= z^p - \sum_{k=p+1}^{\infty} a_k z^k, \quad g_n(z) = (-1)^n \sum_{k=p}^{\infty} b_k z^k, \\ &\quad a_k, b_k \geq 0. \end{aligned} \quad (6)$$

We note that $\mathcal{H}_{p,1}^-(n; \alpha) = \mathcal{H}_p^-(n; \alpha)$, where the class $\mathcal{H}_p^-(n; \alpha)$ was defined and studied by Cotirla [5].

In this paper, we obtain coefficient characterization of the classes $\mathcal{H}_{p,\lambda}^-(n; \alpha)$ and $\mathcal{H}_{p,\lambda}^-(n; \alpha)$. We also obtain extreme points and distortion bounds for functions in the class $\mathcal{H}_{p,\lambda}^-(n; \alpha)$.

2. Coefficient Characterization

Unless otherwise mentioned, we assume throughout this paper that $p \in \mathbb{N}$, $n \in \mathbb{N}_0$, $0 \leq \alpha < 1$, $a_p = 1$, and $\lambda > 0$. We begin with a sufficient condition for functions in $\mathcal{H}_{p,\lambda}^-(n; \alpha)$.

Theorem 1. Let $f = h + \bar{g}$ so that h and g are given by (2). Furthermore, let

$$\sum_{k=p}^{\infty} \{ \Psi_{p,\lambda}(n, k, \alpha) |a_k| + \Phi_{p,\lambda}(n, k, \alpha) |b_k| \} \leq 2, \quad (7)$$

where

$$\begin{aligned} \Psi_{p,\lambda}(n, k, \alpha) &= \left(\left(\frac{p}{p + \lambda(k-p)} \right)^n \right. \\ &\quad \left. - \alpha \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} \right) \\ &\quad \times (1 - \alpha)^{-1}, \end{aligned} \quad (8)$$

$$\begin{aligned} \Phi_{p,\lambda}(n, k, \alpha) &= \left(\left(\frac{p}{p + \lambda(k-p)} \right)^n \right. \\ &\quad \left. + \alpha \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} \right) \\ &\quad \times (1 - \alpha)^{-1}. \end{aligned} \quad (9)$$

Then, f is sense preserving in U and $f \in \mathcal{H}_{p,\lambda}^-(n; \alpha)$.

Proof. According to (2) and (3), we only need to show that

$$\operatorname{Re} \left\{ \frac{I_{p,\lambda}^n f(z) - \alpha I_{p,\lambda}^{n+1} f(z)}{I_{p,\lambda}^{n+1} f(z)} \right\} \geq 0 \quad (z \in U). \quad (10)$$

It follows that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{I_{p,\lambda}^n f(z) - \alpha I_{p,\lambda}^{n+1} f(z)}{I_{p,\lambda}^{n+1} f(z)} \right\} \\ = \operatorname{Re} \{ ((1 - \alpha) z^p \end{aligned}$$

$$\begin{aligned} &+ \sum_{k=p+1}^{\infty} \times \left[\left(\frac{p}{p + \lambda(k-p)} \right)^n \right. \\ &\quad \left. - \alpha \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} \right] a_k z^k \Big) \\ &\times \left(z^p + \sum_{k=p+1}^{\infty} \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} a_k z^k \right. \\ &\quad \left. + (-1)^{n+1} \sum_{k=p}^{\infty} \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} \bar{b}_k \bar{z}^k \right)^{-1} \\ &+ \left((-1)^n \sum_{k=p}^{\infty} \left[\left(\frac{p}{p + \lambda(k-p)} \right)^n \right. \right. \\ &\quad \left. \left. + \alpha \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} \right] \bar{b}_k \bar{z}^k \right) \\ &\times \left(z^p + \sum_{k=p+1}^{\infty} \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} a_k z^k \right. \\ &\quad \left. + (-1)^{n+1} \sum_{k=p}^{\infty} \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} \bar{b}_k \bar{z}^k \right)^{-1} \Big\} \\ &= \operatorname{Re} \{ ((1 - \alpha) \\ &\quad + \sum_{k=p+1}^{\infty} \times \left[\left(\frac{p}{p + \lambda(k-p)} \right)^n \right. \\ &\quad \left. - \alpha \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} \right] a_k z^{k-p} \Big) \\ &\times \left(1 + \sum_{k=p+1}^{\infty} \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} a_k z^{k-p} \right. \\ &\quad \left. + (-1)^{n+1} \sum_{k=p}^{\infty} \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} \bar{b}_k \bar{z}^k z^{-p} \right)^{-1} \\ &+ \left((-1)^n \sum_{k=p}^{\infty} \left[\left(\frac{p}{p + \lambda(k-p)} \right)^n \right. \right. \\ &\quad \left. \left. + \alpha \left(\frac{p}{p + \lambda(k-p)} \right)^{n+1} \right] \right. \\ &\quad \left. \times \bar{b}_k \bar{z}^k z^{-p} \right) \end{aligned}$$

$$\begin{aligned}
& \times \left(1 + \sum_{k=p+1}^{\infty} \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \right. \\
& \quad \times a_k z^{k-p} + (-1)^{n+1} \sum_{k=p}^{\infty} \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \\
& \quad \times \bar{b}_k \bar{z}^k z^{-p} \Big)^{-1} \Big\} \\
& = \operatorname{Re} \left\{ \frac{1-\alpha+A(z)}{1+B(z)} \right\}.
\end{aligned} \tag{11}$$

For $z = re^{i\theta}$, we have

$$\begin{aligned}
A(re^{i\theta}) &= \sum_{k=p+1}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)} \right)^n \right. \\
& \quad \left. - \alpha \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \right] a_k r^{k-p} e^{i(k-p)\theta} \\
& \quad + (-1)^n \sum_{k=p}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)} \right)^n \right. \\
& \quad \left. + \alpha \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \right] \\
& \quad \times \bar{b}_k r^{k-p} e^{-i(k+p)\theta}, \\
B(re^{i\theta}) &= \sum_{k=p+1}^{\infty} \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} a_k r^{k-p} e^{i(k-p)\theta} \\
& \quad + (-1)^{n+1} \sum_{k=p}^{\infty} \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \bar{b}_k r^{k-p} e^{-i(k+p)\theta}.
\end{aligned} \tag{12}$$

Setting that

$$\frac{1-\alpha+A(z)}{1+B(z)} = (1-\alpha) \frac{1+w(z)}{1-w(z)}, \tag{13}$$

the proof will be complete if we can show that $|w(z)| < 1$. Using the condition (7), we can write

$$\begin{aligned}
|w(z)| &= \left| \frac{A(z) - (1-\alpha)B(z)}{A(z) + (1-\alpha)B(z) + 2(1-\alpha)} \right| \\
&= \left| \left(\sum_{k=p+1}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)} \right)^n \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \quad \left. - \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \right] a_k r^{k-p} e^{i(k-p)\theta} \Big) \\
& \quad \times \left(2(1-\alpha) + \sum_{k=p+1}^{\infty} c_k a_k r^{k-p} e^{i(k-p)\theta} \right. \\
& \quad \left. + (-1)^n \sum_{k=p}^{\infty} d_k \bar{b}_k r^{k-p} e^{-i(k+p)\theta} \right)^{-1} \\
& \quad + \left((-1)^n \sum_{k=p}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)} \right)^n \right. \right. \\
& \quad \left. \left. + \alpha \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \right] \right. \\
& \quad \left. \times \bar{b}_k r^{k-p} e^{-i(k+p)\theta} \right) \\
& \quad \times \left(2(1-\alpha) + \sum_{k=p+1}^{\infty} c_k a_k r^{k-p} e^{i(k-p)\theta} \right. \\
& \quad \left. + (-1)^n \sum_{k=p}^{\infty} d_k \bar{b}_k r^{k-p} e^{-i(k+p)\theta} \right)^{-1} \Big| \\
& \leq \left| \left(\sum_{k=p+1}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)} \right)^n \right. \right. \right. \\
& \quad \left. \left. - \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \right] |a_k| r^{k-p} \right) \right. \\
& \quad \times \left(2(1-\alpha) - \sum_{k=p+1}^{\infty} c_k |a_k| r^{k-p} \right. \\
& \quad \left. - \sum_{k=p}^{\infty} d_k |b_k| r^{k-p} \right)^{-1} \\
& \quad + \left(\sum_{k=p}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)} \right)^n \right. \right. \\
& \quad \left. \left. + \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \right] |b_k| r^{k-p} \right) \\
& \quad \times \left(2(1-\alpha) - \sum_{k=p+1}^{\infty} c_k |a_k| r^{k-p} \right. \\
& \quad \left. - \sum_{k=p}^{\infty} d_k |b_k| r^{k-p} \right)^{-1} \Big| \\
& = \left| \left(\sum_{k=p+1}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)} \right)^n \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\left(\frac{p}{p+\lambda(k-p)}\right)^{n+1} \Big] |a_k| r^{k-p} \Big) \\
& \times \left(4(1-\alpha) - \sum_{k=p}^{\infty} \{c_k |a_k| + d_k |b_k|\} r^{k-p} \right)^{-1} \\
& + \left(\sum_{k=p}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)}\right)^n \right. \right. \\
& \quad \left. \left. + \left(\frac{p}{p+\lambda(k-p)}\right)^{n+1} \right] |b_k| r^{k-p} \right) \\
& \times \left(4(1-\alpha) - \sum_{k=p}^{\infty} \{c_k |a_k| + d_k |b_k|\} r^{k-p} \right)^{-1} \Big| \\
& < \left| \left(\sum_{k=p+1}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)}\right)^n \right. \right. \right. \\
& \quad \left. \left. - \left(\frac{p}{p+\lambda(k-p)}\right)^{n+1} \right] |a_k| \right) \right. \\
& \times \left(4(1-\alpha) - \sum_{k=p}^{\infty} \{c_k |a_k| + d_k |b_k|\} \right)^{-1} \\
& + \left(\sum_{k=p}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)}\right)^n \right. \right. \\
& \quad \left. \left. + \left(\frac{p}{p+\lambda(k-p)}\right)^{n+1} \right] |b_k| \right) \\
& \times \left(4(1-\alpha) - \sum_{k=p}^{\infty} \{c_k |a_k| + d_k |b_k|\} \right)^{-1} \Big| \\
& \leq 1,
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
c_k &= \left(\frac{p}{p+\lambda(k-p)}\right)^n + (1-2\alpha) \left(\frac{p}{p+\lambda(k-p)}\right)^{n+1}, \\
d_k &= \left(\frac{p}{p+\lambda(k-p)}\right)^n - (1-2\alpha) \left(\frac{p}{p+\lambda(k-p)}\right)^{n+1}.
\end{aligned} \tag{15}$$

The harmonic functions are as follows:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{1}{\Psi_{p,\lambda}(n, k, \alpha)} x_k z^k$$

$$+ \sum_{k=p}^{\infty} \frac{1}{\Phi_{p,\lambda}(n, k, \alpha)} \overline{y_k z^k}, \tag{16}$$

where $\sum_{k=p+1}^{\infty} |x_k| + \sum_{k=p}^{\infty} |y_k| = 1$ show that the coefficient bound given by (7) is sharp. The functions of the form (8) are in the class $\mathcal{H}_{p,\lambda}(n; \alpha)$ because

$$\begin{aligned}
& \sum_{k=p}^{\infty} \{ \Psi_{p,\lambda}(n, k, \alpha) |a_k| + \Phi_{p,\lambda}(n, k, \alpha) |b_k| \} \\
& = 1 + \sum_{k=p+1}^{\infty} |x_k| + \sum_{k=p}^{\infty} |y_k| = 2.
\end{aligned} \tag{17}$$

This completes the proof of Theorem 1. $\square$

In the following theorem, it is shown that the condition (7) is also necessary for functions $f_n = h + \overline{g_n}$, where h and g_n are of the form (6).

Theorem 2. Let $f_n = h + \overline{g_n}$, where h and g_n are given by (6). Then, $f_n \in \mathcal{H}_{p,\lambda}^-(n; \alpha)$ if and only if

$$\sum_{k=p}^{\infty} \{ \Psi_{p,\lambda}(n, k, \alpha) a_k + \Phi_{p,\lambda}(n, k, \alpha) b_k \} \leq 2, \tag{18}$$

where $\Psi_{p,\lambda}(n, k, \alpha)$ and $\Phi_{p,\lambda}(n, k, \alpha)$ are given by (8) and (9), respectively.

Proof. Since $\mathcal{H}_{p,\lambda}^-(n; \alpha) \subset \mathcal{H}_{p,\lambda}(n; \alpha)$, we only need to prove the “only if” part of the theorem. To this end, for functions $f_n = h + \overline{g_n}$, where h and g_n are given by (6), we notice that the condition $\operatorname{Re}\{I_{p,\lambda}^n f(z)/I_{p,\lambda}^{n+1} f(z)\} > \alpha$ is equivalent to

$$\operatorname{Re}\{((1-\alpha)z^p$$

$$\begin{aligned}
& - \sum_{k=p+1}^{\infty} \times \left[\left(\frac{p}{p+\lambda(k-p)}\right)^n \right. \\
& \quad \left. - \alpha \left(\frac{p}{p+\lambda(k-p)}\right)^{n+1} \right] a_k z^k \Big) \\
& \times \left(z^p - \sum_{k=p+1}^{\infty} \left(\frac{p}{p+\lambda(k-p)}\right)^{n+1} a_k z^k \right. \\
& \quad \left. + (-1)^{2n} \sum_{k=p}^{\infty} \left(\frac{p}{p+\lambda(k-p)}\right)^{n+1} b_k \overline{z^k} \right)^{-1}
\end{aligned}$$

$$\begin{aligned}
& + \left((-1)^{2n-1} \sum_{k=p}^{\infty} \left(\frac{p}{p+\lambda(k-p)} \right)^n \right. \\
& \quad \times \left[\left(\frac{p}{p+\lambda(k-p)} \right)^n \right. \\
& \quad \quad \left. + \alpha \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \right] b_k \bar{z}^k \Bigg) \\
& \times \left(z^p - \sum_{k=p+1}^{\infty} \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} a_k z^k \right. \\
& \quad \left. + (-1)^{2n} \sum_{k=p}^{\infty} \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \bar{b}_k \bar{z}^k \right)^{-1} \Bigg\} \\
& \geq 0.
\end{aligned} \tag{19}$$

The previous required condition (19) must hold for all values of z in U . Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\begin{aligned}
& \left((1-\alpha) - \sum_{k=p+1}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)} \right)^n \right. \right. \\
& \quad \left. \left. - \alpha \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \right] a_k r^{k-p} \right) \\
& \times \left(1 - \sum_{k=p+1}^{\infty} \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} a_k r^{k-p} \right. \\
& \quad \left. + \sum_{k=p}^{\infty} \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} b_k r^{k-p} \right)^{-1} \\
& + \left(- \sum_{k=p}^{\infty} \left[\left(\frac{p}{p+\lambda(k-p)} \right)^n \right. \right. \\
& \quad \left. \left. + \alpha \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} \right] b_k r^{k-p} \right) \\
& \times \left(1 - \sum_{k=p+1}^{\infty} \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} a_k r^{k-p} \right. \\
& \quad \left. + \sum_{k=p}^{\infty} \left(\frac{p}{p+\lambda(k-p)} \right)^{n+1} b_k r^{k-p} \right)^{-1} \\
& \geq 0.
\end{aligned} \tag{20}$$

If the condition (18) does not hold, then the numerator in (20) is negative for r sufficiently close to 1. Hence there exists

$z_0 = r_0$ in $(0, 1)$ for which the quotient in (20) is negative. This contradicts the required condition for $f_n \in \mathcal{H}_{p,\lambda}^-(n; \alpha)$, and so the proof of Theorem 2 is completed. $\square$

3. Extreme Points and Distortion Theorem

Our next theorem is on the extreme points of convex hulls of the class $\mathcal{H}_{p,\lambda}^-(n; \alpha)$ denoted by $clco\mathcal{H}_{p,\lambda}^-(n; \alpha)$.

Theorem 3. Let $f_n = h + \bar{g}_n$, where h and g_n are given by (6). Then, $f_n \in \mathcal{H}_{p,\lambda}^-(n; \alpha)$ if and only if

$$f_n(z) = \sum_{k=p}^{\infty} [x_k h_k(z) + y_k g_{n_k}(z)], \tag{21}$$

where

$$h_1(z) = z^p, \quad h_k(z) = z^p - \frac{1}{\Psi_{p,\lambda}(n, k, \alpha)} z^k$$

$$(k = p+1, p+2, p+3, \dots),$$

$$g_{n_k}(z) = z^p + (-1)^{n-1} \frac{1}{\Phi_{p,\lambda}(n, k, \alpha)} \bar{z}^k \tag{22}$$

$$(k = p, p+1, p+2, \dots),$$

$$x_k, y_k \geq 0, \quad x_p = 1 - \sum_{k=p+1}^{\infty} x_k - \sum_{k=p}^{\infty} y_k.$$

In particular, the extreme points of the class $\mathcal{H}_{p,\lambda}^-(n; \alpha)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. Suppose that

$$\begin{aligned}
f_n(z) &= \sum_{k=p}^{\infty} (x_k h_k(z) + y_k g_{n_k}(z)) \\
&= \sum_{k=p}^{\infty} (x_k + y_k) z^p - \sum_{k=p+1}^{\infty} \frac{1}{\Psi_{p,\lambda}(n, k, \alpha)} x_k z^k \\
&\quad + (-1)^{n-1} \sum_{k=1}^{\infty} \frac{1}{\Phi_{p,\lambda}(n, k, \alpha)} y_k \bar{z}^k.
\end{aligned} \tag{23}$$

Then,

$$\begin{aligned}
& \sum_{k=p+1}^{\infty} \Psi_{p,\lambda}(n, k, \alpha) \left(\frac{1}{\Psi_{p,\lambda}(n, k, \alpha)} x_k \right) \\
& + \sum_{k=1}^{\infty} \Phi_{p,\lambda}(n, k, \alpha) \left(\frac{1}{\Phi_{p,\lambda}(n, k, \alpha)} y_k \right) \\
& = \sum_{k=p+1}^{\infty} x_k + \sum_{k=p}^{\infty} y_k = 1 - x_p \leq 1,
\end{aligned} \tag{24}$$

and so $f_n \in clco\mathcal{H}_{p,\lambda}^-(n; \alpha)$.

Conversely, if $f_n \in clco\mathcal{H}_{p,\lambda}^-(n; \alpha)$, then

$$a_k \leq \frac{1}{\Psi_{p,\lambda}(n, k, \alpha)}, \quad b_k \leq \frac{1}{\Phi_{p,\lambda}(n, k, \alpha)}. \quad (25)$$

Set that

$$\begin{aligned} x_k &= \Psi_{p,\lambda}(n, k, \alpha) a_k \quad (k = p+1, p+2, p+3, \dots), \\ y_k &= \Phi_{p,\lambda}(n, k, \alpha) b_k \quad (k = p, p+1, p+2, \dots). \end{aligned} \quad (26)$$

Then note that by Theorem 2, $0 \leq x_k \leq 1$, ($k = p+1, p+2, p+3, \dots$), and $0 \leq y_k \leq 1$, ($k = p, p+1, p+2, \dots$). We define that $x_p = 1 - \sum_{k=p+1}^{\infty} x_k - \sum_{k=p}^{\infty} y_k$ and note that by Theorem 2, $x_p \geq 0$. Consequently, we obtain $f_n(z) = \sum_{k=p}^{\infty} \{x_k h_k(z) + y_k g_k(z)\}$ as required. $\square$

The following theorem gives the distortion bounds for functions in the class $\mathcal{H}_{p,\lambda}^-(n; \alpha)$ which yields a covering result for this class.

Theorem 4. Let $f_n(z) \in \mathcal{H}_{p,\lambda}^-(n; \alpha)$. Then, for $|z| = r < 1$, we have

$$\begin{aligned} (1 - b_p) r^p - \{\Gamma_{p,\lambda}(n, \alpha) - \Delta_{p,\lambda}(n, \alpha)\} r^{p+1} &\leq |f_n(z)| \\ &\leq (1 + b_p) r^p + \{\Gamma_{p,\lambda}(n, \alpha) - \Delta_{p,\lambda}(n, \alpha) b_p\} r^{p+1}, \end{aligned} \quad (27)$$

where

$$\begin{aligned} \Gamma_{p,\lambda}(n, \alpha) &= \frac{1 - \alpha}{(p/(p+\lambda))^n - \alpha(p/(p+\lambda))^{n+1}}, \\ \Delta_{p,\lambda}(n, \alpha) &= \frac{1 + \alpha}{(p/(p+\lambda))^n - \alpha(p/(p+\lambda))^{n+1}}. \end{aligned} \quad (28)$$

The result is sharp.

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f_n(z) \in \mathcal{H}_{p,\lambda}^-(n; \alpha)$. Taking the absolute value of f_n , we have

$$\begin{aligned} |f_n(z)| &\leq (1 + b_p) r^p + \sum_{k=p+1}^{\infty} (a_k + b_k) r^k \\ &\leq (1 + b_p) r^p + \sum_{k=p+1}^{\infty} (a_k + b_k) r^{p+1} \\ &= (1 + b_p) r^p + \Gamma_{p,\lambda}(n, \alpha) \\ &\quad \times \sum_{k=p+1}^{\infty} \frac{1}{\Gamma_{p,\lambda}(n, \alpha)} (a_k + b_k) r^{p+1} \\ &\leq (1 + b_p) r^p + \Gamma_{p,\lambda}(n, \alpha) r^{p+1} \end{aligned}$$

$$\begin{aligned} &\times \sum_{k=p+1}^{\infty} \{\Psi_{p,\lambda}(n, k, \alpha) a_k + \Phi_{p,\lambda}(n, k, \alpha) b_k\} \\ &\leq (1 + b_p) r^p \\ &\quad + \{\Gamma_{p,\lambda}(n, k, \alpha) - \Delta_{p,\lambda}(n, k, \alpha) b_p\} r^{p+1}. \end{aligned} \quad (29)$$

The bounds given in Theorem 4 for functions $f_n = h + \bar{g}_n$, where h and g_n of form (6), also hold for functions of form (2) if the coefficient condition (7) is satisfied. The upper bound given for $f \in \mathcal{H}_{p,\lambda}^-(n; \alpha)$ is sharp, and the equality occurs for the functions

$$\begin{aligned} f(z) &= z^p + b_p \bar{z}^p \\ &\quad - \left(\frac{1 - \alpha}{(p/(p+\lambda))^n - \alpha(p/(p+\lambda))^{n+1}} \right. \\ &\quad \left. - \frac{1 - \alpha}{(p/(p+\lambda))^n + \alpha(p/(p+\lambda))^{n+1}} b_p \right) \bar{z}^{p+1}, \end{aligned} \quad (30)$$

$$\begin{aligned} f(z) &= z^p + b_p \bar{z}^p \\ &\quad - \left(\frac{1 - \alpha}{(p/(p+\lambda))^n - \alpha(p/(p+\lambda))^{n+1}} \right. \\ &\quad \left. - \frac{1 - \alpha}{(p/(p+\lambda))^n + \alpha(p/(p+\lambda))^{n+1}} b_p \right) z^{p+1} \end{aligned} \quad (31)$$

showing that the bounds given in Theorem 4 are sharp. $\square$

Remark 5. (i) Putting $\lambda = 1$ in the previous results, we obtain the results of Cotirla [5].

(ii) Putting $\lambda = 1$ in the previous results, we obtain the results of Cotirla [6], when $\beta = 0$.

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